M. Bottkol
March 28, 2022

Preliminaries

- Abstractly, we denote the Poincaré group by $\mathcal{P}$ and the Lorentz group by $\mathcal{L}$, so $\mathcal{L} \subset \mathcal{P}$. $\mathcal{L}$ acts on 4-dimensional Minkowski space, denoted by $\mathbb{R}^{3,1}$, as a group of 4×4 matrices (this is called the *fundamental* representation of $\mathcal{L}$). These matrices will generally be denoted by Λ. The Poincaré group acts on $\mathbb{R}^{3,1}$ is a group of inhomogeneous linear transformations

$$\mathbf{x} \;\rightarrow\; \Lambda\mathbf{x} + \mathbf{a}$$

  where the 4-vector $\mathbf{a}$ defines a spacetime translation. Elements of $\mathcal{P}$ in this representation may be then written as $p(\Lambda\mathbf{x}, \mathbf{a})$.

- $\mathcal{P}$ has an infinite dimensional (reducible) unitary representation as a group of operators acting on the Hilbert space

$$\mathscr{H} = L^2(\mathbb{R}^{3,1}, d^4\mathbf{x})$$

  Here $d^4\mathbf{x}$ is viewed as a positive measure, invariant under that action of $\mathcal{P}$. The action of an element $(\Lambda, a) \in \mathcal{P}$ on a function $\psi \in \mathscr{H}$ is defined by group translation, $T_p : \psi(\mathbf{x}) \to \psi(p^{-1}\mathbf{x})$:

$$\rho(\Lambda, a) : \psi(\mathbf{x}) \to \; \psi'(\mathbf{x}) = \psi(\Lambda^{-1}(\mathbf{x} - \mathbf{a})) = \psi(\mathbf{x}')$$

  This representation is unitary because $\det\Lambda = 1$, which is why we say that $d^4\mathbf{x}$ is a Poincaré invariant measure:

$$\int d^4\mathbf{x}\; \psi(\mathbf{x}) = \int d^4\mathbf{x}\; \psi(\Lambda^{-1}(\mathbf{x} - \mathbf{a})) = \int d^4\mathbf{x}'\; \psi(\mathbf{x}')$$

- The unitary infinite dimensional action of $\mathcal{P}$ on $\mathscr{H} = L^2(\mathbb{R}^{3,1}, d^4\mathbf{x})$ is generated by a 10-dimensional Lie algebra. Generators of this Lie algebra can be represented as 10 first-order linear differential operators, in their Hermitian form given by

$$\mathrm{J}_1 = \quad i\left(x^3\partial_2 - x^2\partial_3\right) \quad \mathrm{J}_2 = \quad i\left(x^1\partial_3 - x^3\partial_1\right) \quad \mathrm{J}_3 = \quad i\left(x^2\partial_1 - x^1\partial_2\right)$$

$$\mathrm{K}_1 = -i\left(x^1\partial_0 + x^0\partial_1\right) \quad \mathrm{K}_2 = -i\left(x^2\partial_0 + x^0\partial_2\right) \quad \mathrm{K}_3 = -i\left(x^3\partial_0 + x^0\partial_3\right)$$

$$\mathrm{H} = i\partial_0 \qquad \mathrm{P}_1 = -i\partial_1 \qquad \mathrm{P}_2 = -i\partial_2 \qquad \mathrm{P}_3 = -i\partial_3$$

  Thus, for example, the action of a rotation about the x_3 axis by angle θ is given by

$$\psi(\mathbf{x}) \to \; \psi'(\mathbf{x}) = \psi\left(e^{-iJ_3\theta}\mathbf{x}\right) = \psi(x^0, x^1\cos\theta + x^2\sin\theta, -x^1\sin\theta + x^2\cos\theta, x^3)$$

  The Lie algebra generators $(J_\ell, K_\ell, H, P_\ell)$ satisfy a canonical set of commutation relations, which will be discussed in more detail below. These commutation relations must necessarily be satisfied by any generators of a Poincaré group representation.

Irreducibility: spin 0 representations of mass m

The infinite dimensional unitary representation $\rho(\Lambda, \mathbf{a})$ of the Poincaré group on $L^2(\mathbb{R}^{3,1}, d^4\mathbf{x})$, given above, is *not* irreducible. The purpose of everything that follows is to construct an irreducible representation.

Reducibility means the following. ρ acting on $\mathscr{H}$ is said to be *reducible* if there are subspaces $\mathscr{H}_m \subset \mathscr{H}$ which are invariant under every operator $\rho(\Lambda, \mathbf{a})$ in the representation. If ρ is reducible, and if we restrict ρ to an invariant subspace $\mathscr{H}_m \subset \mathscr{H}$ we get a *subrepresentation* of ρ. If there are no further invariant subspaces in $\mathscr{H}_m$ we say the the subrepresentation is *irreducible*.

Determining irreducibility turns out to be closely tied to determining a set of *Casimir operators*. Casimir operators are defined as operators which commute with every element of the Lie algebra.

Lemma 1 *(Schur's Lemma) A representation ρ is irreducible if and only if every Casimir of its Lie algebra is a multiple of the identity operator.*

Lemma 2 *Let $\{J_\ell, K_m, H, P_k\}$ be the generators of the Poincaré group representation ρ given above, acting on the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^{3,1}, d^4\mathbf{x})$. Define the operator*

$$M^2 = H^2 - P_1^2 - P_2^2 - P_3^2 \tag{1}$$

Then

1. *M^2 commutes with all of the generators $\{J_\ell, K_m, H, P_k\}$, i.e. M^2 is a Casimir for the representation ρ*
2. *for every $m > 0$, M^2 has an infinite dimensional eigenspace*

$$\mathscr{H}_m = \{\psi \in \mathscr{H} : M^2\psi = m^2\psi\} \subset \mathscr{H}$$

It follows that $\mathscr{H}_m$ is invariant under the action of ρ, and so restricting ρ to $\mathscr{H}_m$ gives us a subrepresentation.

See discussion of the Klein-Gordon equation below. It is not claimed that the subrepresentation ρ_m is irreducible. It will turn out however that ρ_m is in fact irreducible. This will follow from

Lemma 3 *The representation ρ generated by $\{J_\ell, K_m, H, P_k\}$ on $\mathscr{H} = L^2(\mathbb{R}^{3,1})$ is spin 0. This means that the operator*

$$S^2 = W^\mu W_\mu \tag{2}$$

vanishes identically on $\mathscr{H}$, and is therefore (trivially) a Casimir operator for the representation ρ. Here W_μ is the Pauli-Lubanski vector

$$W_\mu = \begin{bmatrix} \mathbf{J} \cdot \mathbf{P} \\ H\mathbf{J} - \mathbf{P} \times \mathbf{K} \end{bmatrix}$$

Each of the 4 operators in this vector vanishes identically on $\mathscr{H}$.

Casimirs

In the present case, $S^2 = 0$ is a trivial Casimir, and only the M^2 Casimir has influence. It turns out there are no other Casimirs in our representation of the Poincaré Lie group. This is essentially because the representation acts on scalar functions, rather than spinors.

Lemma 4 *M^2 is the only Casimir for the Poincaré group Lie algebra representation given above. Consequently (by Schur's lemma), the subrepresentation ρ_m acts irreducibly on the M^2 eigenspace $\mathscr{H}_m \subset \mathscr{H}$.*

Summary

According to the above discussion, we have now found irreducible representations of the Poincaré group $\mathcal{P}$ indexed by $m > 0$. Namely, $\mathcal{P}$ acts irreducibly on the subspace $\mathscr{H}_m$, the eigenspace of M^2 with eigenvalue m^2. In the Wigner classification these are the spin 0 representations.

But now comes the interesting part! We can give an explicit construction of this representation. This will make use of the mass shell in momentum space.

The Invariant Subspace $\mathscr{H}_m \subset \mathscr{H}$

In the scalar case, eigenstates of the operator (1) are functions satisfying the Klein-Gordon equation

$$-M^2\psi_m = \left(\frac{\partial^2}{\partial t^2} - \Delta\right)\psi_m = -m^2\psi_m \tag{3}$$

where Δ is the spatial Laplace operator on $\mathbb{R}^3$. Fixing the eigenvalue m^2, the linear eigenspace $\mathscr{H}_m \subset L^2(\mathbb{R}^{3,1})$ of M^2 is invariant under the action of the Poincaré group, and this action is irreducible. $\mathscr{H}_m$ is spanned by dispersive plane-wave solutions of the Klein-Gordon equation,

$$\psi_m(t, \mathbf{x}) = e^{i(Et - \mathbf{p}\cdot\mathbf{x})} \quad E^2 - |\mathbf{p}|^2 = m^2 \tag{4}$$

and the general function in $\mathscr{H}_m$ is representable as a superposition of these plane waves, whose wave 4-vectors $k = (E, p)$ are confined to a mass shell.

Integration over a Mass Shell

The eigenspace $\mathscr{H}_m$ is spanned by dispersive plane-waves (4). The dispersion relation $E^2 - |\mathbf{p}|^2 = m^2$ has a geometrical interpretation: the energy-momentum 4-vector $p = (E, \mathbf{p})$ defining the plane wave is confined to a 3-dimensional hyperboloid sheet, or *mass shell* in Minkowski space,

$$\Sigma(m) = \{(E, \mathbf{p}) \in \mathbb{R}^4 : E^2 - |\mathbf{p}| = m^2 > 0\} \subset \mathbb{R}^{3,1}$$

A general function in $\mathscr{H}_m$ is represented as a superposition of plane waves of mass m,

$$\psi(t, \mathbf{x}) = \int_{(E,\mathbf{p})\in\Sigma(m)} \hat{\psi}(E, \mathbf{p}) e^{i(Et - \mathbf{p}\cdot\mathbf{x})} d\mu$$

where $\hat{\psi}(E, \mathbf{p})$ is a weight function and $d\mu$ is a positive measure on the mass shell $\Sigma(m)$. Introducing $\mathbf{p}$ as coordinates on the mass shell, it is natural to take the nice Lorentz-invariant measure (see Appendix I)

$$d\mu = \frac{d^3\mathbf{p}}{E_\mathbf{p}} \qquad E_\mathbf{p} = \sqrt{m^2 + |\mathbf{p}|^2}$$

which lets us write the general function in $\mathscr{H}_m$ as a type of Fourier integral over 3-space:

$$\psi(t, \mathbf{x}) = \int \frac{d^3\mathbf{p}}{E_\mathbf{p}}\ e^{i(E_\mathbf{p} t - \mathbf{p}\cdot\mathbf{x})}\hat{\psi}(\mathbf{p}) \tag{5}$$

This gives an expression for the general element $\psi(t, \mathbf{x})$ in the invariant subspace $\mathscr{H}_m \subset \mathscr{H}$ in terms of a "weight function" $\hat{\psi}(\mathbf{p})$ defined on the mass shell $\Sigma(m)$.

The next step is essentially to throw away the $\psi(t, \mathbf{x})$ functions and work just with the weight functions $\hat{\psi}(\mathbf{p})$ on $\Sigma(m)$. These functions can be taken to lie in the Hilbert space $\hat{\mathscr{H}}_m$ of functions on $\Sigma(m)$ with invariant inner product

$$\langle\hat{\psi}_1|\hat{\psi}_2\rangle = \int \frac{d^3\mathbf{p}}{E_p}\ \hat{\psi}_1^*(\mathbf{p})\hat{\psi}_2(\mathbf{p})$$

We now define an irreducible representation of the Poincaré group on the Hilbert space $\hat{\mathscr{H}}_m$..

The Poincaré Group $\mathcal{P}$ acting on mass shell Hilbert space $\hat{\mathscr{H}}_m$

Above we defined 10 generators $\{J_\ell, K_m, H, P_k\}$ for the Lie algebra of the Poincaré group acting on $\mathscr{H} = L^2(\mathbb{R}^{3,1}, d^4\mathbf{x})$. These operators satisfy a set of commutation relations

$$
\begin{array}{lll}
[\mathrm{J}_2, \mathrm{J}_3] = i\mathrm{J}_1 & [\mathrm{K}_2, \mathrm{K}_3] = -i\mathrm{J}_1 & [\mathrm{J}_2, \mathrm{K}_3] = i\mathrm{K}_1 \\
[\mathrm{J}_3, \mathrm{J}_1] = i\mathrm{J}_2 & [\mathrm{K}_3, \mathrm{K}_1] = -i\mathrm{J}_2 & [\mathrm{J}_3, \mathrm{K}_1] = i\mathrm{K}_2 \\
[\mathrm{J}_1, \mathrm{J}_2] = i\mathrm{J}_3 & [\mathrm{K}_1, \mathrm{K}_2] = -i\mathrm{J}_3 & [\mathrm{J}_1, \mathrm{K}_2] = i\mathrm{K}_3
\end{array}
$$

$$
\begin{array}{lll}
[\mathrm{J}_2, \mathrm{P}_3] = i\mathrm{P}_1 & [\mathrm{K}_1, \mathrm{P}_1] = -i\mathrm{H} & [\mathrm{K}_1, \mathrm{H}] = -i\mathrm{P}_1 \\
[\mathrm{J}_3, \mathrm{P}_1] = i\mathrm{P}_2 & [\mathrm{K}_2, \mathrm{P}_2] = -i\mathrm{H} & [\mathrm{K}_2, \mathrm{H}] = -i\mathrm{P}_2 \\
[\mathrm{J}_1, \mathrm{P}_2] = i\mathrm{P}_3 & [\mathrm{K}_3, \mathrm{P}_3] = -i\mathrm{H} & [\mathrm{K}_3, \mathrm{H}] = -i\mathrm{P}_3
\end{array}
$$

all other commutators being zero. It can be shown that any representation of the Poincaré group has a basis of generators satisfying these commutation relations.

The 10 generators of the Lie algebra of $\mathcal{P}$ acting on the mass shell Hilbert space $\hat{\mathscr{H}}_m$ are as follows (Appendix II)

$$
\mathrm{J}_1 = i\left(p^3\partial_2 - p^2\partial_3\right) \quad \mathrm{J}_2 = i\left(p^1\partial_3 - p^3\partial_1\right) \quad \mathrm{J}_3 = i\left(p^2\partial_1 - p^1\partial_2\right)
$$

$$
\mathrm{K}_1 = -iE_p\partial_1 \qquad \mathrm{K}_2 = -iE_p\partial_2 \qquad \mathrm{K}_3 = -iE_p\partial_3
$$

$$
\mathrm{H} = E_p \qquad \mathrm{P}_1 = -p_1 \qquad \mathrm{P}_2 = -p_2 \qquad \mathrm{P}_3 = -p_3
$$

here $\partial_k = \partial/\partial p_k,\ k = 1,2,3$ and the spacetime translation generators (H, P_k) are Hermitian operators which act by multiplication,

$$
H\psi = E_p \cdot \psi \quad P_k\psi = -p_k \cdot \psi
$$

One checks that the Hermitian operators $\{J_\ell, K_m, H, P_k\}$ defined here do indeed satisfy the canonical commutations relations of the Poincaré group given above.

Appendix I: Mass Shell Geometry

The mass shell $\Sigma(m)$ is representable as 3-dimensional parametrized surface in Minkowski momentum space

$$\begin{bmatrix} E \\ p^1 \\ p^2 \\ p^3 \end{bmatrix} = \begin{bmatrix} \sqrt{m^2+|\mathbf{p}|^2} \\ p^1 \\ p^2 \\ p^3 \end{bmatrix}$$

Taking $\mathbf{p} = (p^1, p^2, p^3)$ as surface parameters, by differentiating we get 3 tangent basis vectors at each point $\sigma(\mathbf{p}) \in \Sigma(m)$:

$$\mathbf{w}_1(\mathbf{p}) = \begin{bmatrix} p^1/\sqrt{m^2+|\mathbf{p}|^2} \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{w}_2(\mathbf{p}) = \begin{bmatrix} p^2/\sqrt{m^2+|\mathbf{p}|^2} \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{w}_3(\mathbf{p}) = \begin{bmatrix} p^3/\sqrt{m^2+|\mathbf{p}|^2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Taking the Lorentz metric with signature $(-.+,+,+)$, the Lorentz metric is positive definite on a tangent space to $\Sigma(m)$, given in our basis vectors by the 3×3 metric tensor

$$g_{ij}(\mathbf{p}) = (\mathbf{w}_i, \mathbf{w}_j)_{\text{Lorentz}} \quad \Leftrightarrow \quad \mathrm{G} = \mathrm{I}_{3\times 3} - \frac{\mathbf{p}\mathbf{p}^{\mathrm{T}}}{m^2+|\mathbf{p}|^2}$$

This makes the mass shell $\Sigma(m)$ into a Riemannian manifold[1] with volume form

$$\mathrm{vol}_{\Sigma(m)} = \sqrt{\det \mathrm{G}}\; dp^1 \wedge dp^2 \wedge dp^3 = \frac{m}{\sqrt{m^2+|\mathbf{p}|^2}}\; dp^1 \wedge dp^2 \wedge dp^3$$

This volume form on $\Sigma(m)$ is used to construct a Lebesgue measure $d\mu_m$ on $\Sigma(m)$ and a Hilbert space $\hat{\mathcal{H}}_m = L^2(\Sigma(m), d\mu_m)$ of functions on $\Sigma(m)$ with inner product

$$\langle \hat\psi_1 | \hat\psi_2 \rangle = \int \hat\psi_1^*(\mathbf{p}) \hat\psi_2(\mathbf{p}) d\mu_m \qquad d\mu_m = \frac{d^3\mathbf{p}}{E_p} \quad E_p = \sqrt{m^2+|\mathbf{p}|^2}$$

The Lorentz group leaves the mass shell Σ_m invariant and, since the measure $d\mu_m$ is Lorentz invariant, acts on functions in $\mathcal{H}_m$ as an infinite dimensional unitary representation of the Lorentz group. In has the following Lie algebra generators, given in Hermitian form by (see Appendix II)

$$\mathrm{J}_1 = i\left(p^3\partial_2 - p^2\partial_3\right) \qquad \mathrm{J}_2 = i\left(p^1\partial_3 - p^3\partial_1\right) \qquad \mathrm{J}_3 = i\left(p^2\partial_1 - p^1\partial_2\right)$$

$$\mathrm{K}_1 = -i\sqrt{m^2+|\mathbf{p}|^2}\;\partial_1 \quad \mathrm{K}_2 = -i\sqrt{m^2+|\mathbf{p}|^2}\;\partial_2 \quad \mathrm{K}_3 = -i\sqrt{m^2+|\mathbf{p}|^2}\;\partial_3$$

These satisfy the canonical commutation relations of the Lie algebra of the Lorentz group

$$\begin{array}{lll} [\mathrm{J}_2, \mathrm{J}_3] = i\mathrm{J}_1 & [\mathrm{K}_2, \mathrm{K}_3] = -i\mathrm{J}_1 & [\mathrm{J}_2, \mathrm{K}_3] = i\mathrm{K}_1 \\ [\mathrm{J}_3, \mathrm{J}_1] = i\mathrm{J}_2 & [\mathrm{K}_3, \mathrm{K}_1] = -i\mathrm{J}_2 & [\mathrm{J}_3, \mathrm{K}_1] = i\mathrm{K}_2 \\ [\mathrm{J}_1, \mathrm{J}_2] = i\mathrm{J}_3 & [\mathrm{K}_1, \mathrm{K}_2] = -i\mathrm{J}_3 & [\mathrm{J}_1, \mathrm{K}_2] = i\mathrm{K}_3 \end{array}$$

[1] indeed, a Riemannian manifold with a transitive group of isometries, and so having constant curvature. I haven't done the work, but feel this must be a manifold of constant negative Riemannian curvature.

Appendix II: Action of $\mathcal{P}$ on $\hat{\mathscr{H}}_m$, and its generators

Boost along x_1 axis acting on $\Sigma(m) \subset \mathbb{R}^{3,1}$:

$$\begin{aligned} E_p \to E'_p = &= \sqrt{m^2 + |\mathbf{p}|^2}\cosh\alpha + p^1 \sinh\alpha \\ \mathbf{p} \to \mathbf{p}' &= \begin{bmatrix} \sqrt{m^2 + |\mathbf{p}|^2}\sinh\alpha + p^1\cosh\alpha \\ p^2 \\ p^3 \end{bmatrix} \end{aligned}$$

The transformation $\mathbf{p} \to \mathbf{p}'$ preserves the measure $d\mu_m = d^3\mathbf{p}/\sqrt{m^2 + |\mathbf{p}|^2}$ on Σ_m:

$$\begin{aligned} dp'_1 &= \frac{\mathbf{p}\cdot d\mathbf{p}}{\sqrt{m^2 + |\mathbf{p}|^2}}\sinh\alpha + \cosh\alpha dp^1 \\ &= \left(\frac{p_1}{\sqrt{m^2 + |\mathbf{p}|^2}}\sinh\alpha + \cosh\alpha\right)dp^1 + \frac{1}{\sqrt{m^2 + |\mathbf{p}|^2}}(p^2 dp_2 + p^3 dp_3) \end{aligned}$$

$$dp'_1 \wedge dp'_2 \wedge dp'_3 = \frac{\sqrt{m^2 + |\mathbf{p}|^2}\cosh\alpha + p^1\sinh\alpha}{\sqrt{m^2 + |\mathbf{p}|^2}} dp^1 \wedge dp^2 \wedge dp^3 = \frac{E'_p}{E_p} dp_1 \wedge dp_2 \wedge dp_3$$

or

$$\frac{dp_1 \wedge dp_2 \wedge dp_3}{E_p} \to \frac{dp'_1 \wedge dp'_2 \wedge dp'_3}{E'_p}$$

Lie Algebra Generators for the Lorentz Group

As an example, compute the infinitesimal boost operator acting on $L^2(\Sigma_m, d\mu_m)$

$$iK_1\psi = \frac{d}{d\alpha}\Big|_{\alpha=0}\psi(\sqrt{m^2 + |\mathbf{p}|^2}\sinh\alpha + p^1\cosh\alpha, p^2, p^3) = \sqrt{m^2 + |\mathbf{p}|^2}\ \partial_1\psi$$

Infinitesimal rotation operators have the usual representations, e.g. $\mathrm{J}_3 = i\left(p^2\partial_1 - p^1\partial_2\right)$. We can write down 6 Hermitian generators for the Lie algebra representation:

$$\mathrm{J}_1 = i\left(p^3\partial_2 - p^2\partial_3\right) \quad \mathrm{J}_2 = i\left(p^1\partial_3 - p^3\partial_1\right) \quad \mathrm{J}_3 = i\left(p^2\partial_1 - p^1\partial_2\right)$$

$$\mathrm{K}_1 = -i\sqrt{m^2 + |\mathbf{p}|^2}\ \partial_1 \quad \mathrm{K}_2 = -i\sqrt{m^2 + |\mathbf{p}|^2}\ \partial_2 \quad \mathrm{K}_3 = -i\sqrt{m^2 + |\mathbf{p}|^2}\ \partial_3$$

One checks that these operators do indeed satisfy the commutation relations

$$\begin{array}{lll} [\mathrm{J}_2, \mathrm{J}_3] = i\mathrm{J}_1 & [\mathrm{K}_2, \mathrm{K}_3] = -i\mathrm{J}_1 & [\mathrm{J}_2, \mathrm{K}_3] = i\mathrm{K}_1 \\ [\mathrm{J}_3, \mathrm{J}_1] = i\mathrm{J}_2 & [\mathrm{K}_3, \mathrm{K}_1] = -i\mathrm{J}_2 & [\mathrm{J}_3, \mathrm{K}_1] = i\mathrm{K}_2 \\ [\mathrm{J}_1, \mathrm{J}_2] = i\mathrm{J}_3 & [\mathrm{K}_1, \mathrm{K}_2] = -i\mathrm{J}_3 & [\mathrm{J}_1, \mathrm{K}_2] = i\mathrm{K}_3 \end{array} \tag{6}$$

Lie Algebra Generators for Spacetime Translation

We make use of equation (5).

$$\begin{aligned} \psi(t + \Delta t, \mathbf{x}) &= \int \frac{d^3\mathbf{p}}{E_\mathbf{p}}\ e^{i(E_\mathbf{p}t - \mathbf{p}\cdot\mathbf{x})}\left(e^{iE_p\Delta t}\hat{\psi}(\mathbf{p})\right) \\ \psi(t, \mathbf{x} + \Delta\mathbf{x}) &= \int \frac{d^3\mathbf{p}}{E_\mathbf{p}}\ e^{i(E_\mathbf{p}t - \mathbf{p}\cdot\mathbf{x})}\left(e^{-i\mathbf{p}\cdot\Delta\mathbf{x}}\hat{\psi}(\mathbf{p})\right) \end{aligned}$$

Thus spacetime translations act on functions in $\hat{\mathscr{H}}_m$ by multiplication by a phase factor. The corresponding infinitesimal generators, given Hermitian character, become

$$\mathrm{H} = E_p \quad \mathrm{P}_1 = -p^1 \quad \mathrm{P}_2 = -p^2 \quad \mathrm{P}_3 = -p^3$$

For instance H is the Hermitian operator $H : \psi(\mathbf{p}) \to E_p\psi(\mathbf{p})$. These translation generators are found to satisfy the following commutation relations

$$\begin{array}{lll} [\mathrm{J}_2, \mathrm{P}_3] = i\mathrm{P}_1 & [\mathrm{K}_1, \mathrm{P}_1] = -i\mathrm{H} & [\mathrm{K}_1, \mathrm{H}] = -i\mathrm{P}_1 \\ [\mathrm{J}_3, \mathrm{P}_1] = i\mathrm{P}_2 & [\mathrm{K}_2, \mathrm{P}_2] = -i\mathrm{H} & [\mathrm{K}_2, \mathrm{H}] = -i\mathrm{P}_2 \\ [\mathrm{J}_1, \mathrm{P}_2] = i\mathrm{P}_3 & [\mathrm{K}_3, \mathrm{P}_3] = -i\mathrm{H} & [\mathrm{K}_3, \mathrm{H}] = -i\mathrm{P}_3 \end{array}$$