Making sense of QFT

Lecture 3: Dirac's interactions

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Outline

1. Multiparticle Hilbert space $\mathcal{H}_N$
2. Non-interacting representation of the Poincaré group in $\mathcal{H}_N$
3. Dirac's forms of dynamics
4. Interaction in the instant form
5. N-particle theories of Bakamjian-Thomas and Coester-Polyzou

Hilbert spaces of multiparticle systems

- One-particle Hilbert space = space $\hbar_i$ of a unitary irreducible representation of the Poincaré group
- N-particle Hilbert space $\mathcal{H}_N$ is obtained as a tensor product (with proper symmetrization or antisymmetrization) of one-particle spaces

$$\mathcal{H}_N = \hbar_1 \otimes \hbar_2 \otimes \cdots \otimes \hbar_N$$

- N-particle Hilbert space $\mathcal{H}_N$ contains operators of basic observables for each particle, satisfying Poincaré commutators individually

$$(\boldsymbol{p}_1, \boldsymbol{j}_1, \boldsymbol{k}_1, h_1)$$
$$(\boldsymbol{p}_2, \boldsymbol{j}_2, \boldsymbol{k}_2, h_2)$$
$$\ldots$$
$$(\boldsymbol{p}_N, \boldsymbol{j}_N, \boldsymbol{k}_N, h_N)$$

- Operators of different particles commute, e.g.,

$$[\boldsymbol{p}_1, \boldsymbol{k}_2] = 0$$

Unitary representation of the Poincaré group in $\mathcal{H}_N$

- According to Wigner, a quantum theory in $\mathcal{H}_N$ will be relativistic if we managed to construct 10 Hermitian operators $(\boldsymbol{P}, \boldsymbol{J}, \boldsymbol{K}, H)$ satisfying Poincaré commutators.
- One approach that seems natural: build $(\boldsymbol{P}, \boldsymbol{J}, \boldsymbol{K}, H)$ as functions of $(\boldsymbol{p}_i, \boldsymbol{j}_i, \boldsymbol{k}_i, h_i)$
- One obvious solution:

$$\boldsymbol{P}_0 = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \cdots + \boldsymbol{p}_N$$
$$\boldsymbol{J}_0 = \boldsymbol{j}_1 + \boldsymbol{j}_2 + \cdots + \boldsymbol{j}_N$$
$$\boldsymbol{K}_0 = \boldsymbol{k}_1 + \boldsymbol{k}_2 + \cdots + \boldsymbol{k}_N$$
$$H_0 = h_1 + h_2 + \cdots + h_N$$

  corresponds to N non-interacting particles
- Next important problem: *construct the interacting representation* $(\boldsymbol{P}, \boldsymbol{J}, \boldsymbol{K}, H)$ *of the Poincaré Lie algebra in* $\mathcal{H}_N$. Generally

$$(\boldsymbol{P}, \boldsymbol{J}, \boldsymbol{K}, H) \neq (\boldsymbol{P}_0, \boldsymbol{J}_0, \boldsymbol{K}_0, H_0)$$

Unitary representation of the Poincaré group in $\mathcal{H}_N$

- Important! Once we constructed representation $(\boldsymbol{P}, \boldsymbol{J}, \boldsymbol{K}, H)$, we know *everything* about our N-particle system! We can answer all questions about physics. For example:

  - Diagonalize H and obtain energies and wave functions of bound states
  - Calculate S-matrix and scattering amplitudes (more about that later)
  - Find time evolution of particle observables:

$$\boldsymbol{p}_2(t) = e^{iH\mathrm{t}/\hbar}\boldsymbol{p}_2 e^{-iH\mathrm{t}/\hbar} = \boldsymbol{p}_2 + \frac{it}{\hbar}[H, \boldsymbol{p}_2] - \frac{t^2}{2!\hbar^2}[H,[H, \boldsymbol{p}_2]] + \cdots$$

  - Find boost transformations of observables:

$$x_1(\theta) = e^{-icK_y\theta/\hbar}\, x_1 e^{icK_y\theta/\hbar} = x_1 - \frac{ic\theta}{\hbar}[K_y, x_1] - \frac{c^2\theta^2}{2!\hbar^2}\left[K_y, [K_y, x_1]\right] + \cdots$$

  - etc, etc.

Representation of the Poincaré Lie algebra in $\mathcal{H}_N$

- In non-relativistic physics, interaction V was added to the Hamiltonian:

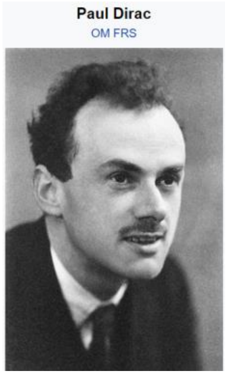
$$\boldsymbol{P} = \boldsymbol{P}_0$$
$$\boldsymbol{J} = \boldsymbol{J}_0$$
$$\boldsymbol{K} = \boldsymbol{K}_0$$
$$H = H_0 + V$$

- This method does not work in relativistic physics, because it violates Poincaré commutators, e.g.

$$[P_x, K_x] = [P_{0x}, K_{0x}] = \frac{i\hbar}{c^2} H_0 \neq \frac{i\hbar}{c^2}(H_0 + V) = \frac{i\hbar}{c^2} H$$

- This means that "interactions" should be present either in momentum $\boldsymbol{P}$ or in boost $\boldsymbol{K}$ or in both
- In addition to "potential energy" V we have to consider "potential momentum" or/and "potential boost".

Dirac's relativistic forms of dynamics (1949)



- *Point form*: interaction is present in the generator of translations:

$$\boldsymbol{P} = \boldsymbol{P}_0 + \boldsymbol{W}$$
$$\boldsymbol{J} = \boldsymbol{J}_0$$
$$\boldsymbol{K} = \boldsymbol{K}_0$$
$$H = H_0 + V$$

- *Instant form*: interaction is present in the generator of boosts:

$$\boldsymbol{P} = \boldsymbol{P}_0$$
$$\boldsymbol{J} = \boldsymbol{J}_0$$
$$\boldsymbol{K} = \boldsymbol{K}_0 + \boldsymbol{Z}$$
$$H = H_0 + V$$

- Instant form is preferable, because space translations are known experimentally to be simple, universal and interaction-independent.

Attempts to construct instant form dynamics in $\mathcal{H}_N$

- *Idea:* build potential energy V and potential boost $\boldsymbol{Z}$ operators as functions of one-particle observables $(\boldsymbol{p}_i, \boldsymbol{j}_i, \boldsymbol{k}_i, h_i)$.
- For relativistic invariance V and $\boldsymbol{Z}$ must satisfy a non-trivial set of commutation relations with $(\boldsymbol{P}_0, \boldsymbol{J}_0, \boldsymbol{K}_0, H_0)$

$$[\boldsymbol{J}_0, V] = [\boldsymbol{P}_0, V] = 0, \tag{6.26}$$

$$[Z_i, P_{0j}] = -\frac{i\hbar\delta_{ij}}{c^2} V, \tag{6.27}$$

$$[J_{0i}, Z_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} Z_k, \tag{6.28}$$

$$[K_{0i}, Z_j] + [Z_i, K_{0j}] + [Z_i, Z_j] = 0, \tag{6.29}$$

$$[\boldsymbol{Z}, H_0] + [\boldsymbol{K}_0, V] + [\boldsymbol{Z}, V] = 0. \tag{6.30}$$

Attempts to construct instant form dynamics in $\mathcal{H}_N$

- Bakamjian-Thomas (1953), Coester-Polyzou (1982), many others
- Multiple problems:
  - difficult to ensure cluster separability of interaction
  - mathematical complexity
  - difficult to describe systems with variable numbers of particles
- In real physical systems the number of particles is often not conserved (decays, reactions, light emission and absorption, etc). So, it is not wise to limit our theory to a fixed number of particles in $\mathcal{H}_N$.
- Can we build a general Hilbert space, where systems with all possible numbers of particles coexist?

Thank you!