Making sense of QFT

Lecture 4: Weinberg's fields

by Eugene Stefanovich

Outline

1. Fock space $\mathcal{H}_{Fock}$
2. Creation-annihilation operators
3. Non-interacting representation of the Poincaré group in $\mathcal{H}_{Fock}$
4. Weinberg's trick
5. The role of quantum fields

Fock space

Vladimir Fock

- Fock space is built as a direct sum of N-particle spaces:

$$\mathcal{H}_{Fock} = |vac\rangle \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$$

- It contains states of any system with any number of particles.
- A general state $|\Psi\rangle \in \mathcal{H}_{Fock}$ is a superposition of states with different number of particles, e.g.,
  (atom in an excited state) + (atom in the ground state and a photon)

Fock space

- Each *j*-particle sector $\mathcal{H}_j$ has a natural non-interacting representation of the Poincaré group U_g^j. Then one can easily build the non-interacting representation in the entire Fock space

$$U_g^0 = 1 \oplus U_g^1 \oplus U_g^2 \oplus \cdots$$

- *Good news*: we have a complete description of systems with any number of non-interacting particles.
- *Bad news*: a very cumbersome notation for operators of observables.
  For example, the total energy is written as

$$H_0 = 0 \oplus \sqrt{p^2c^2 + m^2c^4} \oplus \left(\sqrt{p_1^2c^2 + m^2c^4} + \sqrt{p_2^2c^2 + m^2c^4}\right) \oplus \cdots$$

  This is a theoretical nightmare.

Fock space (creation and annihilation operators)

- *Creation operator* $a^{\dagger}_{\boldsymbol{p}\sigma}$ acts on $|\Psi\rangle$ and adds to this state one particle with momentum $\boldsymbol{p}$ and spin projection/helicity σ. (If the particle is a fermion and the state $|\boldsymbol{p}\sigma\rangle$ was already present in $|\Psi\rangle$, then $a^{\dagger}_{\boldsymbol{p}\sigma}|\Psi\rangle = 0$).
- *Annihilation operator* $a_{\boldsymbol{p}\sigma}$ acts on $|\Psi\rangle$ and removes from this state one particle with momentum $\boldsymbol{p}$ and spin projection/helicity σ. (If the state $|\boldsymbol{p}\sigma\rangle$ was not present in $|\Psi\rangle$, then $a_{\boldsymbol{p}\sigma}|\Psi\rangle = 0$).

Fock space (creation and annihilation operators)

- Creation/annihilation operators for fermions (e.g., electrons) satisfy anticommutators

$$\{a^\dagger_{\boldsymbol{p}\sigma}, a^\dagger_{\boldsymbol{p}'\sigma'}\} = \{a_{\boldsymbol{p}\sigma}, a_{\boldsymbol{p}'\sigma'}\} = 0$$
$$\{a^\dagger_{\boldsymbol{p}\sigma}, a_{\boldsymbol{p}'\sigma'}\} = \delta(\boldsymbol{p}-\boldsymbol{p}')\delta_{\sigma\sigma'}$$

- Creation/annihilation operators for bosons (e.g., photons) satisfy commutators

$$[c^\dagger_{\boldsymbol{p}\tau}, c^\dagger_{\boldsymbol{p}'\tau'}] = [c_{\boldsymbol{p}\tau}, c_{\boldsymbol{p}'\tau'}] = 0$$
$$[c^\dagger_{\boldsymbol{p}\tau}, c_{\boldsymbol{p}'\tau'}] = \delta(\boldsymbol{p}-\boldsymbol{p}')\delta_{\tau\tau'}$$

Some physical operators in the Fock space

- Number of electrons with momentum $\boldsymbol{p}$ and spin σ

$$\left(a_{\boldsymbol{p}\sigma}^{\dagger}a_{\boldsymbol{p}\sigma}\right)|\Psi\rangle = |\Psi\rangle \qquad \text{if electron state } (\boldsymbol{p}\sigma) \text{ was present in } |\Psi\rangle$$

$$\left(a_{\boldsymbol{p}\sigma}^{\dagger}a_{\boldsymbol{p}\sigma}\right)|\Psi\rangle = 0 \qquad \text{if electron state } (\boldsymbol{p}\sigma) \text{ was not present in } |\Psi\rangle$$

- Total number of electrons

$$N_e = \sum_{\sigma} \int d\boldsymbol{p}\, a_{\boldsymbol{p}\sigma}^{\dagger}a_{\boldsymbol{p}\sigma}$$

- Total energy of non-interacting electrons (Poincaré generator)

$$H_0 = \sum_{\sigma} \int d\boldsymbol{p} \sqrt{p^2c^2 + m^2c^4}\, a_{\boldsymbol{p}\sigma}^{\dagger}a_{\boldsymbol{p}\sigma}$$

- This is much better than what we had before

$$H_0 = 0 \oplus \sqrt{p^2c^2 + m^2c^4} \oplus \left(\sqrt{p_1^2c^2 + m^2c^4} + \sqrt{p_2^2c^2 + m^2c^4}\right) \oplus \cdots$$

Non-interacting generators in the Fock space

- Similar formulas exist for $\boldsymbol{P}_0, \boldsymbol{J}_0, \boldsymbol{K}_0$.
- By using exponential functions of the 10 generators we can build the non-interacting unitary representation of the Poincaré group
$$U_g = U_0(\Lambda, \tilde{a})$$
- Conclusion: we have a complete description for any number of non-interacting particles in a convenient notation.

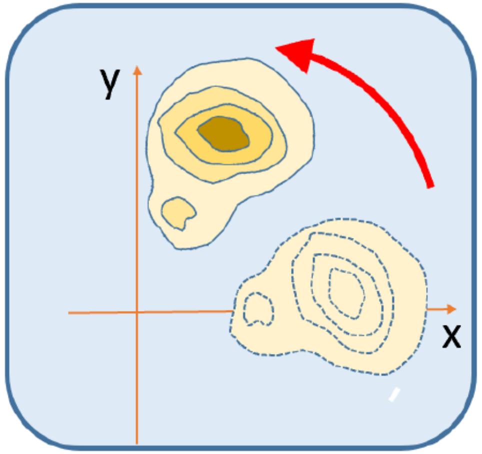
Fock space (interaction)

- Assume that the Fock space interaction is in Dirac's instant form. Then the full interacting representation of the Poincaré Lie algebra in $\mathcal{H}_{Fock}$ may be written as

$$\boxed{\begin{aligned} \boldsymbol{P} &= \boldsymbol{P}_0 \\ \boldsymbol{J} &= \boldsymbol{J}_0 \\ \boldsymbol{K} &= \boldsymbol{K}_0 + \boldsymbol{Z} \\ H &= H_0 + V \end{aligned}}$$

- Most important next step: get expressions for interaction operators V and $\boldsymbol{Z}$ *through creation and annihilation operators*.
- All physics has been reduced to the construction of only 4 operators V, $\boldsymbol{Z}$!
- This problem is very challenging mathematically, because the operators V and $\boldsymbol{Z}$ must satisfy a non-trivial set of commutation relations.

Weinberg's method



- Suppose that we managed to build an operator function (called *potential energy density*)

$$V(t, x, y, z) \equiv V(\tilde{x})$$

with the following properties:

  - It transforms as a scalar with respect to the noninteracting representation of the Poincaré group

$$U_0(\Lambda; \tilde{a}) V(\tilde{x}) U_0^{-1}(\Lambda; \tilde{a}) = V(\Lambda(\tilde{x} + \tilde{a})). \tag{3.8}$$

  - It commutes with itself at space-like intervals, i.e.,

$$[V(t, \boldsymbol{x}), V(t, \boldsymbol{y})] = 0, \quad \text{if } \boldsymbol{x} \neq \boldsymbol{y}. \tag{3.9}$$

Weinberg's method

- Then the interacting Hamiltonian and boost satisfying all requirements, can be constructed as

$$H = H_0 + V = H_0 + \int d\boldsymbol{x} V(0, \boldsymbol{x}), \tag{3.10}$$

$$\boldsymbol{K} = \boldsymbol{K}_0 + \boldsymbol{Z} = \boldsymbol{K}_0 - \frac{1}{c^2} \int d\boldsymbol{x} \boldsymbol{x} V(0, \boldsymbol{x}). \tag{3.11}$$

- For a proof see Appendix E.1 in *E.S. EPT, Volume 2*
- How can we build $V(0, \boldsymbol{x})$?

Weinberg's method (quantum fields)

- Suppose that for each particle-antiparticle species we defined an operator function $\phi_\alpha(t, \boldsymbol{x})$ (quantum field) with the following properties:
  - it is linear in creation and annihilation operators of the particle and antiparticle
  - it has a simple transformation law with respect to the non-interacting representation of the Poincaré group

    $$U_0(\Lambda; \tilde{a})\phi_\alpha(\tilde{x})U_0^{-1}(\Lambda; \tilde{a}) = \sum_\beta D_{\alpha\beta}(\Lambda^{-1})\phi_\beta(\Lambda(\tilde{x} + \tilde{a})) \qquad (3.1)$$

    where $D_{\alpha\beta}(\Lambda^{-1})$ are matrices of some finite-dimensional representation of the Lorentz subgroup
  - We require following anticommutators for fermion fields

    $$\{\psi_\alpha(t, \boldsymbol{x}), \psi_\beta^\dagger(t, \boldsymbol{y})\} = \delta(\boldsymbol{x} - \boldsymbol{y})\delta_{\alpha\beta}, \qquad (3.2)$$

    $$\{\psi_\alpha(t, \boldsymbol{x}), \psi_\beta(t, \boldsymbol{y})\} = \{\psi_\alpha^\dagger(t, \boldsymbol{x}), \psi_\beta^\dagger(t, \boldsymbol{y})\} = 0. \qquad (3.3)$$

Weinberg's method (quantum fields)

- and commutators for boson fields

$$[\phi_\alpha(t,\mathbf{x}),\phi_\beta^\dagger(t,\mathbf{y})] = \delta(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}, \qquad (3.4)$$

$$[\phi_\alpha(t,\mathbf{x}),\phi_\beta(t,\mathbf{y})] = [\phi_\alpha^\dagger(t,\mathbf{x}),\phi_\beta^\dagger(t,\mathbf{y})] = 0. \qquad (3.5)$$

- Then it can be proven that the potential energy density operator with all required properties can be build as a product of several quantum fields at the same "space-time points"

$$V(t,\mathbf{x}) = \sum_{\alpha,\beta,\gamma\ldots} G_{\alpha\beta\gamma\ldots}\phi_\alpha(t,\mathbf{x})\psi_\beta(t,\mathbf{x})\chi_\gamma(t,\mathbf{x})\cdots,$$

  with properly selected coefficients *G.*
- For explicit construction of field operators and proofs see
  S. Weinberg "The quantum theory of fields". Vol. 1, Chapter 5
  E.S. EPT Vol. 2, Appendices B,C

Example of Weinberg's method: Quantum electrodynamics

- **electron-positron quantum field**:

$$\psi_a(\tilde{x}) \equiv \psi_a(t, \boldsymbol{x})$$
$$= \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^{3/2}} \sqrt{\frac{m_e c^2}{\omega_{\boldsymbol{p}}}} \sum_{s_z} (e^{-\frac{i}{\hbar}\tilde{p}\cdot\tilde{x}} u_a(\boldsymbol{p}, s_z) a_{\boldsymbol{p}s_z} + e^{\frac{i}{\hbar}\tilde{p}\cdot\tilde{x}} v_a(\boldsymbol{p}, s_z) b^\dagger_{\boldsymbol{p}s_z}). \qquad \text{(B.34)}$$

u_a and v_a are 4-component bispinors, which are carefully selected to satisfy all requirements for the fields.

- **photon quantum field:**

$$\mathcal{A}_\mu(\tilde{x}) \equiv \mathcal{A}_\mu(t, \boldsymbol{x})$$
$$= \frac{\hbar c}{(2\pi\hbar)^{3/2}} \int \frac{d\boldsymbol{p}}{\sqrt{2pc}} \sum_{\tau} [e^{-\frac{i}{\hbar}\tilde{p}\cdot\tilde{x}} e_\mu(\boldsymbol{p}, \tau) c_{\boldsymbol{p}\tau} + e^{\frac{i}{\hbar}\tilde{p}\cdot\tilde{x}} e^*_\mu(\boldsymbol{p}, \tau) c^\dagger_{\boldsymbol{p}\tau}], \qquad \text{(C.2)}$$

e_μ is a 4-component polarization function,

- **Potential energy density of QED**

$$V(\tilde{x}) = -e\overline{\psi}(\tilde{x})\gamma^\mu \psi(\tilde{x}) A_\mu(\tilde{x})$$

is relativistically invariant in the Wigner-Dirac-Weinberg sense.

Example of Weinberg's method: Quantum electrodynamics

- Now, according to Wigner-Dirac principles, we have a full interacting representation of the Poincaré Lie algebra in the Fock space of electrons, positrons and photons (protons etc. can be added in a similar way),

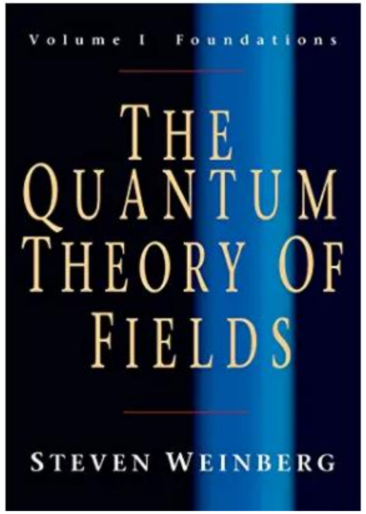
$$H = H_0 + V = H_0 + \int d\boldsymbol{x} V(0, \boldsymbol{x}), \tag{3.10}$$

$$\boldsymbol{K} = \boldsymbol{K}_0 + \boldsymbol{Z} = \boldsymbol{K}_0 - \frac{1}{c^2} \int d\boldsymbol{x}\boldsymbol{x} V(0, \boldsymbol{x}). \tag{3.11}$$

- Now we should be able to answer <u>any</u> question related to the physics of such systems: bound states, scattering, decays, reactions, etc, etc.
- Weinberg's method leads to the same quantum field theory as in all other textbooks, but it offers a completely different intepretation and physical picture.

Two alternative ways to interpret Quantum Field Theory

Steven Weinberg



interpretation of QFT	particles	quantum fields	mathematical description
standard way	"excitations" of fields	primary ingredients of Nature	field Lagrangians and field equations
Weinberg's way	primary ingredients of Nature	useful mathematical tools	unitary representation of the Poincaré group

Electron-positron quantum field

- satisfies Dirac equation

$$\left(\gamma^0 \frac{\partial}{\partial t} + c\boldsymbol{\gamma} \frac{\partial}{\partial \boldsymbol{x}}\right) \psi(t, \boldsymbol{x}) = \frac{imc^2}{\hbar} \psi(t, \boldsymbol{x})$$

- $\psi(t, \boldsymbol{x})$ should not be interpreted as a wave function promoted to operator ("second quantization"):
  - quantum field $\psi(t, \boldsymbol{x})$ has 4 components while electron (positron) wave functions have only two components (spin up and spin down).
  - quantum field $\psi(t, \boldsymbol{x})$ does not have probabilistic interpretation
  - Poincaré transformations of the quantum field $\psi(t, \boldsymbol{x})$ are non-unitary. They are very different from unitary transformations of Wigner's wave functions.
- Dirac equation should not be interpreted as a relativistic analog of the Schrödinger equation.
  - good results for the spectrum of the hydrogen atom are likely to be a coincidence.

Weinberg's method (conclusions)

- Quantum fields $\psi(t, x, y, z)$ are just auxiliary mathematical tools, similar to annihilation $a_{\boldsymbol{p}\sigma}$ and creation $a_{\boldsymbol{p}\sigma}^{\dagger}$ operators.
- Quantum fields are need only for one purpose - construction of the potential energy density $V(\tilde{x}) = -e\overline{\psi}(\tilde{x})\gamma^{\mu}\psi(\tilde{x})A_{\mu}(\tilde{x})$. All other tasks can be accomplished without resorting to fields.
- Currently, there is no other regular method to construct a non-trivial representation of the Poincaré group in the Fock space. That's how all existing QFT models are constructed today.
- However, there is no proof that Weinberg's way is the only way to build a quantum relativistic interacting theory in the Fock space. We will see in future lectures that other possibilities exist.

Few more words about interpretation

- Are we sure that arguments of quantum fields $\psi(t,x,y,z)$ and energy densities $V(t,x,y,z)$ must be interpreted as time and space coordinates?
- My answer is **no**, because the only purpose of these quantities is to help with construction of space-time independent interaction operators

$$H = H_0 + V = H_0 + \int d\boldsymbol{x} V(0, \boldsymbol{x}), \tag{3.10}$$

$$\boldsymbol{K} = \boldsymbol{K}_0 + \boldsymbol{Z} = \boldsymbol{K}_0 - \frac{1}{c^2} \int d\boldsymbol{x}\boldsymbol{x} V(0, \boldsymbol{x}). \tag{3.11}$$

where "time" is set to zero and "space coordinates" are dummy integration variables.

- Our theory has true physical time as a parameter of Poincaré time shifts and Newton-Wigner coordinates of particles in each Fock sector. These quantities are unrelated to the arguments (t,x,y,z) of quantum fields.

Thank you!