Making sense of QFT

Lecture 2: Wigner's particles. Q&A, Exercises

by Eugene Stefanovich

Outline:

Q1: What is the physical meaning of *irreducible* representations of the Poincaré group?

Q2: What is the physical meaning of *reducible* representations of the Poincaré group?

Q3: What is the physical meaning of Casimir operators?

Q4: How come that electron's wave function has 2 components, but Dirac's field has 4 components?

Exercise 1: Prove $e^{tB}A\, e^{-tB} = A + t[B, A] + \frac{t^2}{2!}\big[B, [B, A]\big] + \cdots$

Exercise 2: How do components of momentum transform under rotations?

Exercise 3: Show that energy-momentum $\left(H, cP_x, cP_y, cP_z\right)$ form components of a relativistic 4-vector.

Exercise 4: Show that mass squared $M^2 = (H^2 - P^2c^2)/c^4$ is a Casimir operator.

Exercise 5: Build Hilbert space $\mathcal{H}$ of a massive spinless representation of the Poincaré group.

Q1: What is the physical meaning of *irreducible* representations of the Poincaré group?

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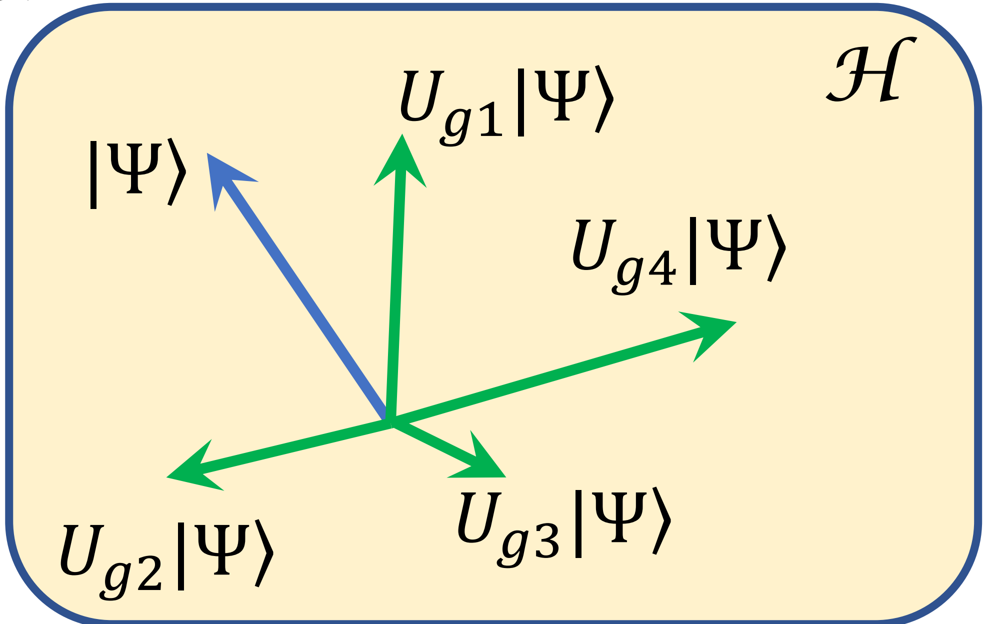
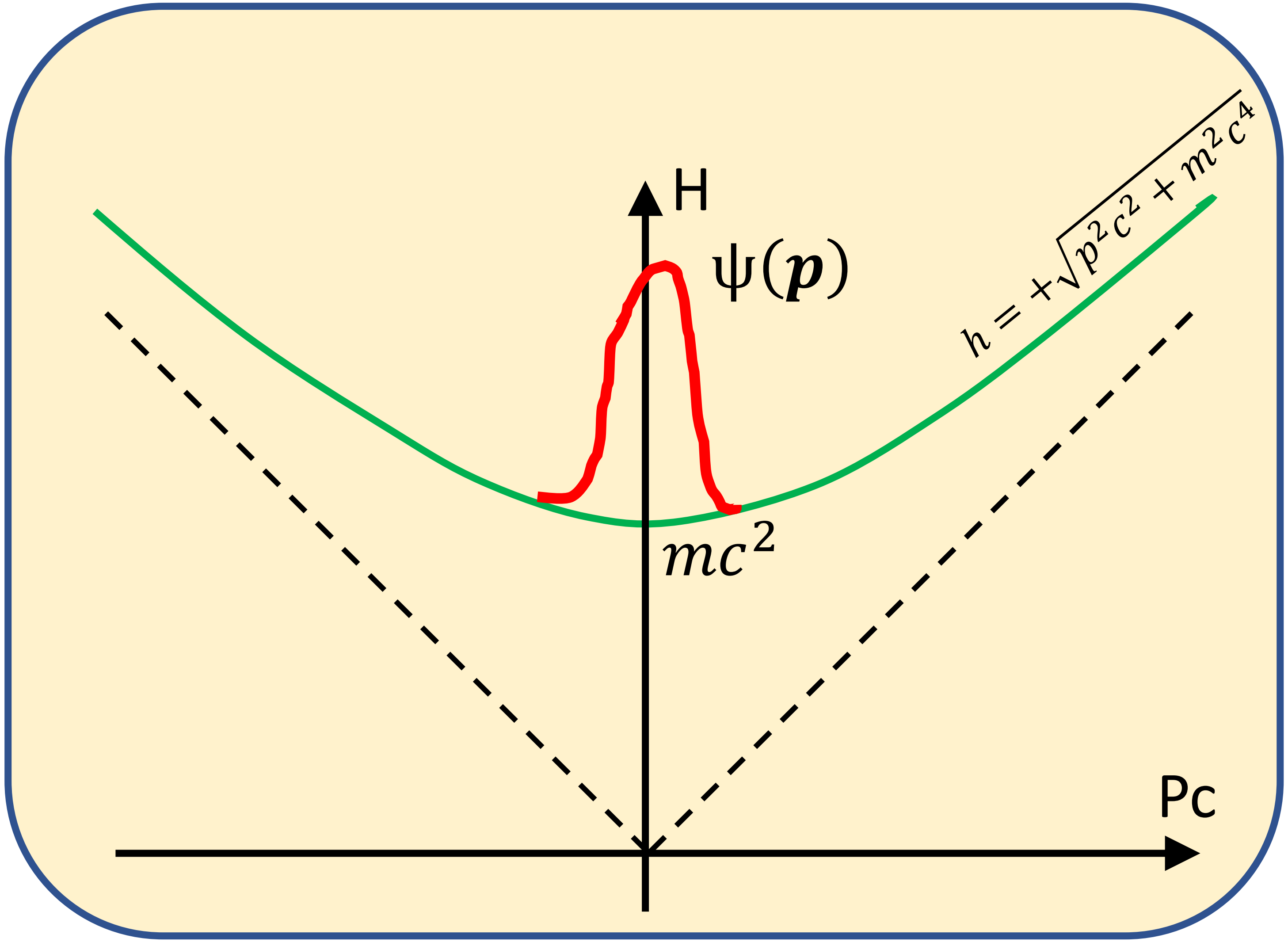
- Theorem: Unitary representations U_g of a group is irreducible if and only if it is *cyclic.*
- The meaning of a *cyclic* representation:
  - if we take any vector $|\Psi\rangle \in \mathcal{H}$ from the representation Hilbert space,
  - apply to $|\Psi\rangle$ representation operators U_g for all group elements g,
  - form a linear span (all possible linear combinations) of all images $U_g|\Psi\rangle$,
  - then we obtain the entire Hilbert space $\mathcal{H}$.



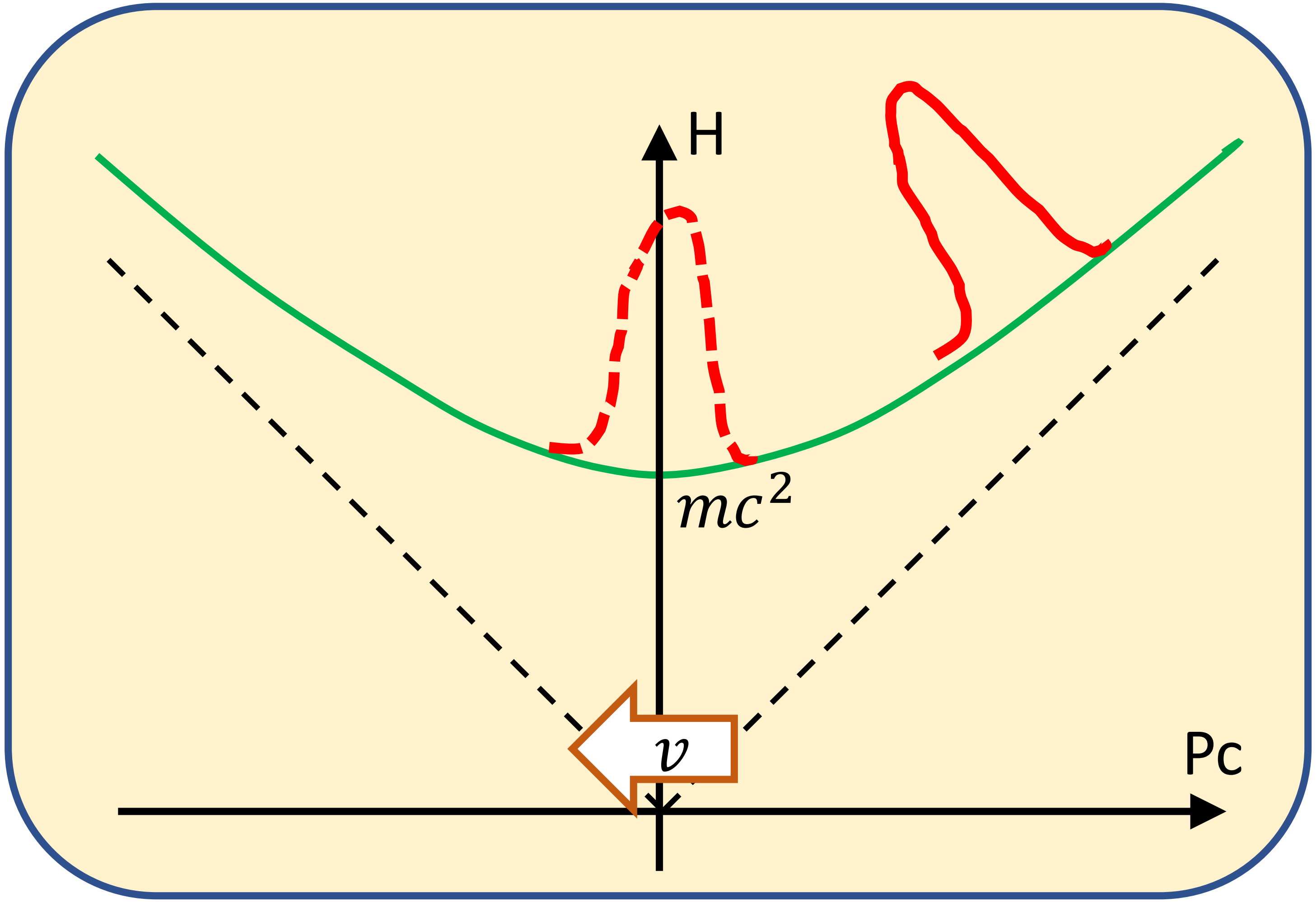
Q1: What is the physical meaning of *irreducible* representations of the Poincaré group? (continued)

- Example: Spinless massive particle



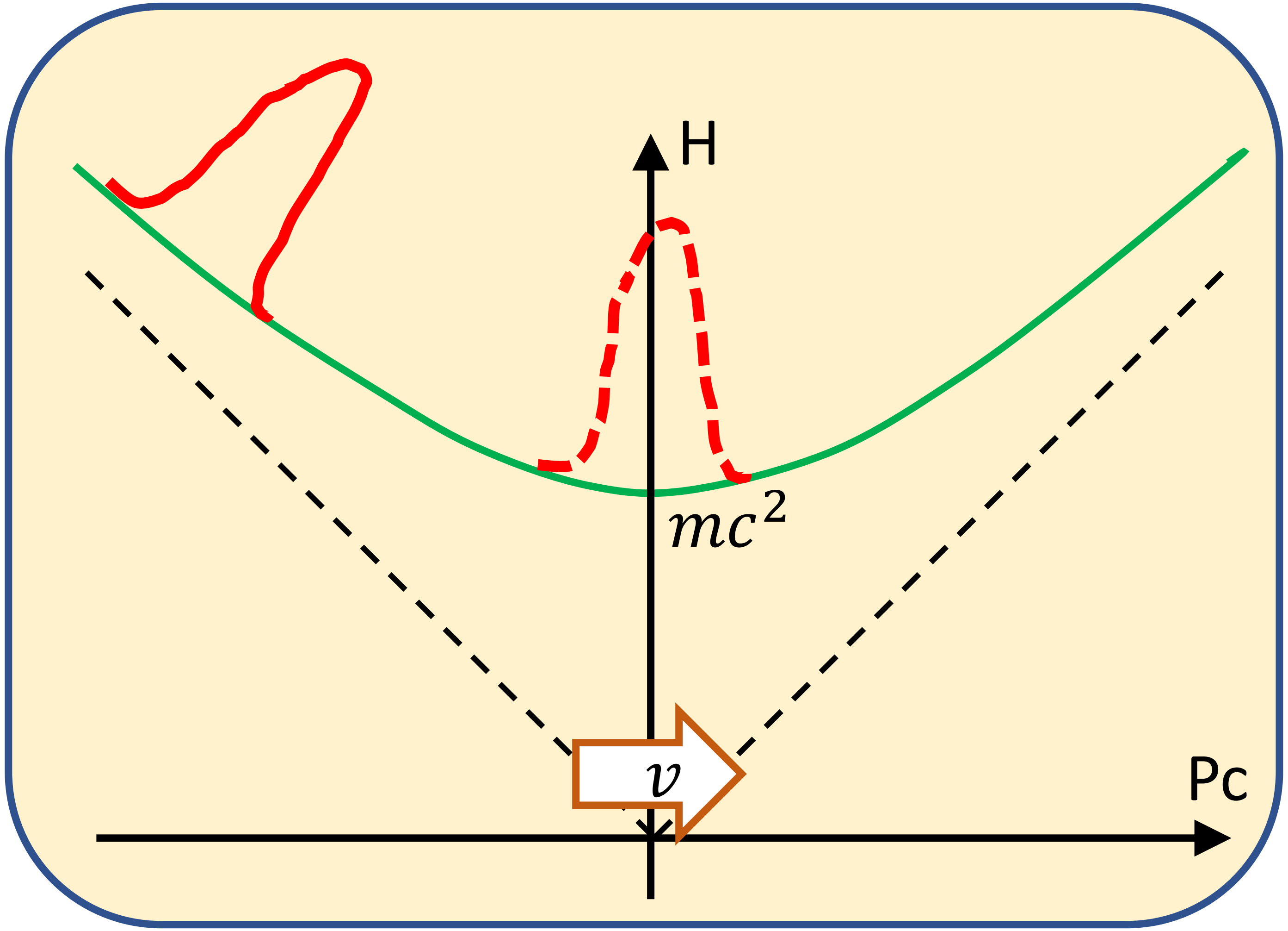
Q1: What is the physical meaning of *irreducible* representations of the Poincaré group? (continued)

- Example: Spinless massive particle
- Observer is moving to the left



Q1: What is the physical meaning of *irreducible* representations of the Poincaré group? (continued)

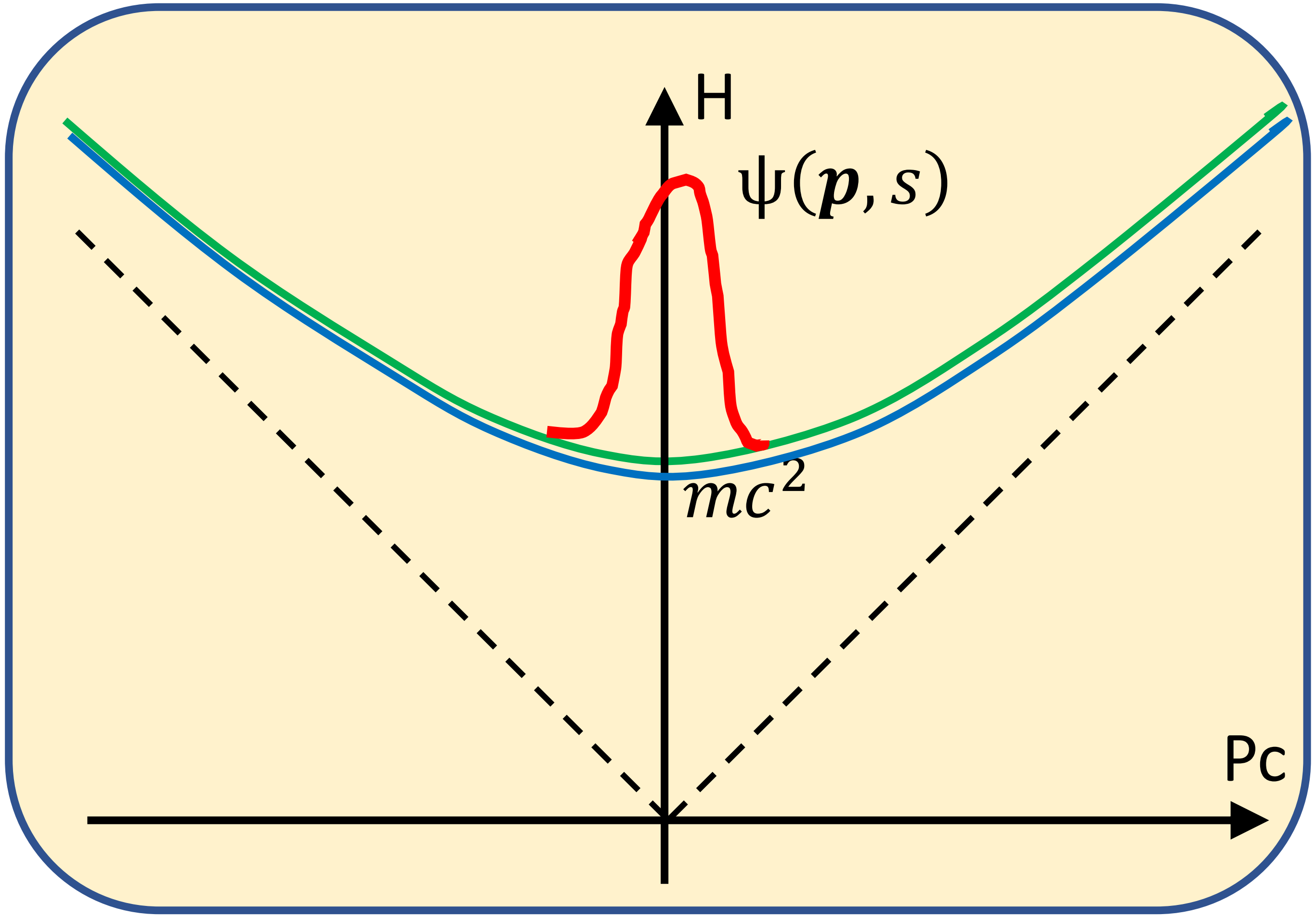
- Example: Spinless massive particle
- Observer is moving to the right.



Conclusion: changes of observer allow to explore the entire Hilbert space => representation is cyclic => representation is *irreducible*

Q1: What is the physical meaning of *irreducible* representations of the Poincaré group? (continued)

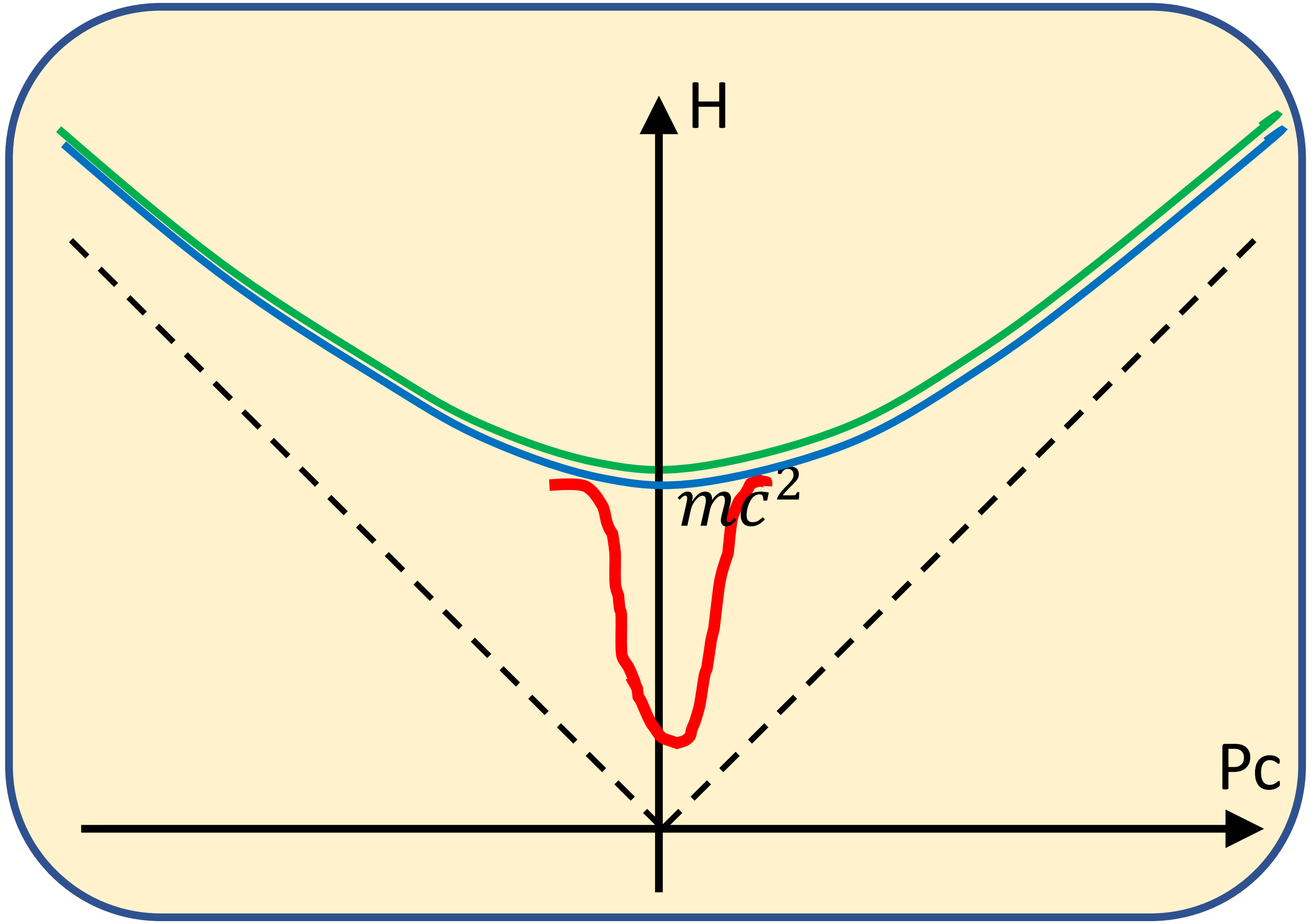
- Example: Massive particle with spin 1/2



spin up $\boldsymbol{s = 1/2}$
spin down $s = -1/2$

Q1: What is the physical meaning of *irreducible* representations of the Poincaré group? (continued)

- Example: Massive particle with spin 1/2
- After rotation by 180 degrees the spin projection changes its sign.



spin up $s = 1/2$

spin down $s = -1/2$

Conclusion: changes of observer allow to explore the entire Hilbert space => representation is cyclic => representation is *irreducible*

Q2: What is the physical meaning of *reducible* representations of the Poincaré group?

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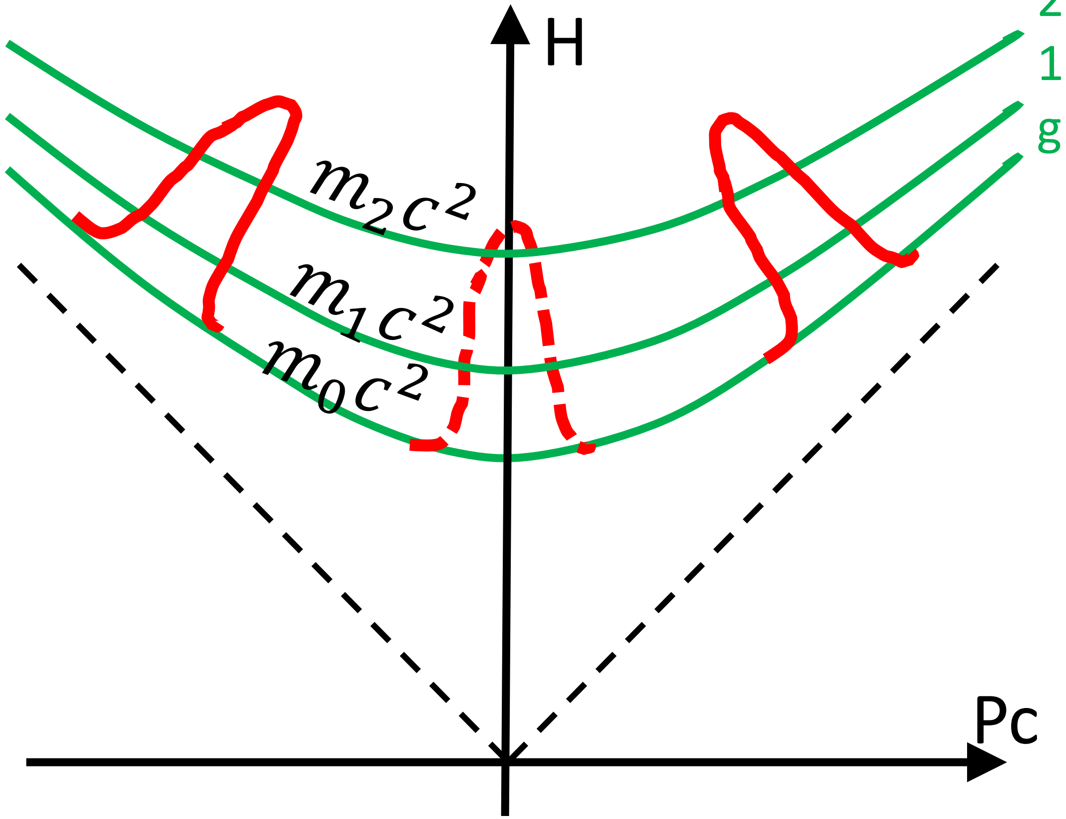
- Example: Multiparticle bound state (e.g., hydrogen atom) is described by a *reducible representation* of the Poincaré group.
- If the system is in its ground state, then this is true for all inertial observers.
- By changing observer, one cannot "excite" the atom.



Conclusions:

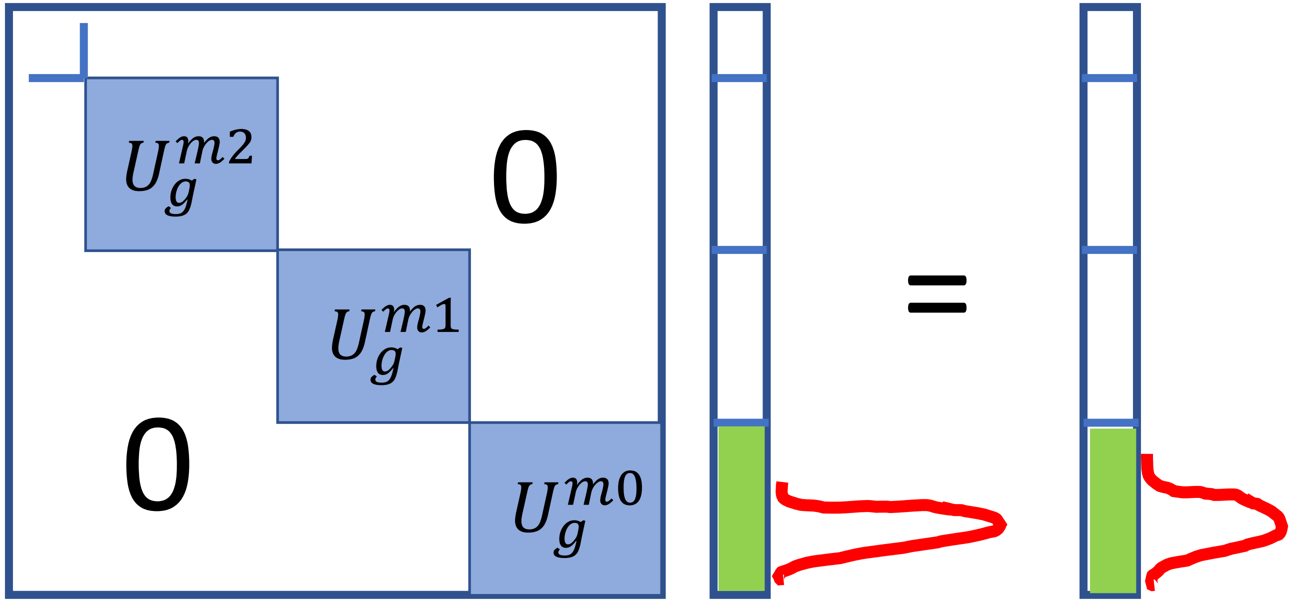
- Representation in the Hilbert space of the hydrogen atom is a direct sum of irreducible ones:

$$U_g = U_g^{m0} \oplus U_g^{m1} \oplus U_g^{m2} \oplus \ldots$$

- One mass eigenstate of the atom behaves as an "elementary" system.

Q2: What is the physical meaning of *reducible* representations of the Poincaré group? (continued)

- Representation matrices have block-diagonal form, so state vector cannot leave irreducible subspace.



$$U_g|\Psi\rangle \quad = \quad |\Psi'\rangle$$

Q3: What is the physical meaning of Casimir operators?

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- Casimir operators M^2 (mass squared) and S^2 (spin squared) commute with all 10 generators of the Poincaré group.
- This means that these operators remain invariant under all inertial transformations, e.g.

$$e^{-i\boldsymbol{P}\cdot\mathbf{x}/\hbar}\, M^2\, e^{i\boldsymbol{P}\cdot\mathbf{x}/\hbar} = M^2 \qquad \text{(space translations)}$$

$$e^{-i\boldsymbol{J}\cdot\boldsymbol{\phi}/\hbar} M^2 e^{i\boldsymbol{J}\cdot\boldsymbol{\phi}/\hbar} = M^2 \qquad \text{(rotations)}$$

$$e^{-ic\boldsymbol{K}\cdot\boldsymbol{\theta}/\hbar}\, M^2\, e^{ic\boldsymbol{K}\cdot\boldsymbol{\theta}/\hbar} = M^2 \qquad \text{(boosts)}$$

$$e^{iH\mathrm{t}/\hbar}\, M^2\, e^{-iH\mathrm{t}/\hbar} = M^2 \qquad \text{(time translations)}$$

- This means that observables M^2 and S^2 have the same values (spectrum) for all inertial observers.
- These are true intrinsic observer-independent properties of the system.
- There are other intrinsic properties, like charge, flavor, etc. But they are not related to space and time symmetries. They are related to the way particles interact with each other.

Q4: How come that electron's wave function has 2 components, but Dirac's field has 4 components?

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- Electron's wave function $\psi_\sigma\,(\boldsymbol{p})$ (or its position-space analog $\psi_\sigma(\boldsymbol{r}, t)$) and Dirac's quantum field $\Psi_i(\boldsymbol{x}, t)$ are two completely different objects. They are different both physically and mathematically.
- Let's return to this question in Lecture 4 after studying Weinberg's method for introducing quantum fields.

Exercise 1: Prove that identity

$$e^{tB} A\, e^{-tB} = A + t[B, A] + \frac{t^2}{2!}\big[B, [B, A]\big] + \cdots$$

is valid for any two operators A, B and number t

Exercise 1: Prove $e^{tB} A\, e^{-tB} = A + t[B, A] + \frac{t^2}{2!}[B, [B, A]] + \cdots$

Plan:

1. First check that both sides of this identity satisfy the same differential equation

$$dX/dt = [B, X] = BX - XB \quad (1)$$

2. Check that both sides coincide at initial condition $t = 0$.

1a. Take t-derivative of the left hand side

$$d/dt\,(e^{tB} A\, e^{-tB}) = d/dt\,(e^{tB}) A\, e^{-tB} + e^{tB} A\, d/dt\,(e^{-tB})$$
$$= (Be^{tB})\, A\, e^{-tB} - e^{tB} A (e^{-tB} B)$$
$$= B(e^{tB} A\, e^{-tB}) - (e^{tB} A\, e^{-tB}) B$$

1b. Take t-derivative of the right hand side

$$d/dt\left(A + t[B, A] + \frac{t^2}{2!}[B, [B, A]] + \cdots\right) = [B, A] + t\big[B, [B, A]\big] + \cdots$$
$$= \left[B, \left(A + t[B, A] + \frac{t^2}{2!}[B, [B, A]] + \cdots\right)\right]$$

Exercise 1: Prove $e^{tB} A\, e^{-tB} = A + t[B, A] + \frac{t^2}{2!}\left[B, [B, A]\right] + \cdots$

2. Check that both sides coincide at initial condition $t = 0$.

2a. $e^{tB} A\, e^{-tB} \xrightarrow[t \to 0]{} A$

2b. $A + t[B, A] + \frac{t^2}{2!}\left[B, [B, A]\right] + \cdots \xrightarrow[t \to 0]{} A$

Exercise 2: How do components of momentum transform under rotations?

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- Generator of rotations about z-axis is the z-component of the total angular momentum J_z.
- The x-component of momentum P_x transforms as

$$P'_x = e^{-iJ_z\Phi/\hbar} P_x e^{iJ_z\Phi/\hbar}$$

- Useful formula valid for any non-commuting A and B

$$\boxed{A' = e^{iB} A\, e^{-iB} = A + i[B, A] - \frac{1}{2!}\big[B, [B, A]\big] + \cdots}$$

  We will use this formula very often!

- Then

$$P'_x = P_x - \frac{i\Phi}{\hbar}[J_z, P_x] - \frac{\Phi^2}{2!\hbar^2}\big[J_z, [J_z, P_x]\big] + \frac{i\Phi^3}{3!\hbar^3}\Big[J_z, \big[J_z, [J_z, P_x]\big]\Big] \ldots$$

Exercise 2: How do components of momentum transform under rotations? (continued)

- The necessary commutators will be taken from the Lie algebra representation

$$[J_i, P_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} P_k, \qquad (3.49)$$

- Here we have the Levi-Civita symbol

$$\epsilon_{xyz} = \epsilon_{yzx} = \epsilon_{zxy} = 1$$
$$\epsilon_{yxz} = \epsilon_{xzy} = \epsilon_{zyx} = -1$$
$$\epsilon_{ijk} = 0 \text{ in all other cases}$$

- Therefore

$$[J_z, P_x] = i\hbar\epsilon_{zxy}P_y = i\hbar P_y$$
$$\left[J_z, P_y\right] = i\hbar\epsilon_{zyx}P_x = -i\hbar P_x$$

Exercise 2: How do components of momentum transform under rotations? (continued)

$$
\begin{aligned}
P'_x &= P_x + \phi P_y - \frac{\phi^2}{2!} P_x - \frac{\phi^3}{3!} P_y + \cdots \\
&= P_x \left(1 - \frac{\phi^2}{2!} + \cdots \right) + P_y \left(\phi - \frac{\phi^3}{3!} + \cdots \right) \\
&= P_x \cos\phi + P_y \sin\phi
\end{aligned}
$$

- Conclusion: momentum transforms as a vector under rotations.

Exercise 3: Show that energy-momentum $\left(H, cP_x, cP_y, cP_z\right)$ form components of a relativistic 4-vector.

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- Check transformations with respect to the boost along x-axis

$$H' = e^{-icK_x\theta/\hbar} H e^{icK_x\theta/\hbar}$$
$$P_x' = e^{-icK_x\theta/\hbar} P_x e^{icK_x\theta/\hbar}$$
$$P_y' = e^{-icK_x\theta/\hbar} P_y e^{icK_x\theta/\hbar} = P_y$$
$$P_z' = e^{-icK_x\theta/\hbar} P_z e^{icK_x\theta/\hbar} = P_z$$

- Use Poincaré commutators

$$[K_i, P_j] = -\frac{i\hbar}{c^2} H\delta_{ij}, \tag{3.54}$$
$$[K_i, H] = -i\hbar P_i. \tag{3.55}$$

$$[K_x, P_x] = -i\hbar/c^2 H$$
$$[K_x, H] = -i\hbar P_x$$

Exercise 3: Show that energy-momentum (H, cP_x, cP_y, cP_z) form components of a relativistic 4-vector. (continued)

- Use our standard formula

$$H(\theta) = e^{-icK_x\theta/\hbar}\, He^{icK_x\theta/\hbar}$$

$$= H - \frac{ic\theta}{\hbar}[K_x, H] - \frac{c^2\theta^2}{2!\hbar^2}[K_x, [K_x, H]] + \frac{ic^3\theta^3}{3!\hbar^3}\left[K_x, [K_x, [K_x, H]]\right] + \cdots$$

$$= H - ic\theta(-i)\, P_x - \frac{c^2\theta^2}{2!}(-i)(-i/c^2)H + \frac{ic^3\theta^3}{3!}(-i)^2(-i/c^2)P_x + \cdots$$

$$= H\left(1 + \frac{\theta^2}{2!} + \cdots\right) - cP_x\left(\theta + \frac{\theta^3}{3!} + \cdots\right)$$

$$= H\cosh\theta - cP_x\sinh\theta$$

$$cP_x(\theta) = ce^{-icK_x\theta/\hbar}\, P_x\, e^{icK_x\theta/\hbar}$$

$$= cP_x\cosh\theta - H\sinh\theta$$

Exercise 3: Show that energy-momentum (H, cP_x, cP_y, cP_z) form components of a relativistic 4-vector. (continued)

- Take into account that

$$\tanh\theta = v/c$$

$$\cosh\theta = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\sinh\theta = \frac{v/c}{\sqrt{1 - v^2/c^2}}$$

- Then transformations assume a more familiar Lorentz form

$$H(\theta) = \frac{H - vP_x}{\sqrt{1 - v^2/c^2}}$$

$$cP_x(\theta) = \frac{cP_x - H\,v/c}{\sqrt{1 - v^2/c^2}}$$

Exercise 4: Show that mass squared $M^2 = (H^2 - P^2c^2)/c^4$ is a Casimir operator.

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- Check commutators with translation generators

$$[P_i, P_j] = [J_i, H] = [P_i, H] = 0, \qquad (3.52)$$

Therefore $[M^2, P_i] = [M^2, H\ ] = 0$.

Exercise 4: Show that mass squared $M^2 = (H^2 - P^2c^2)/c^4$ is a Casimir operator

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Therefore $[M^2, P_i] = [M^2, H\,] = 0$.

- Check commutators with generators of rotations:

$$[P_i, P_j] = [J_i, H] = [P_i, H] = 0, \tag{3.52}$$

$$[J_i, P_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} P_k, \tag{3.49}$$

Therefore $\boldsymbol{P}$ is a 3-vector and P^2 is a scalar w.r.t. rotations.

$$[M^2, J_i] = 0$$

Exercise 4: Show that mass squared $M^2 = (H^2 - P^2c^2)/c^4$ is a Casimir operator (continued).

- Check boost transformation:

$$
\begin{aligned}
c^4 M^2(\theta) &= e^{-icK_x\theta/\hbar}(H^2 - P^2c^2)e^{icK_x\theta/\hbar} \\
&= H(\theta)^2 - P(\theta)^2c^2 \\
&= (H\cosh\theta - cP_x\sinh\theta)^2 - (cP_x\cosh\theta - H\sinh\theta)^2 - c^2P_y^2 - c^2P_z^2 \\
&= H^2(\cosh^2\theta - \sinh^2\theta) - c^2P_x^2(\cosh^2\theta - \sinh^2\theta) - c^2P_y^2 - c^2P_z^2 \\
&= H^2 - P^2c^2 = c^4M^2
\end{aligned}
$$

- This proves that

$$[M^2, K_i] = 0$$

Exercise 5: Build Hilbert space $\mathscr{H}$ of a massive spinless representation of the Poincaré group.

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- Among Poincaré algebra generators there are three mutually commuting operators P_x, P_y, P_z, H

$$[P_i, P_j] = [J_i, H] = [P_i, H] = 0, \qquad (3.52)$$

- They have common sets of eigenvectors forming an orthonormal basis in the Hilbert space $|p_x, p_y, p_z\rangle$.
- Next, let us prove that the spectrum of (p_x, p_y, p_z) occupies the entire 3D momentum space

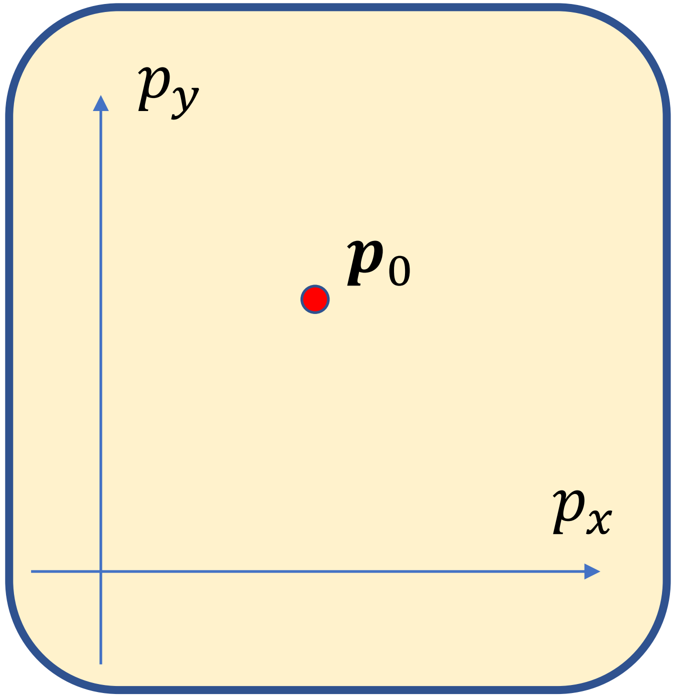
Exercise 5: Build Hilbert space $\mathscr{H}$ of a massive spinless representation of the Poincaré group.

- Suppose that there is one eigenvector $|p_{0x}, p_{0y}, p_{0z}\rangle \in \mathscr{H}$ such that

$$P_x|p_{0x}, p_{0y}, p_{0z}\rangle = p_{0x}|p_{0x}, p_{0y}, p_{0z}\rangle$$
$$P_y|p_{0x}, p_{0y}, p_{0z}\rangle = p_{0y}|p_{0x}, p_{0y}, p_{0z}\rangle$$
$$P_z|p_{0x}, p_{0y}, p_{0z}\rangle = p_{0z}|p_{0x}, p_{0y}, p_{0z}\rangle$$

- Act on this vector by the rotation operator

$$e^{iJ_z\Phi/\hbar}|p_{0x}, p_{0y}, p_{0z}\rangle$$



Exercise 5: Build Hilbert space $\mathcal{H}$ of a massive spinless representation of the Poincaré group.

- The resulting vector is also an eigenvector of momentum

$$P_x\left(e^{iJ_z\phi/\hbar}|p_{0x},p_{0y},p_{0z}\rangle\right)=\left(e^{iJ_z\phi/\hbar}e^{-iJ_z\phi/\hbar}\right)P_x e^{iJ_z\phi/\hbar}|p_{0x},p_{0y},p_{0z}\rangle$$

$$=e^{iJ_z\phi/\hbar}\left(\ e^{-iJ_z\phi/\hbar}P_x e^{iJ_z\phi/\hbar}\right)|p_{0x},p_{0y},p_{0z}\rangle$$

$$=e^{iJ_z\phi/\hbar}\ \left(P_x\cos\phi+P_y\sin\phi\right)|p_{0x},p_{0y},p_{0z}\rangle$$

$$=e^{iJ_z\phi/\hbar}\ \left(p_{0x}\cos\phi+p_{0y}\sin\phi\right)|p_{0x},p_{0y},p_{0z}\rangle$$

$$=\left(p_{0x}\cos\phi+p_{0y}\sin\phi\right)\left(e^{iJ_z\phi/\hbar}|p_{0x},p_{0y},p_{0z}\rangle\right)$$

$$P_y\left(e^{iJ_z\phi/\hbar}|p_{0x},p_{0y},p_{0z}\rangle\right)=\left(p_{0x}\sin\phi+p_{0y}\cos\phi\right)\left(e^{iJ_z\phi/\hbar}|p_{0x},p_{0y},p_{0z}\rangle\right)$$

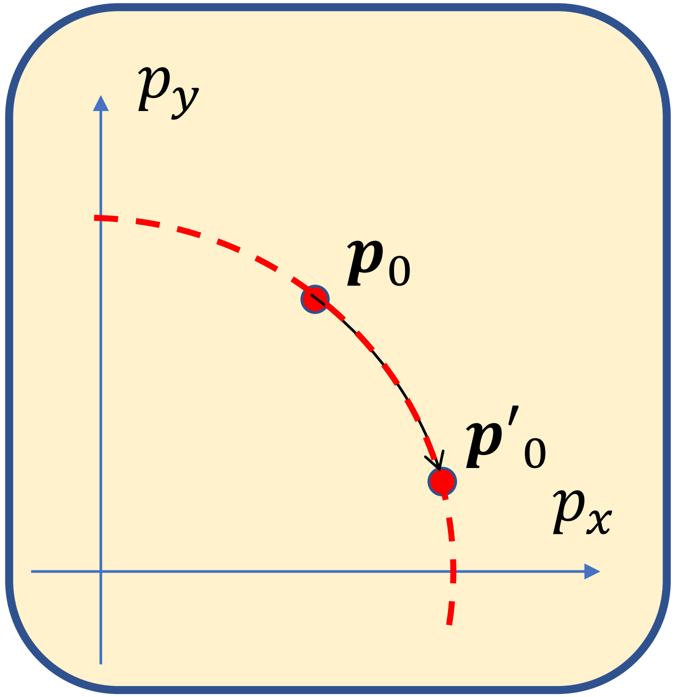
$$P_z\left(e^{iJ_z\phi/\hbar}|p_{0x},p_{0y},p_{0z}\rangle\right)=p_{0z}\left(e^{iJ_z\phi/\hbar}|p_{0x},p_{0y},p_{0z}\rangle\right)$$

Exercise 5: Build Hilbert space $\mathscr{H}$ of a massive spinless representation of the Poincaré group.

- Vector $|\boldsymbol{p}'_0\rangle = e^{iJ_z\phi/\hbar}|\boldsymbol{p}_0\rangle$ belongs to the same irreducible space $\mathscr{H}$
- Then all vectors on the sphere with radius $|\boldsymbol{p}_0|$ also belong to the subspace $\mathscr{H}$ of the irreducible representation
- Using the same arguments for vectors

$$|\boldsymbol{p}''_0\rangle = e^{icK_x\theta/\hbar}\,|\boldsymbol{p}_0\rangle$$

one can prove that all possible eigenvectors of $\boldsymbol{P}$ are in the Hilbert space $\mathscr{H}$



Exercise 5: Build Hilbert space $\mathcal{H}$ of a massive spinless representation of the Poincaré group.

- Hamiltonian H also commutes with the three components of momentum

$$[P_i, P_j] = [J_i, H] = [P_i, H] = 0, \qquad (3.52)$$

- however h is not an independent eigenvalue, because in an irreducible representation $(m = const > 0)$

$$h = +\sqrt{p^2c^2 + m^2c^4}$$

- Wave functions $\psi(p_x, p_y, p_z)$ in the momentum representation can be regarded as functions on the "mass shell hyperboloid".
- We will see how these wave functions transform w.r.t. Poincaré group elements in a separate exercise.

Thank you!