Thomson Chapters 7 and 8 – QED continued

Bill Celmaster

May 23, 2022

1 Review of the Big Picture

You start with the rules – **Quantum Field Theory**. Then you identify the objects of interest (particles) and their interactions (**The Standard Model**). Then you do arithmetic.

1.1 The rules (Quantum Field Theory)

- Quantum Mechanics
- Relativity (a.k.a Poincaré invariance), which strongly limits the time-space relationships
- The Schrodinger equation

$$i\frac{\partial \phi}{dt} = (\hat{H}_0 + \hat{V})\phi(x) \tag{1}$$

where ϕ is the wavefunction of the system of interest, and the operator $\hat{V}$ describes *the stuff that changes the free behavior ($\hat{H}_0$) of the object of interest.*

1.2 The objects and interactions (The Standard Model)

- The objects of interest are the particles and these can be mathematically described by fields (which are operators).
- The interactions of interest are represented by the quantity that Eugene called $\hat{V}$ and which is a function of the fields.
- $\hat{V}$ is obtained by finding the simplest function consistent with observed patterns in processes.

– **In particular, the interactions $\hat{V}$ are often taken to be sums of simple low-order monomials in the fields – for example (Eugene's notes),**

$$\hat{V}(\tilde{x}) = -e\bar{\psi}(\tilde{x})\gamma^{\mu}\psi(\tilde{x})A_{\mu}(\tilde{x}) \tag{2}$$

– In this example, ψ and A_{μ} are fields and $\hat{V}$ is a product of 3 fields.

2 Review of Feynman diagrams through Chapter 6

2.1 General construction and interpretation of a Feynman diagram

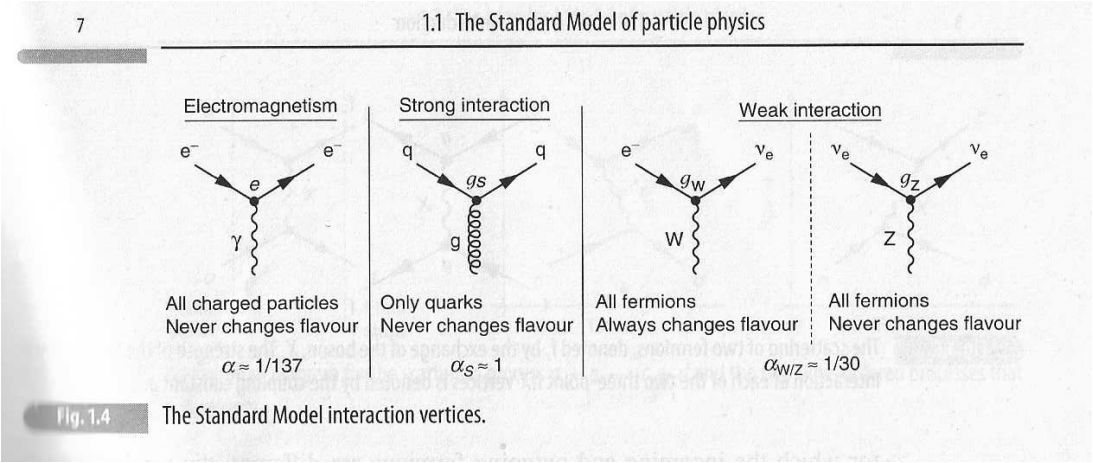


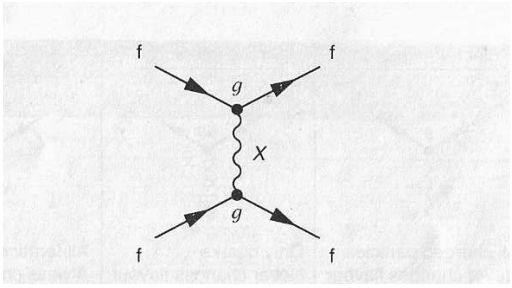
Fig. 1.4 The Standard Model interaction vertices.

- The vertices are schematics for the monomials in $\hat{V}$ above. A product of N fields is schematically represented as a vertex with N lines that represent fields.

- We use these vertices to compose schematics known as "Feynman diagrams". **There are schematic rules that tell us how to compute the rate of the process depicted – either a scattering process or a decay process.**

  – The end of a line (straight or squiggly) can either be connected to a vertex or it can be "free". In the above diagrams, all lines have one free end and one end connected to a vertex. We call these

"free lines". If both ends are connected, we call it a "connected line". See the squiggly line below.



- We describe the schematics with jargon:
  - A free line is called a "real particle" and it is either a **initial particle** or a **final particle**
  - A connected line is sometimes described as a "virtual particle" but when referring to the mathematical rules it is known as **a propagator**.
- The diagram is interpreted as time-ordered for real particles: Real particles on the left are *initial* (e.g. they are colliding with one another or decaying); real particles on the right are *final* (the products of a collision or decay).
- Virtual particles are not time-ordered, even if they are depicted as going from left to right. We are free to regard a virtual particle as either entering a vertex, or departing a vertex.
- The real particles describe the experiment. We must specify properties such as momentum, polarization and spin.
- A vertex determines whether or not a certain set of particles can collide and produce another set of particles. If a schematic can't be created for that scenario, then the scenario isn't possible. Also, at a vertex, momentum is conserved.
- Virtual particles can be regarded as a weighted average over all possibilities consistent with the vertex rules. Virtual particles also "carry" a momentum which appear in the propagators.

2.2 Feynman rules for QED (Chapter 5)

We proceed from schematics (Feynman diagrams) to dynamics by using techniques for solving and interpreting the Schrodinger equation for a system of

particles. The prototypical theory is **quantum electrodynamics** (QED)– the theory of charged particles interacting with an electromagnetic field. I will describe this theory using the Lagrangian formulation rather than the Hamiltonian formulation of Eugene's notes. Then $\hat{V}$ becomes a term in the Lagrangian rather than the Hamiltonian.

- The lines of a vertex represent fields (therefore particles) of the Lagrangian. For example, consider the Lagrangian for QED .

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{EM} \tag{3}$$

  where $\mathcal{L}_0$ is the part of the Lagrangian which is quadratic (this part is called the free Lagrangian) in the electron field ψ_i^e or the electromagnetic (photon) field A_μ, and $\mathcal{L}_{EM}$ is the electromagnetic interaction term

$$\mathcal{L}_{EM} = -e \sum_{i,j,k,\mu} \psi_i^{e\dagger} \gamma_{ij}^0 \gamma_{jk}^\mu A_\mu \psi_k^e \tag{4}$$

  This interaction term can be seen to be a sum of products of 3 field terms, $\psi_i^{e\dagger}$, A_μ, and ψ_k^e multiplied by complex coefficients formed from products of components of γ matrices. The coefficients include an overall factor of e, which is known as the coupling constant.

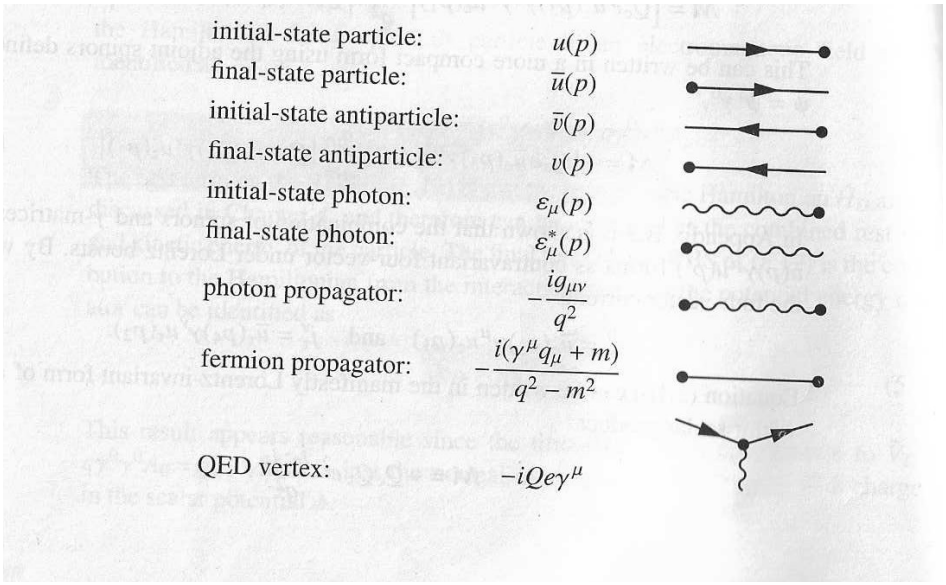
- When the process of interest is either a scattering or decay process, there is a series of rules– **Feynman rules** – for calculations based on schematics. The Feynman rules are obtained from the Lagrangian by any one of several techniques. My favorite technique is to use the Lagrangian in a path integral whose integrand includes a term

$$\begin{aligned} e^{i\int d^4x(\mathcal{L}_0+\mathcal{L}_{EM}+i\epsilon)} &= e^{i\int d^4x(\mathcal{L}_0+i\epsilon)} e^{i\int d^4x\mathcal{L}_{EM}} \\ &= e^{i\int d^4x(\mathcal{L}_0+i\epsilon)} \left(1 + i\int d^4y_1 \mathcal{L}_{EM} + i^2 \frac{1}{2!} \int d^4y_1 \mathcal{L}_{EM} \int d^4y_2 \mathcal{L}_{EM} + ...\right) \end{aligned} \tag{5}$$

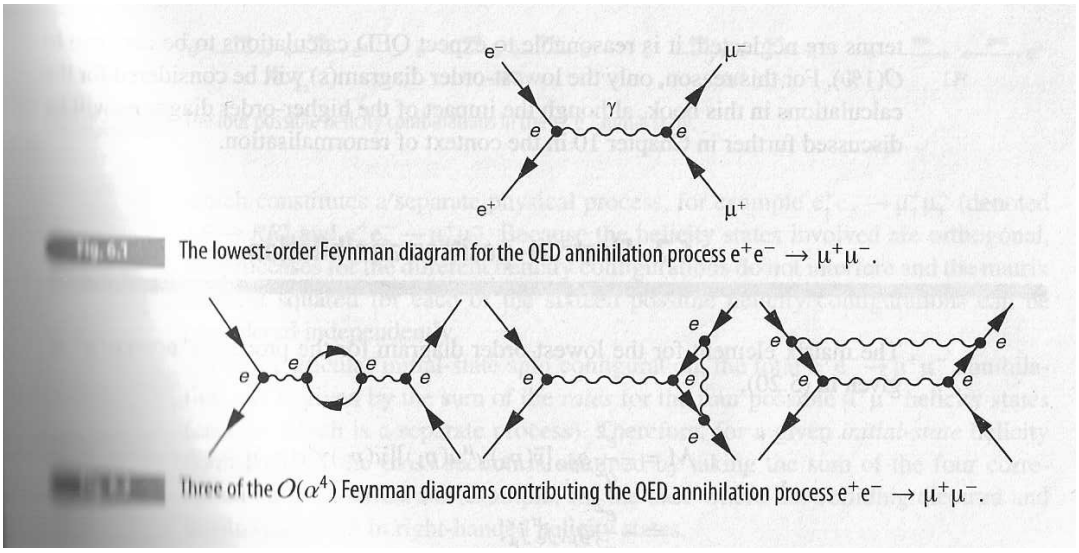
  where we've Taylor expanded the second exponential, using the assumption that $\mathcal{L}_{EM}$ is small. The path integral generally also includes other multiplicative terms in the integrand, which are fields representing the initial and final states of the interaction.

We construct Feynman diagrams by joining permissible vertices to one another, and to "free lines" (real particles) representing the initial or final particles of a scattering or decay process. Each free line is associated with a particular spin or polarization and all lines are labeled by a momentum. Momentum is conserved at vertices. The rules for the Feynman diagram tell

you how to compute the amplitude $\mathcal{M}_{fi}$ or sometimes just $\mathcal{M}$ (the subscripts simply mean 'initial to final').



2.3 An example: Electron-positron annihilation (Chapter 6)



In the example above of electron-positron annihilation, the first diagram (on top) has two vertices joined by a connected line (virtual particle / propagator). The diagrams on the bottom illustrate higher orders in perturbation

theory. They each have 4 vertices (and are therefore of order 4 in the coupling constant).

Following the Feynman rules for the first diagram, we obtain

$$\mathcal{M} = -\frac{e^2}{q^2} g_{\mu\nu} [\bar{v}(p_2)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma^\nu v(p_4)], \tag{6}$$

where the incoming particles have momenta p_1 and p_2, and the momentum q of the virtual photon, is $q = p_1 + p_2$, the center-of-mass 4-momentum. **We often describe this diagram as an s-channel diagram** because the virtual particle has a momentum $q = p_1 + p_2$, and q^2 is known as the Mandelstam variable "s".

Once we know the amplitude $\mathcal{M}$, we can sum over initial and final states of interest (for example, the solid angles of incidence) and after some geometrical calculations we arrive at expressions that give scattering probabilities. Some of these expressions are derived in Chapter 3. For example, the differential cross-section (the amount of scattering per unit of solid angle) is

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s} \frac{p_f^*}{p_i^*} |\mathcal{M}_{fi}|^2, \tag{7}$$

where the asterisk denotes that the quantities (initial solid angle, initial momentum, final momentum) are in the center of mass frame.

The differential cross-section can be integrated. If we substitute the amplitude computed in eq. (6) we obtain equation 6.23, and to obtain the total cross-section we can integrate over the angle to obtain

$$\sigma = \frac{4\pi\alpha^2}{3s} \tag{8}$$

leading to graphs like

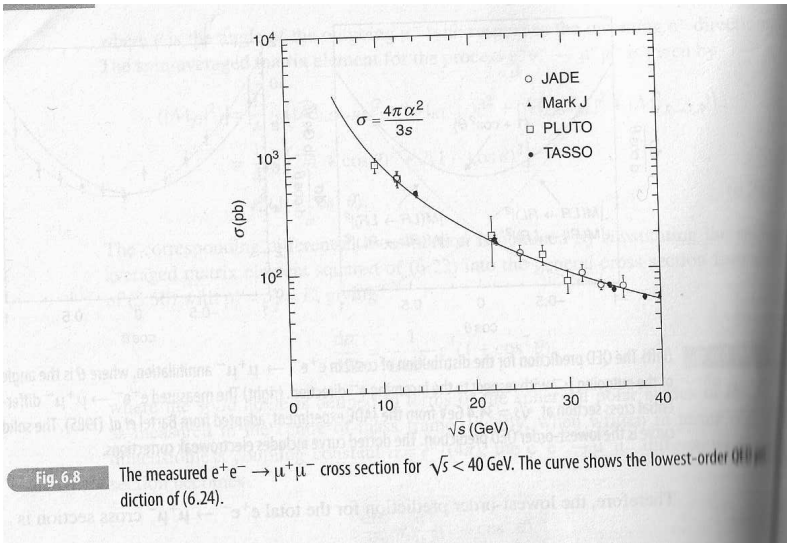


Fig. 6.8 The measured $e^+e^- \to \mu^+\mu^-$ cross section for $\sqrt{s} < 40$ GeV. The curve shows the lowest-order QED prediction of (6.24).

3 Electron-proton scattering (Chapters 7 and 8)

Remember in quantum mechanics that $\lambda \propto 1/E$ where λ is the effective wavelength of a particle. Similarly we can characterize the propagator (aka virtual photon) by a characteristic wavelength which gets smaller when the collision-energy gets bigger (see Fig. 7.1 in Thomson). If the wavelength is bigger than the proton (i.e., the energy is low), then you can pretend the proton is small and behaves like an elementary particle. We call that *elastic scattering*. **Think of the collisions of billiard balls.**

If the wavelength is much smaller (i.e., energy is very high) then the interaction becomes sensitive to the proton components and ultimately to the quark constituents. When that happens, the quarks are separated and the proton shatters into a bunch of other particles. We call that *inelastic scattering*. **Think of the collisions of eggs.**

3.1 Elastic scattering

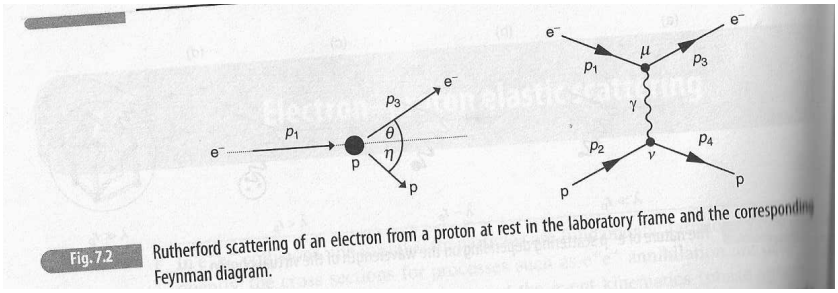


Fig. 7.2 Rutherford scattering of an electron from a proton at rest in the laboratory frame and the corresponding Feynman diagram.

The Feynman rules give us

$$\mathcal{M} = -\frac{e^2}{q^2} g_{\mu\nu}[\bar{u}(p_3)\gamma^\mu u(p_1)][\bar{u}(p_4)\gamma^\nu u(p_2)], \tag{9}$$

where $q = p_1 - p_3$. (See Thomson eq. 7.1.) The components of the spinors are given in Thomson eq. 4.65, for helicity spin-states. For example,

$$u_\uparrow = \sqrt{E+m} \begin{pmatrix} \cos(\frac{\theta}{2}) \\ e^{i\phi}\sin(\frac{\theta}{2}) \\ \frac{p}{E+m}\cos(\frac{\theta}{2}) \\ \frac{p}{E+m}e^{i\phi}\sin(\frac{\theta}{2}) \end{pmatrix}. \tag{10}$$

When the speed of the recoiling proton is very small, we can set $p/(E + m_P) \approx 0$ and $\sqrt{E_P + m_P} \approx \sqrt{2m_P}$. Then, if we want the total spin-averaged amplitude, we average all spin combinations implicit in eq. (9), and assume that $\phi = 0$, we obtain Thomson eq. 7.9.

$$< |\mathcal{M}_{fi}^2| >= \frac{m_p^2 m_e^2 e^4}{P^4 \sin^4(\theta/2)} \left[1 + \frac{v_e^2}{1 - v_e^2}\cos^2(\frac{\theta}{2})\right], \tag{11}$$

where the electron velocity v_e is understood to be in units where $c = 1$ (this is sometimes written as β_e to denote v_e/c).

3.1.1 Rutherford scattering

Rutherford scattering describes collisions of a slow electron with a stationary proton whose recoil velocity is small.The electron momentum is much smaller (in units where $c = 1$) than the electron mass. In particular, $v_e \approx 0$ so, substituting in eq. (11) we get

$$< |\mathcal{M}_{fi}^2| >_{\text{Ruth}}= \frac{m_p^2 m_e^2 e^4}{P^4 \sin^4(\theta/2)}. \tag{12}$$

As before (e.g. electron-positron derivations), we can transform the amplitude to a differential cross-section. See section 7.2.1 of Thomson. The resulting differential cross-section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Ruth}} = \frac{\alpha^2}{16E_K^2 \sin^4(\theta/2)}, \tag{13}$$

where we use the common notation $\alpha = \frac{e^2}{4\pi}$, and E_K is the kinetic energy of the electron. This equation can also be easily derived directly from non-relativistic quantum mechanics of electron scattering in a Coulomb potential.

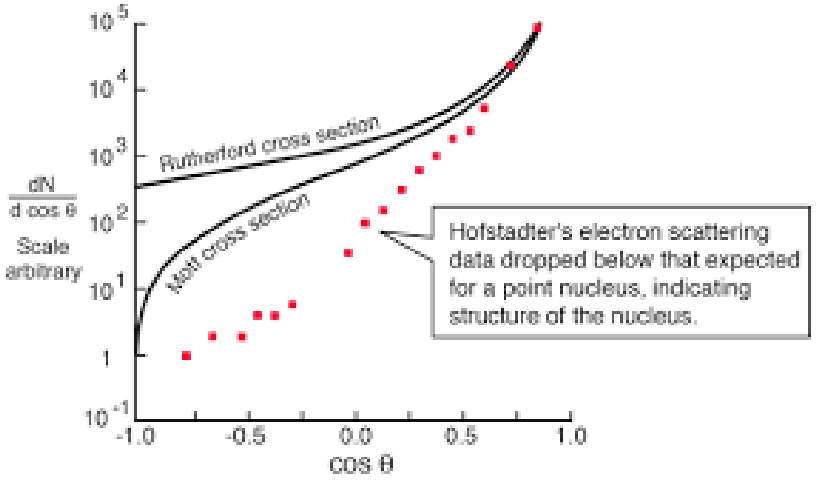
3.1.2 Mott scattering

Rutherford scattering describes collisions of a very fast electron with a stationary proton whose recoil velocity is small. The speed of the electron is sufficiently fast that its total energy is much greater than the electron rest mass. However, since the proton recoil is small, the electron's total energy is also much less than the protons. In this limit, we use eq. (11) to obtain

$$< |\mathcal{M}_{fi}^2| >_{\text{Mott}} = \frac{m_p^2 e^4}{E^2 \sin^4(\theta/2)} \cos^2(\frac{\theta}{2}). \tag{14}$$

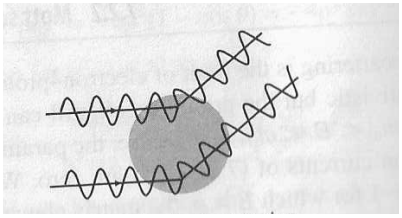
and then

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \cos^2(\frac{\theta}{2}). \tag{15}$$



3.1.3 Proton structure and form factors

Actual experiments show significant deviation from the Mott scattering prediction. This is explained by saying that the proton isn't properly described as an elementary particle. Instead, it is more accurate to describe the proton as a charge distribution of some kind and then to infer the distribution based on a modified version of Mott scattering. **This is sometimes called a phenomenological model.** There are several assumptions here, but historically that's how progress has been made. Thomson describes the assumptions and approach.



The Mott equation is then modified to

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott'}} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \cos^2(\frac{\theta}{2})|F(q^2)|^2, \tag{16}$$

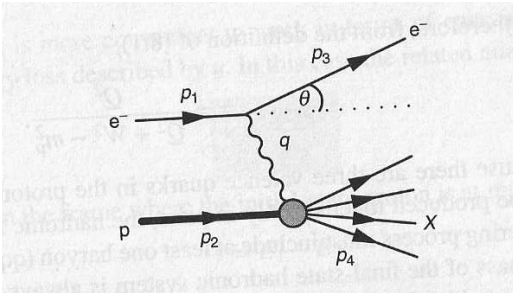
where the function $F(q^2)$ is known as the **form factor**. The form factor is then derived by comparing experiment to the Mott curve.

3.1.4 Don't ignore proton recoil

If $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_3$ isn't small, then things are a bit messier. Check equations 7.30 and 7.31 in Thomson. When this equation is generalized to very high energy, then there are two form factors and not just one. These are known as G_E and G_M. The resulting equation is known as the **Rosenbluth formula** eq. 7.33. In section 7.5.1, Thomson describes how they are measured.

3.2 Deep inelastic scattering and quarks

"Deep" just means "very". Previously we assumed that $E_e << m_p$. When the electron energy exceeds the proton mass, most scattering events are inelastic – i.e., the final state includes other particles (not the one electron and one proton). However, when scattering events are elastic, they are by the Rosenbluth formula.



The vertex bubble terminating in the cluster X looks like a Feynman vertex but we don't have computational details for it. However, on general principles of Lorentz invariance etc., the scattering amplitude can be expressed as a simple generalization of the Rosenbluth formula, introducing generalizations of the form factors. These are **structure functions** $F_1(x, Q^2)$ and $F_2(x, Q^2)$ where the scattering energy $Q^2 \equiv -(p_1 - p_3)^2$ is very large and the "elasticity" $x \equiv \frac{Q^2}{2p_2 \cdot q}$ is equal 1 when the process is elastic (the cluster "X" is then just a single proton). x is also known as *Bjorken* x.

High-precision deep inelastic scattering experiments have been critical to understanding and testing QCD (quantum chromodynamics). Initially, by plotting F_1 and F_2 and related *parton distribution functions*, physicists found relationships that led to the understanding of protons as built out of more fundamental constituents.

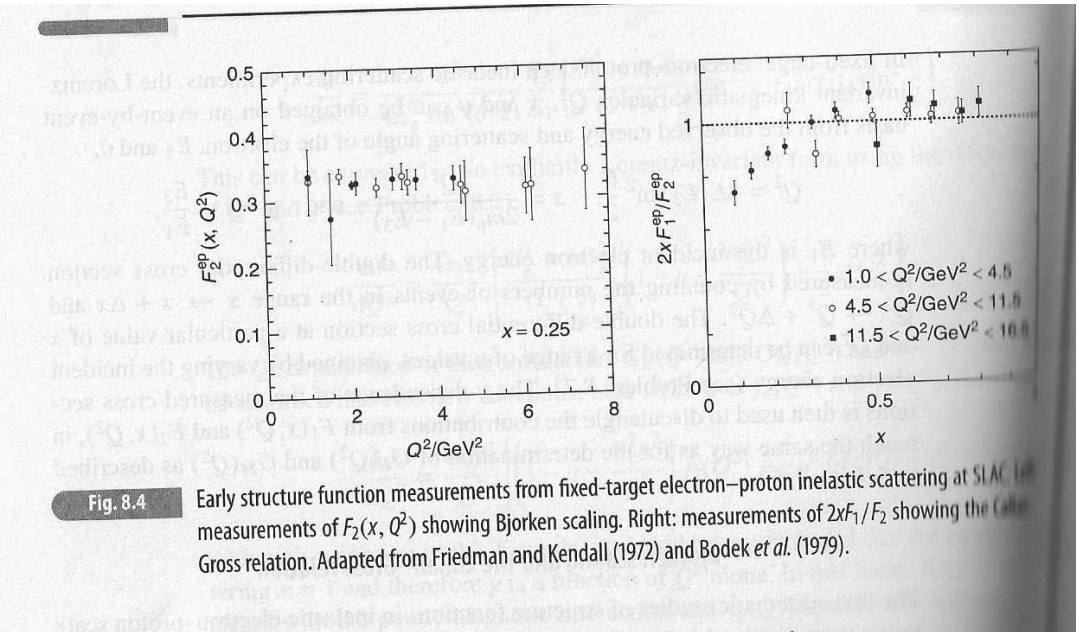


Fig. 8.4 Early structure function measurements from fixed-target electron–proton inelastic scattering at SLAC [illegible] measurements of $F_2(x, Q^2)$ showing Bjorken scaling. Right: measurements of $2xF_1/F_2$ showing the Callan-Gross relation. Adapted from Friedman and Kendall (1972) and Bodek *et al.* (1979).

Later, when a full theory was developed of quarks and gluons, the Q

dependence of the structure functions was predicted to high accuracy and shown to agree with experiments.

Recently it has become possible to compute the x-dependence of F_1 and F_2 from first principles and can be compared to high-precision experiments.

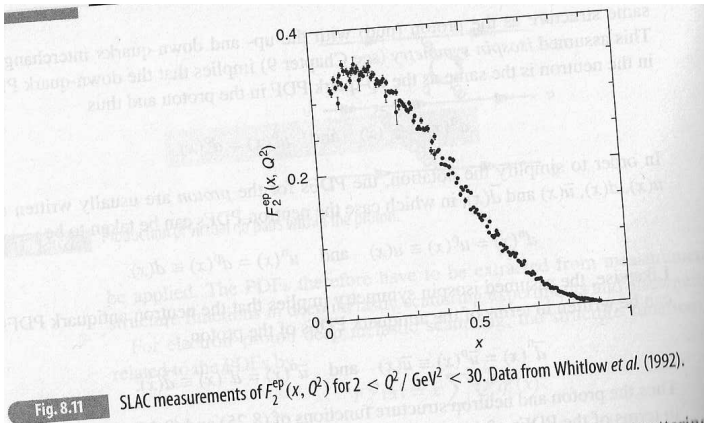


Fig. 8.11 SLAC measurements of $F_2^{ep}(x, Q^2)$ for $2 < Q^2 / \text{GeV}^2 < 30$. Data from Whitlow *et al.* (1992).

The calculations cannot be done using Feynman diagrams, because perturbation theory breaks down. Instead, the calculations are done by simulating QFT on a lattice (*lattice gauge theory*).