Thomson Chapter 9 – Isospin

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1 Overview

- Flavors, quarks and the early history of isospin
- Isospin symmetry transformations
- Up and down quarks
- Baryons
- Mesons

2 Flavors, quarks and the early history of isospin

- 1932: James Chadwick's experiments show that atomic cores include new neutral particles that Chadwick calls **neutrons** and which have the same mass as protons.
- 1932: Heisenberg hypothesizes that a neutron and proton are two quantum states of a single particle, the *nucleon*. This is a non-trivial statement as shown here:
  - It means we can construct two **different** quantum states (symmetric and antisymmetric)

    $$|\psi_S\rangle = |p\rangle + |n\rangle \tag{1}$$

    and

    $$|\psi_A\rangle = |p\rangle - |n\rangle. \tag{2}$$

- How can we tell that the two states are different? Consider an observable O. It should be possible to measure its expectation value in each of the two states:

$$\begin{aligned}\langle\psi_S|O|\psi_S\rangle &= (\langle p| + \langle n|)\, O\, (|p\rangle + |n\rangle) \\ &= (\langle p|O|p\rangle + \langle n|O|n\rangle) + (\langle p|O|n\rangle + \langle n|O|p\rangle),\end{aligned} \tag{3}$$

and

$$\begin{aligned}\langle\psi_A|O|\psi_A\rangle &= (\langle p| - \langle n|)\, O\, (|p\rangle - |n\rangle) \\ &= (\langle p|O|p\rangle + \langle n|O|n\rangle) - (\langle p|O|n\rangle + \langle n|O|p\rangle).\end{aligned} \tag{4}$$

Because of the relative minus sign, the two expectation values are different. That's called *coherence* and is different than you'd see with a classical mix of two different particles.[1]

- A key consequence of this idea, is that there must be **an observable for which the symmetric and antisymmetric states have different eigenvalues**. This consequence is thoroughly baked into the quantum mechanics of every two-particle state regardless of the particle properties. In that sense, every particle is an alternate "state" of every other particle. But in 1932, these ideas were new and it seemed as though the proton and neutron were more or less "the same" particle. In modern times, we've come to understand this "similarity" as a manifestation of symmetry between states.

- 1932 - ≈1948:
  - Experiments and theoretical analysis show that interactions involving neutrons and protons, as inferred from nuclear energy states for various atoms, and also from scattering data, obey the *isospin* symmetry.
  - Wigner clarifies the group structure.
  - Yukawa hypothesizes (1935) that neutrons and protons are bound by a strong force whose origin is a meson field, with a meson of mass around 100 MeV. The mesons are discovered in the late 1940's and called pions. They are assigned the isospin values required for isospin symmetry. Subsequently, pion-pion and pion-nucleon interactions are shown to obey the constraints of isospin symmetry.

[1] Notice that this argument fails if we can't find an operator O so that $\langle n|O|p\rangle \neq 0$. This situation would be called a *superselection rule* and is a fancy way of saying something does NOT behave in a quantum way.

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- 1944-1953: Various particle tracks are found and analyzed to be different (and heavier) than pions, muons etc. By 1953, these become known as K-mesons. In 1953, Gell-Mann and collaborators hypothesize that the properties of K-meson interactions can be explained by a new conservation law that conserves a quantity they call *strangeness*.

- 1961: Gell-Mann and (independently) Ne'eman propose that strangeness-conservation is associated with a symmetry embedded in a larger symmetry group that includes isospin. The larger symmetry group is $SU(3)$. This symmetry is used to predict the existence of a new particle named the Ω^-. That is experimentally discovered in 1964.

- 1964: Gell-Mann and (independently) Zweig hypothesize that the particle Hilbert space should include the fundamental representation of $SU(3)$ in much the way that the rotation group admits a spin-half representation (up to a phase). The representation is comprised of 3 particles that Gell-Mann called quarks.

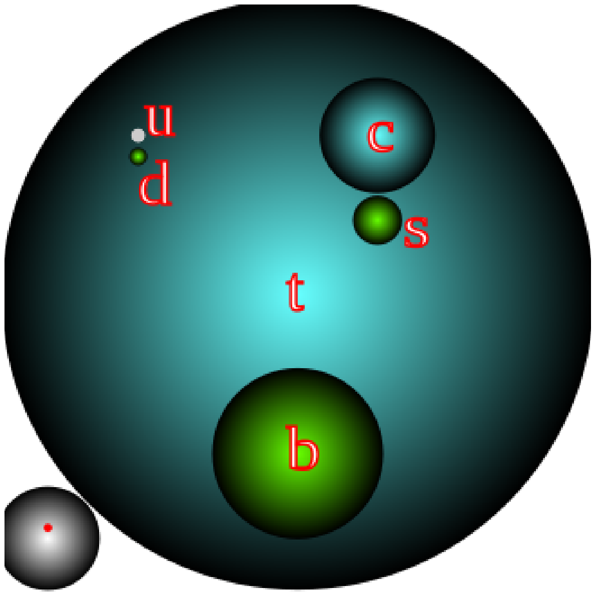
- 1964: The 3 quarks are called "up", "down" and "strange", and each has an associated anti-quark. All hadrons are composed of combinations of those quarks and their anti-quarks. (The term "baryon" refers to hadrons made of 3 quarks or 3 anti-quarks and the term "meson" refers to hadrons made of a quark and anti-quark.) Most properties that can be inferred from the quark picture can also be inferred from the SU(3) symmetry

- 1964-1970: Glashow et al. hypothesize a symmetry relationship between quarks and leptons (electrons, neutrinos, muons) but which requires a 4th quark that they call "charm". Charmed mesons are experimentally observed in 1974.

- 1973: Politzer, Gross and Wilczek discover that QCD, which is the prevailing candidate field theory of strong interactions, has the property of *asymptotic freedom*. In brief, this property makes it possible to apply QCD perturbation theory for processes involving large energy exchanges (e.g. deep inelastic scattering). For the first time, it becomes possible to predict and measure the hypothesis that quarks are constituents of nucleons etc. Since then, high precision experiments, combined with high-precision perturbative expansions, have shown a fantastic of the quark-constituent model.

- 1973: Kobayashi and Maskawa, as a result of analyzing weak interactions between quarks and leptons, predict two more quarks, "top" and "bottom".

The type of quark (there are 6 types) is known as FLAVOR. The original SU(3) symmetry has become an SU(6) symmetry. At this time, based on the bandwidth of the Z-meson, it is believed that there are no other flavors. The original isospin is associated with the up and down quarks.



3 Isospin symmetry transformations

3.1 Thomson 9.1 – Brief review of symmetries in QM

- A symmetry group is a set of transformations that preserve the form of equations of motion or a Lagrangian.

- In QM, a symmetry g is implemented as unitary transformation

$$|\psi\rangle \to U(g)|\psi\rangle. \tag{5}$$

  and the symmetry group is represented by the rule

$$U(g_1 \circ g_2) = U(g_1)U(g_2) \tag{6}$$

- By convention, an *internal* symmetry is a symmetry that commutes with all the Poincaré symmetries. In particular,

$$[H, U(g)] = 0. \tag{7}$$

This commutation relationship implies that system-evolution (remember, the system evolves with H via the Schrodinger equation) obeys symmetry relationships, and also that energy/mass eigenvalues are grouped in patterns related to the group representations.

- When it makes sense to speak of "closeness" of two group elements (for example, rotation by 0.001 radians is less than rotation by 0.1 radians), then (subject to certain mathematical assumptions) we can write

$$U(\epsilon) = I + i\epsilon G \tag{8}$$

for some self-adjoint operator G known as an *infinitesimal generator* of the group. In general, the collection of G's is known as the Lie Algebra of the group. A basis can be found for the Lie Algebra and the commutation relations of that basis can be used to find relevant relationships between symmetry transformations.

3.2 Thomson 9.2 – Isospin-1/2 representations

3.2.1 Neutrons and protons

Represent the proton and neutron in the same form we represent spin up and spin down for an electron. Namely,

$$|\mathrm{p}\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\mathrm{n}\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The superposition of a proton and neutron is then represented by a linear combination of the two vectors. **Thomson's notation, although common, is mathematically ambiguous. I'll use different notation, but see Thomson for the more common approach.** A general nucleon state can be thought of as $\alpha|\mathrm{p}\rangle + \beta|\mathrm{n}\rangle$ and is then represented by $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. An SU(2) symmetry is implemented as

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = U \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{9}$$

where U is a special (determinant 1) 2 x 2 unitary matrix.

We studied these transformations when we were studying spin (hence the name "isospin") and learned that the most general such matrix can be written as

$$U = e^{i\alpha\cdot\mathbf{T}} \tag{10}$$

where $\alpha \cdot \mathbf{T} \equiv \alpha_1\mathbf{T}_1 + \alpha_2\mathbf{T}_2 + \alpha_3\mathbf{T}_3$ and $\mathbf{T}_i = \frac{\sigma_i}{2}$. The σ_i are the usual Pauli spin matrices.

In summary, a neutron and proton are two states of an isospin doublet (analogous to spin-up/spin-down). In field theory, the neutron and proton fields are two components of an isospin doublet.

3.2.2 Consequences of isospin symmetry

Remember that all of this started by the desire to construct a theory of strong interactions, which is invariant under isospin transformations. From our general theory of symmetries, that means $[H, U] = 0$. From this it can be shown that all states in the doublet representation (i.e., combinations of neutron and proton, or combinations of up and down) have the same energy/mass.

In particular, the neutron and proton have the same mass. Also the up and down have the same mass.

Note that only the strong interactions are isospin-invariant. However, weak and electromagnetic interactions are not. So there are perturbative corrections (because weak and EM are both small) to any isospin-based conclusions.

WARNING: The last few sentences of Thomson 9.2.1 make some claims that I dispute. In particular, Thomson says that "it does not make sense to form states which are linear combinations of the two ...". I don't agree, and think that he is confusing some concepts here.

3.2.3 Isospin algebra and isospin labels

The isospin algebra is just the SU(2) algebra which we've encountered when studying the rotation group. When we spoke of rotations, we used the angular momentum symbol **J**. When we speak of isospin, we'll use the symbol **T**. Mathematically they are identical, but we use different letters to distinguish context.

When dealing with rotations, there were two operators used to characterize all states in an irreducible representation (such as a spin-1/2 representation). The total isospin operator is the group Casimir operator (see Eugene's notes for more about Casimir operators). It is

$$\mathbf{T}^2 = \mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2. \tag{11}$$

The other operator used for characterizing states is $\mathbf{T}_3$. We then write the isospin content of any state as $|...I, I_3>$ such that

$$\begin{aligned} \mathbf{T}^2|...I, I_3> &= I(I+1)|...I, I_3> \\ \mathbf{T}_3|...I, I_3> &= I_3|...I, I_3> . \end{aligned} \tag{12}$$

The dots indicate other quantum numbers such as 4-momenta or angular momenta etc.

This notation is reminisce of the notation we use for angular momentum. The neutron and proton both have I = 1/2, meaning that they belong to the same 2-dimensional (spin 1/2) multiplet. The proton has $I_3 = +1/2$ and the neutron has $I_3 = -1/2$. Similarly, the up quark has $I_3 = +1/2$ and the down quark has $I_3 = -1/2$.

3.2.4 Up and Down quarks

Recall that quark fields were introduced by Gell-Mann and Ne'eman as members of the fundamental representation of SU(3) (and later, with the discovery of more flavors, they were deemed members of the representation of SU(N) where N is the number of quark flavors).

The "up" and "down" quarks transform into one another under the SU(2) subgroup of SU(3). **This subgroup is the isospin group**. So we can follow precisely the same steps as for the neutron and proton. A general up-down quark state can be thought of as $\alpha|\mathrm{u}\rangle + \beta|\mathrm{d}\rangle$ and is then represented by $\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$. An SU(2) symmetry for quarks is implemented as

$$\begin{pmatrix}\alpha' \\ \beta'\end{pmatrix} = U \begin{pmatrix}\alpha \\ \beta\end{pmatrix} . \tag{13}$$

3.3 Building protons and neutrons from quarks

The isospin quantum numbers of the nucleons are

- p: $(I, I_3) = (\frac{1}{2}, +\frac{1}{2})$,
- n: $(I, I_3) = (\frac{1}{2}, -\frac{1}{2})$.

Similarly, the isospin quantum numbers of the up and down quarks are

- u: $(I, I_3) = (\frac{1}{2}, +\frac{1}{2})$,
- d: $(I, I_3) = (\frac{1}{2}, -\frac{1}{2})$.

The up and down quark have almost the same mass. Each nucleon is a bound state of 3 quarks. The proton is "uud" and the neutron is "ddu". If you add up the I_3 quantum numbers for up and down quarks, you see that the proton and neutron end up with the correct values of I_3. The small difference in mass between the neutron and proton is, in part, due to the small mass difference between the up and down quarks.

3.3.1 Quarks or SU(N)-flavor symmety?

I added this section after a Physics Group discussion on the topic. First a definition: What I've been describing up to now as SU(3) and then later SU(N) for N flavors of quark, I'll now call SU(N)-flavor. This nomenclature is distinct from an entirely different set of symmetries whose strong-interaction subgroup is SU(3). We call that symmetry SU(3)-color, but just like SU(3)-flavor, modern physics has expanded SU(3)-color to larger groups such as SU(5) and SO(10). For the time being, there is no relationship between the color and flavor symmetries.

I made the comment that after the introduction of the quark idea, it was no longer necessary to think about SU(2)-flavor or SU(3)-flavor, because everything that could be derived from the group theoretic analysis, could be more easily derived just by considering quarks. After further discussion, I've come to the conclusion that I mis-spoke. I now realize that my views are a consequence of the path my research has taken over the years. It may be worth my while, some day, to carefully examine alternatives. For now, I think the following may be helpful, especially to those of you who are fond of group theory!

Consider a quantum theory of quarks that can be described by a field theory whose interaction terms are independent of flavor. This theory, whatever it is, can be examined in the non-relativistic limit in much the same way that QED can be examined in the non-relativistic limit. Whatever calculations are performed can then be organized into terms that involve the quark masses, and terms that don't. Those that don't, are entirely independent of quark flavor. Those that do, can be further organized into those terms whose mass-dependencies are identical to one another. Imagine, for example, a Taylor expansion in the quark-mass. Then flavor-symmetry would amount to saying that the coefficients of the Taylor expansion are flavor-independent.

Thus flavor-symmetry constrains the possible forms of bound-state energy distributions, and the possible-form of scattering amplitudes. This kind of thing can be analyzed using group representation facts, or can be analyzed by brute force. In just the same way that spherical symmetry and representations have been used for more than a century as a way of simplifying the solutions of differential equations, so isospin (and SU(N)-flavor) symmetry representations can be used to simplify the QFT solutions for flavor interactions.

3.4 What can we learn from Feynman diagrams?

In the period from 1932 to about 1948, the theory of strong interactions was brought into modern field-theoretic form. Initially, the Hamiltonian was described using a matrix that transformed the neutron-proton doublet state. In modern notation, we would write the most general interaction Hamiltonian transformation in shorthand form as

$$H_I = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \tag{14}$$

so that

$$\begin{pmatrix} p' \\ n' \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} p \\ n \end{pmatrix} \tag{15}$$

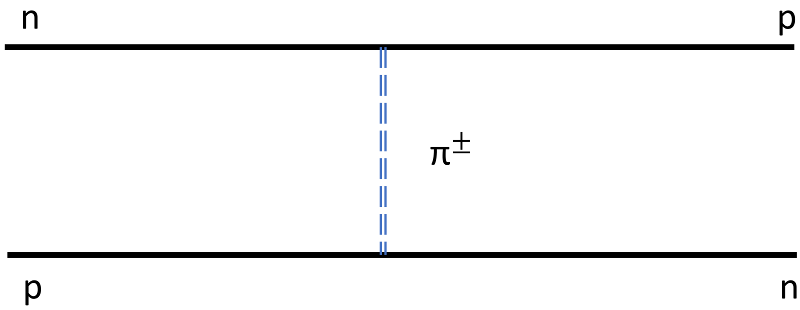
If our only experimental information is that the Hamiltonian transforms the pure proton state into a pure neutron state, in the same way that it transforms a pure neutron state into a pure proton state, then we could describe this in field-theoretic notation as

$$H_I^0 = \alpha \bar{n} p + \alpha^* \bar{p} n \tag{16}$$

where the operators n and p are Dirac fields representing the neutron and proton and the α is a coefficient. Now, by analogy with QED, we might speculate that there is another field that is responsible for the strong force, and that this field would appear in the interaction Hamiltonian, in a way similar to how the electromagnetic field appears in the interaction Hamiltonian for QED. Putting this altogether, and writing it in Lagrangian language, we obtain an interaction Lagrangian

$$\mathcal{L}_I^0(x) = -\alpha \bar{n} p \pi^- - \alpha^* \bar{p} n \pi^+ \tag{17}$$

where now we've now replaced the coefficient α with $\alpha\pi^-$ and α^* with $\alpha^* (\pi^-)^*$. We generally rewrite $(\pi^-)^*$ as π^+. The Lagrangian would imply a leading-order Feynman diagram like this:



The blue line, like the photon in EM interactions, is responsible for the strong force. We call π^+ and π^-, the (charged) pion fields. What do we know about the charge pion and its interactions?

- We see that top vertex is the same as the bottom vertex thus implementing the fact that the strong force doesn't distinguish between proton and neutron.
- Since there is an overall charge-conservation law, and each of the top and bottom lines change charge, then the exchanged line must be charged. Hence the notation $\pi^{\pm}$.
- If you scatter two protons, then the vertices (which have one neutron leg and one proton leg) would force the outgoing particles to both be neutrons (and that would violate charge conservation). Thus protons could not scatter strongly from one another (they could only scatter weakly through electromagnetism). More generally, there would be no strong force between one proton and another. None of this agrees with experiment. So this Lagrangian must be incomplete.
- The blue line represents a propagator. We "say" that the line is a "virtual pion exchange" and we then compute the contribution of the propagator. The result can be compared to a scattering experiment.
- Since the particles on top both have spin-1/2, then the group rules for rotational invariance require that the pions either have spin 0 (scalar particle) or spin 1 (vector particles). Scattering behavior can distinguish the two, and we find that the pions are scalars.
- As we saw on page 165 of Thomson, in the discussion of Rutherford scattering, the low-energy scattering calculated from the (massless)

photon propagator, is the same as what can be computed using the Coulomb potential. That is,

$$\alpha \frac{1}{q^2} \leftrightarrow \alpha \frac{1}{r}. \quad (18)$$

More generally, for massive particles, the propagator is proportional to $\frac{1}{q^2-m^2}$, which corresponds to a potential

$$V(r) = \frac{e^{-mr}}{r}. \quad (19)$$

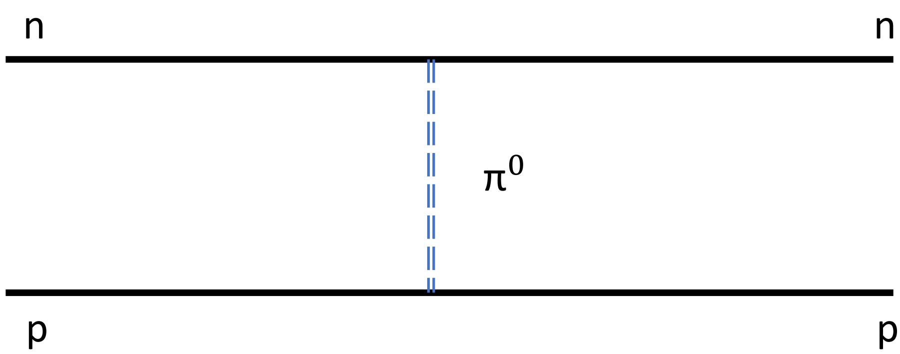
This is a short-range force, and Yukawa deduced from experiments that $m \approx 100$ MeV.

- In summary, the Feynman diagram, together with some experimental data, leads to a hypothesis that the blue line corresponds to the existence of a charged scalar particle whose mass is about 100 MeV. **Experimentally, the pion has a mass of about 140 MeV.**

We concluded above that the Lagrangian was incomplete. Another term which is invariant under the exchange of n with p is

$$H_I^{\pm} = \alpha \left(\bar{n}n + \bar{p}p\right) \pi^0 \quad (20)$$

where both α and π^0 are real. The corresponding Feynman diagram is:



By conservation of charge, we find that π^0 is neutral. Over the period of 1932-1948, experimental data confirmed the invariance of strong interactions under isospin transformations of the neutron-proton states. We see that there are 3 pions and that the general interaction term is a linear combination of terms with 2 nucleons and 1 pion. Thus the strong-interaction vertex is an isospin-invariant function of the form

$$\mathcal{L}_I = \sum_{ijk} \beta_{ijk} \bar{\mathcal{N}}_i \pi_j \mathcal{N}_k + \text{cmplx-conj.} \tag{21}$$

In this expression, the $\mathcal{N}$ indices denote either a neutron or proton, and the π index denotes either a charged or neutral pion. Coefficients β_{ijk} must be chosen so that the Lagrangian is isospin invariant. We already know that the nucleons transform as a 2D representation of the isospin group i.e., the 'spin-1/2' representation of SU(2). By the laws of group representations, the pions must therefore transform in a combination of spin-0 and spin-1 representations of SU(2). Otherwise it wouldn't be possible to find a set of β_{ijk} to make the Lagrangian invariant under SU(2)-isospin. Since we know there are at least 3 pions involved in the strong interaction, we can conclude that they form an isospin-1 representation.

It seems plausible that one could also construct a term involving 2 nucleons and an isospin-0 field different from the pion. In fact, such a particle was discovered and called η. From the above arguments, that particle must be neutral.

3.5 Decay rates and isospin representation theory

This section departs slightly from Thomson but is inspired by Thomson Problem 9.3.

By considering the isospin states, show that the rates for the following strong interaction decays occur in the ratios

$$\Gamma(\Delta^{++} \to \pi^+ p) : \Gamma(\Delta^0 \to \pi^- p) = 3 : 1 \tag{22}$$

etc.

The method we'll use to analyze this[2], is typical of the methods used since the 60's, of using Lagrangian symmetries to find ratios of masses, scattering amplitudes and decay rates. Also notice that these kinds of predictions cannot be obtained (at least not easily) by simply considering that 'ordinary' particles are made of two kinds of quarks (up and down).

[2] I believe that Thomson was intending to have us solve probem 9.3 by considering states rather than fields but the method is similar and I prefer to stick with fields.

3.5.1 How to build isospin-invariant Lagrangians

Recall eq. (21),

$$\mathcal{L}_I = \sum_{ijk} \beta_{ijk} \bar{\mathcal{N}_i} \pi_j \mathcal{N}_k + \text{cmplx-conj}. \tag{23}$$

What are the coefficients β_{ijk} so that $\mathcal{L}_I$ is invariant under isospin transformations? Let's unpack this question for the case where $\mathcal{N}_{-1/2}$ represents a neutron, $\mathcal{N}_{+1/2}$ represents a proton, and π_{-1}, π_0, π_1 represent respectively π^-, π^0, π^{+1}.

- Recall how isospin-$\frac{1}{2}$ and isospin-0 states transform under an isospin transformation. The transformation rules are just like those for spin. The isospin transformations are parametrized by parameters α, $\hat{r}$ which are abstract analogues of a rotation-angle θ and the unit axis-vector $\hat{v}$. So for example,

$$\begin{aligned} U(\alpha, \hat{e}_3) &: \begin{pmatrix} \mathcal{N}_{\frac{1}{2}} \\ \mathcal{N}_{-\frac{1}{2}} \end{pmatrix} \to \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \mathcal{N}_{\frac{1}{2}} \\ \mathcal{N}_{-\frac{1}{2}} \end{pmatrix} \\ U(\alpha, \hat{e}_3) &: \begin{pmatrix} \pi_1 \\ \pi_0 \\ \pi_{-1} \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_0 \\ \pi_{-1} \end{pmatrix} \end{aligned} \tag{24}$$

  In general, the first matrix on the right will be described as $\mathbf{M}^{1/2}(\alpha, \hat{e}_3)$ and the second as $\mathbf{M}^1(\alpha, \hat{e}_3)$. These matrices are known as *isospin representations.*

- If we now apply these transformations to each of the three factors in the terms of $\mathcal{L}_I$, we will arrive at a new expression. If we choose the values of βijk appropriately, the new expression will equal the old one. **This is what we mean by an invariance.**

The systematic way to find the β_{ijk} coefficients is via group representation theory, and these kinds of coefficients are generically known as Clebsch-Gordan coefficients. Often this kind of thing is done pairwise.

- First analyze a product of two factors, for example $\mathcal{N}_i \pi_j$. This is also known as a **tensor product**. In this example, there are a total of 6 pairs of indices (i, j) and therefore one can regard the terms $\mathcal{N}_i \pi_j$ as the 6 components of a 6-vector. An isospin transformation acts for example,

$$\begin{aligned} U(\alpha, \hat{ea}_3) : \mathcal{N}_i \pi_j &\to \sum_{i'j'} \mathbf{M}^{1/2}(\alpha, \hat{e}_3)_{ii'} \mathbf{M}^1(\alpha, \hat{e}_3)_{jj'} \mathcal{N}_{i'} \pi_{j'} \\ &\equiv \sum \zeta_{(ij),(i'j')} \mathcal{N}_{i'} \pi_{j'}. \end{aligned} \tag{25}$$

These are 6 x 6 linear transformations on 6-vectors. It is easy to see that the transformations follow the group transformation properties (e.g $\mathbf{M}^{1/2}(\alpha, \hat{e}_3)\mathbf{M}^{1/2}(\alpha', \hat{e}_3) = \mathbf{M}^{1/2}(\alpha + \alpha', \hat{e}_3)$).

- What we've seen is that a tensor-product of a 2-D and a 3-D representation, is a 6-D representation. Is this representation reducible? That is, is there a subset of the 6 terms so that the isospin transformations only act within that subset? This kind of thing is answered by representation theory as:

$$1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}. \tag{26}$$

  Recall that the dimension of a representation with isospin number I, is $2I + 1$. The left hand side has dimension (3 x 2), and the right side has dimension $(4 + 2)$. So we see that the dimensions add up correctly. What the equation means, is that there is one 4D subspace of the 6D space, which transforms to itself under the isospin transformations (that is $I = 3/2$) and another 2D subspace transforming to itself ($I = 1/2$).

- Now we're ready to consider the third factor in the products that appear in the Lagrangian. Namely, we are now looking at $(\mathcal{N}_i \pi_j)\mathcal{N}_k$. The first two factors are placed in a parenthesis to remind us that we have just finished analyzing them in terms of their transformation properties. We are going to use two facts from representation theory:

$$\begin{aligned} \frac{3}{2} \otimes \frac{1}{2} &= 2 \oplus 1 \\ \frac{1}{2} \otimes \frac{1}{2} &= 0 \oplus 1. \end{aligned} \tag{27}$$

  Putting together eqs. (26) and (27) we get

$$\begin{aligned} (1 \otimes \frac{1}{2}) \otimes \frac{1}{2} &= \left(\frac{3}{2} \oplus \frac{1}{2}\right) \otimes \frac{1}{2} \\ &= (\frac{3}{2} \otimes \frac{1}{2}) \oplus (\frac{1}{2} \otimes \frac{1}{2}) \\ &= 2 \oplus 1 \oplus 0 \oplus 1. \end{aligned} \tag{28}$$

- Thus, in the space of tensor products of two nucleons and a pion, there is one subspace with isospin 0 – that is, a 1D subspace which is therefore invariant under isospin transformations. **This is what we care**

about! In general, when constructing Lagrangians that are symmetry-invariant, we use group-representation theory to discover invariant (aka 1D) subspaces of the tensor-product space. The β_{ijk} coefficients of eq. (21) are precisely what is needed in order to construct the invariant subspace.

3.5.2 Two examples of isospin-invariant Lagrangians

- In the last section, we saw that there was one set of β coefficients (up to an overall constant) which can be used to create an isospin invariant Lagrangian out of two nucleons and a pion. Here's the group-theoretic answer (instead of field notation like ψ_p I'll simply write p for simplicity)

$$\mathcal{L}_I = k\left[\bar{p}n\pi^+ + \bar{n}p\pi^- - \frac{1}{\sqrt{2}}\left(\bar{p}p\pi^0 + \bar{n}n\pi^0\right)\right]. \tag{29}$$

  where k is a constant (ultimately a coupling constant).

  This is more or less what we guessed just from thinking about Feynman diagrams, and from the historical development of isospin symmetry. The coefficient $-\frac{1}{\sqrt{2}}$ is the only thing we hadn't yet guessed.

  EXERCISE: Confirm, by directly doing the transformations, that this Lagrangian is invariant under isospin transformations of each of the three fields.

In this next example, we'll look at the isospin invariant term that involves the tensor product of a nucleon, a pion and a decuplet-nucleon. We haven't previously encountered decuplet-nucleons. They were a set of 10 baryons, heavier than the nucleons, and discovered in scattering experiments.

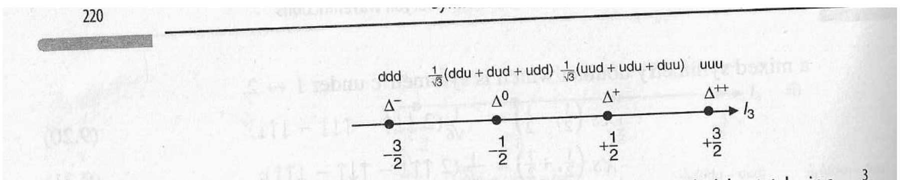


Fig. 9.7 The flavour wavefunctions of the $I = \frac{3}{2}$ light quark Δ-baryons, each of which has total spin $s = \frac{3}{2}$.

The decuplet-nucleons are unstable because they decay into nucleons and pions. We'll use isospin invariance to compute decay ratios. As before, we consider the tensor product of the three kinds of particles. The decuplet is in the isospin representation $I = 3/2$, the nucleons are in the representation, $I = 1/2$ and the pions are in the representation $I = 1$.

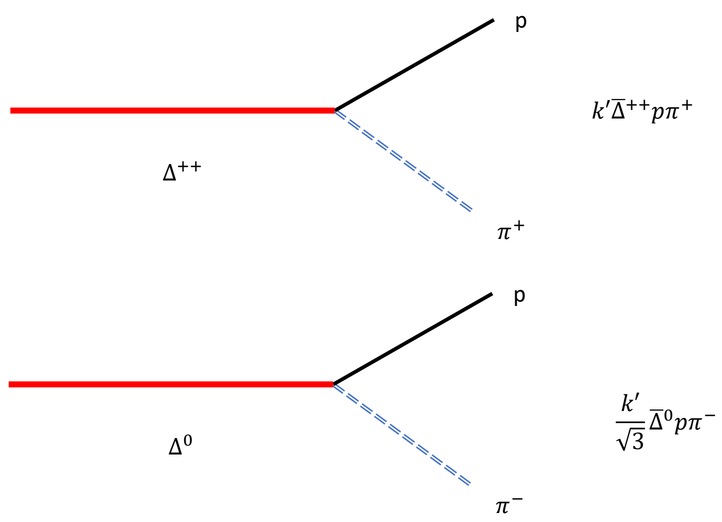
$$\frac{3}{2} \otimes \frac{1}{2} \otimes 1 = 3 \oplus 2 \oplus 2 \oplus 1 \oplus 1 \oplus 0 \tag{30}$$

Again, as before, we only care about the scalar (1D) component with $I = 0$ and we see that there is a unique (up to a constant) combination of terms which is invariant. This leads to

$$\mathcal{L}_I = k' \left[\bar{\Delta}^{++} p \pi^+ - \bar{\Delta}^- n \pi^- - \frac{1}{\sqrt{3}} \left(\bar{\Delta}^+ n \pi^+ - \bar{\Delta}^0 p \pi^- \right) - \frac{\sqrt{2}}{\sqrt{3}} \left(\bar{\Delta}^+ p \pi^0 - \bar{\Delta}^0 n \pi^0 \right) \right]. \tag{31}$$

3.5.3 Decays of Δ's

The Feynman decay diagrams look for example, like this:



These diagrams are similar to one another and describe the decay of a Δ into a pion and a nucleon. What is important is the vertex coefficient. For the Δ^{++}-decay, the coefficient is k' and for the Δ^0-decay, the coefficient is $k'/\sqrt{3}$.

Therefore the ratio of decay amplitudes is $\sqrt{3}$ and the ratio of probabilities is 3.

We therefore predict that the rate of decay of $\Delta^{++} \to p + \pi^+$ is 3 times the rate of decay of $\Delta^0 \to p + \pi^-$.