Thomson Chapter 9 – SU(3)-flavor

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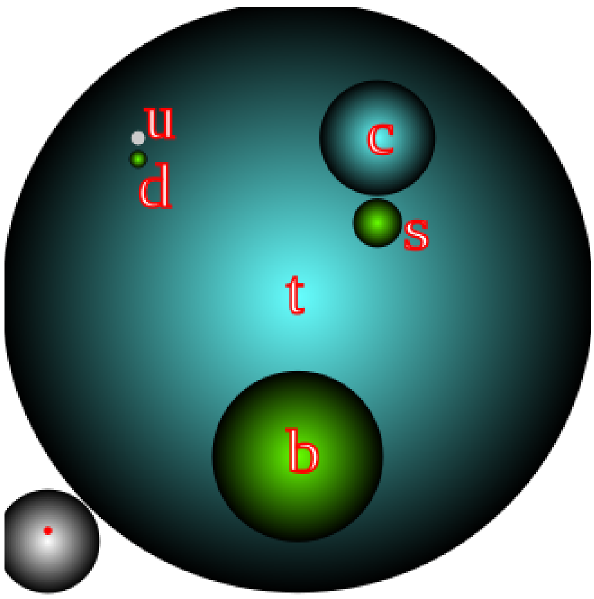
1 Overview

The type of quark (there are 6 types) is known as FLAVOR. The original SU(2)-isospin symmetry (associated with the up and down quarks) has become an SU(3)-flavor symmetry (which adds the strange quark). Although the interactions are symmetric, the quark masses are different and thus break the symmetry – a small amount for up-down particles and a greater amount for particles with strange quarks which are significantly heavier. Although there are 3 more quarks, their large masses break the associated symmetries so much as to be almost useless. At this time, based on the bandwidth of the Z-meson, it is believed that there are no other flavors.



- Isospin recap
- SU(3)-flavor symmetry transformations
- Strange quarks
- Mass dependence of potential energy
- Mesons
- Baryons

2 Isospin recap

2.1 Isospin symmetry

When we say that physics is "isospin symmetric", we mean that the laws of nature look the same after you transform the fields following certain rules known as isospin transformations. For example, if the transformations are linear operators on a vector space, and if $T(g)$ is the linear operator corresponding to the isospin-group element g, then a linear transformation rule, is that $T(g_1)T(g_2) = T(g_1g_2)$.

An immediate consequence of isospin symmetry (or any other symmetry) is that the Hamiltonian – which is responsible for time evolution of the system – commutes with the isospin transformations. It can then be proven that if a system of particles transform among one another according to an *irreducible* representation (a certain specified collection of transformations) then all of their masses will be equal.

The earliest example of an isospin-symmetric system, is the neutron-proton system. The (approximately) equal masses of the neutron and proton are, in fact, a hint that a symmetry is present.

Often, physics is "almost symmetric" and therefore masses are only approximately equal.

The steps to exploitation of isospin symmetry are:

- Characterize different kinds of transformation rules. We will look explicitly here at transformations for 2D and 3D systems.
- Construct symmetric Hamiltonians.

In practice, we don't have a symmetric Hamiltonian. Instead, we construct a symmetric interaction term, which in Lagrangian notation is written $\mathcal{L}_I$. The kinetic (non-interactive) term will have a symmetry-breaking component which – in the quark model – originates purely from the differences in quark masses.

2.2 Isospin-$\frac{1}{2}$ representations

Both the up-down (quark) system, and the neutron-proton system can be described by a 2D vector space using notation of the same form used to represent spin up and spin down for an electron. For example, for neutrons and protons

$$|\text{p}\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\text{n}\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The superposition of a proton and neutron is then represented by a linear combination of the two vectors. A general nucleon state can be thought of as $\alpha|\text{p}\rangle + \beta|\text{n}\rangle$ and can also be written as $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. **A sloppy but convenient notation I'll use, is also $\alpha\mathbf{p}+\beta\mathbf{n}$.** An SU(2) transformation is implemented as

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = U \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{1}$$

where U is a special (determinant 1) 2 x 2 unitary matrix. The most general such matrix can be written as

$$U = e^{i\alpha\cdot\mathbf{T}} \tag{2}$$

where $\alpha \cdot \mathbf{T} \equiv \alpha_1\mathbf{T}_1 + \alpha_2\mathbf{T}_2 + \alpha_3\mathbf{T}_3$ and $\mathbf{T}_i = \frac{\boldsymbol{\sigma}_i}{2}$. The $\boldsymbol{\sigma}_i$ are the usual Pauli spin matrices.

The up and down quark transform in the same way as the proton and neutron, so the superpositions of up and down quark states are in the same representation as the proton-neutron states.

The abstract SU(2) group is actually defined by the above 2D transformations. So the implementation above is both used to define SU(2) and is also used as a 2D representation SU(2). This representation is known as the "fundamental representation" of SU(2) and is denoted the isospin-$\frac{1}{2}$ representation.

2.3 Isospin-1 representations

Another representation is the 3D representation, which is denoted the isospin-1 representation.

The earliest known isospin-1 representation was the system consisting of the three pions π^-, π^0, π^+. These are described by a 3D vector space

$$|\pi^+\rangle \leftrightarrow \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad |\pi^0\rangle \leftrightarrow \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad |\pi^-\rangle \leftrightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}$$

A superposition of pions can be represented in sloppy notation as $\alpha\boldsymbol{\pi}^+ + \beta\boldsymbol{\pi}^0 + \gamma\boldsymbol{\pi}^-$ or in vector notation as

$$\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}. \tag{3}$$

An SU(2) transformation is implemented on this space as

$$\begin{pmatrix}\alpha'\\\beta'\\\gamma'\end{pmatrix} = U \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} \tag{4}$$

where U is can be written as

$$U = e^{i\alpha\cdot\mathbf{T}'} \tag{5}$$

where $\alpha \cdot \mathbf{T}' \equiv \alpha_1\mathbf{T}'_1 + \alpha_2\mathbf{T}'_2 + \alpha_3\mathbf{T}'_3$ and

$$\mathbf{T}'_1 = \frac{\sqrt{2}}{2}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \mathbf{T}'_2 = \frac{\sqrt{2}}{2}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}, \mathbf{T}'_3 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}. \tag{6}$$

2.4 Isospin algebra and isospin labels

The isospin algebra is just the SU(2) algebra which we've encountered when studying the rotation group. When we spoke of rotations, we used the angular momentum symbol $\mathbf{J}$. When we speak of isospin, we'll use the symbol $\mathbf{T}$. Mathematically they are identical, but we use different letters to distinguish context.

When dealing with rotations, there were two operators used to characterize all states in an irreducible representation (such as a spin-1/2 representation). The total isospin operator is the group Casimir operator (see Eugene's notes for more about Casimir operators). It is

$$\mathbf{T}^2 = \mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2. \tag{7}$$

The other operator used for characterizing states is $\mathbf{T}_3$. We then write the isospin content of any state as $|...I, I_3>$ such that

$$\begin{aligned} \mathbf{T}^2|...I, I_3> &= I(I+1)|...I, I_3> \\ \mathbf{T}_3|...I, I_3> &= I_3|...I, I_3> . \end{aligned} \tag{8}$$

The dots indicate other quantum numbers such as 4-momenta or angular momenta etc.

This notation is reminisce of the notation we use for angular momentum. The neutron and proton both have I = 1/2, meaning that they belong to the same 2-dimensional (spin 1/2) multiplet. The proton has $I_3 = +1/2$ and the neutron has $I_3 = -1/2$. Similarly, the up quark has $I_3 = +1/2$ and the down quark has $I_3 = -1/2$.

2.5 Building protons and neutrons from quarks

The isospin quantum numbers of the nucleons are

- p: $(I, I_3) = (\frac{1}{2}, +\frac{1}{2})$,
- n: $(I, I_3) = (\frac{1}{2}, -\frac{1}{2})$.

Similarly, the isospin quantum numbers of the up and down quarks are

- u: $(I, I_3) = (\frac{1}{2}, +\frac{1}{2})$,
- d: $(I, I_3) = (\frac{1}{2}, -\frac{1}{2})$.

The up and down quark have almost the same mass. Each nucleon is a bound state of 3 quarks. The proton is "uud" and the neutron is "ddu". If you add up the I_3 quantum numbers for up and down quarks, you see that the proton and neutron end up with the correct values of I_3. The small difference in mass between the neutron and proton is, in part, due to the small mass difference between the up and down quarks.

2.6 Effective field theories of composite particles

So far, our approach to quantum field theory has emphasized the idea that there are fundamental fields associated with fundamental particles. So, for example, we considered the electron as a fundamental particle associated with a Dirac field, and we considered the photon as a fundamental particle associated with the electromagnetic vector potential field. In the same way, we say that quarks are fundamental particles associated with their own Dirac fields.

However, we could also have imagined that the proton was a fundamental particle whose field theory looks a lot like the field theory of an electron – the only difference being that it has a positive charge and it is much more massive. We would then expect it to have an anti-particle (the anti-proton) and that its dynamics obey the Dirac equation. Indeed, in low-energy experiments, all that is true.

However, unlike an electron, in collisions involving very high energy transfers, the proton appears to behave as though it has constituents, and ultimately this is one of the ways we infer that the proton is actually a composite whose constituents are quarks.

So, in summary, we can predict the low-energy behavior of protons by writing a proton quantum field theory analogous to electrodynamics of electrons, but for much higher energies we describe protons as bound states of more fundamental particles – quarks – which are described by their own **different** quantum field theory.

We refer to the low-energy proton field theory approximation, as *an effective field theory (EFT).* **The importance of isospin and SU(3)-flavor symmetries is a consequence of using effective field theories for the observed baryons in the particle zoo!**

2.7 An isospin-invariant effective field theory of nucleons and pions

Start, for example, with an effective theory of protons, neutrons and pions. In the previous sections, we saw how all of these particles transform under isospin transformations. If we want to create an interaction Lagrangian that is invariant under isospin transformations, and whose terms are each a product of three fields, the unique way to do that[1] is to write (instead of field notation like ψ_p I'll simply write p for simplicity)

$$\mathcal{L}_I = k\left[\bar{p}n\pi^+ + \bar{n}p\pi^- - \frac{1}{\sqrt{2}}\left(\bar{p}p\pi^0 + \bar{n}n\pi^0\right)\right]. \tag{9}$$

where k is a constant (ultimately a coupling constant).

Here is another very similar example. The particle zoo includes baryons that resemble the neutron and proton, but which are heavier. These particles are created in accelerators, and unlike the nucleons, they can strongly decay into other particles including the ordinary proton and neutron, therefore they are unstable.

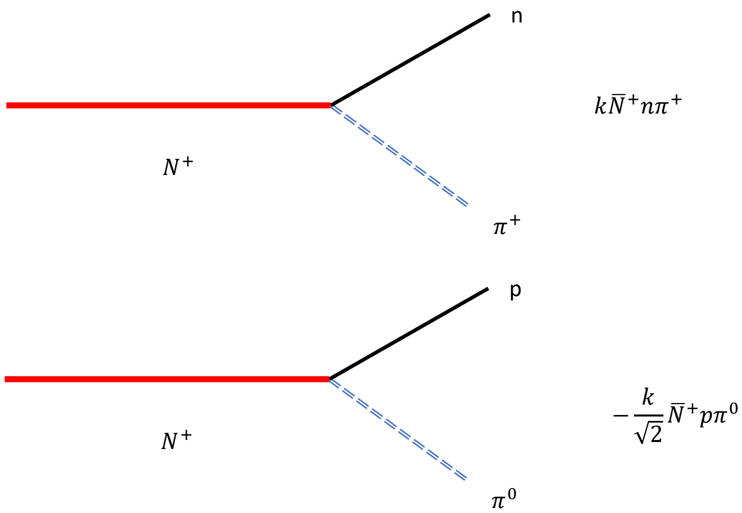
[1]The systematic method of figuring this out is to use the tensor multiplication relations discussed later.

Among these "heavy nucleons" are the so-called Roper resonances N^0 and N^+, whose masses are 1.44 GeV. (Recall that the ordinary nucleons have masses 0.94 GeV and that the pions have masses about 0.14 GeV.) The two Roper resonances form an isospin double just like the neutron-proton system. Therefore if we want to write an isospin-invariant effective Lagrangian that can represent the interactions between Roper-resonances, ordinary nucleons, and pions, we can copy the previous Lagrangian (whose fields all have the same isospin transformation properties) to get

$$\mathcal{L}_I = k\left[\bar{N}^+ n\pi^+ + \bar{N}^- p\pi^- - \frac{1}{\sqrt{2}}\left(\bar{N}^+ p\pi^0 + \bar{N}^0 n\pi^0\right)\right] + \text{herm. conj.}. \quad (10)$$

2.7.1 Prediction of decay ratios

We can then use Feynman diagrams to compute decay rates. For example, here are the Feynman diagrams for positive Roper decays into a nucleon and pion. The Feynman decay diagrams look for example, like this:



These diagrams are similar to one another but what is important is the vertex coefficient. For the first decay, the coefficient is k and for the second decay, the coefficient is $-k/\sqrt{2}$.

Therefore the ratio of decay amplitudes is $-\sqrt{2}$ and the ratio of probabilities is 2.

We therefore predict that the rate of decay of $N^+ \to n + \pi^+$ is 2 times the rate of decay of $N^+ \to p + \pi^0$.

2.8 Tensor products and quark constructions of hadron mass multiplets

Mesons are constructed out of two quarks and baryons are constructed out of three quarks. The overall system Hamiltonian commutes (approximately) with isospin transformations and therefore mesons and baryons appear in (irreducible) isospin multiplets of (approximately) equal mass. Mesons and baryons are therefore classified by I and I_3.

Consider the two-quark meson whose state is

$$|s\rangle = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}). \tag{11}$$

This should remind you of the expression $x^2 + y^2$ which is invariant for any rotation around the z-axis. In a similar way, the state $|s\rangle$ is invariant under SU(2) transformations and therefore is an isospin scalar (a one-dimensional representation). That state is known as the η particle[2].

By contrast, the state $|t\rangle = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$ is NOT invariant under isospin transformations. Instead, it is one state in a 3D space of states (vectors) that transform into one another under isospin transformations. That space is spanned by three basis vectors which are eigenstates of the operator T_3, with eigenvalues -1, 0 and 1. Those eigenstates are known respectively as π^-, π^0, π^+.

We started with a quark and an antiquark, each of which form part of a 2D irreducible representation of isospin. From these pairs, we have described 4 particles – the three pions which form a 3D irreducible representation of isospin and the η which forms a 1D irreducible representation of isospin. Mathematically we say "the tensor product of two 2D reps is equal to the direct sum of a 3D rep and a 1D rep". In symbols, we could write

$$(2) \otimes (2) = (3) \oplus (1). \tag{12}$$

However, more commonly we label the 1D representation with its isospin 0, the 2D representation with its isospin 1/2 and the 3D representation with its isospin 1. The resulting symbolic expression becomes

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1. \tag{13}$$

This kind of expression is known as a tensor-product decomposition, and the coefficients of the state components are known as the Clebsch-Gordan

[2] This is not strictly correct since this particle is also mixed with pairs involving strange and heavier quarks

coefficients. For constructing baryons, the corresponding decomposition is

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2} \tag{14}$$

We've already encountered one of the isospin-$\frac{1}{2}$ baryon families on the RHS of this equation – namely, the neutron-proton family. The isospin-$\frac{3}{2}$ family is 4-dimensional and has the particles $\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-}$.

Although our focus has been on the isospin content of particles, we cannot forget about their angular-momentum content. We know from our studies of the hydrogen atom that a two-particle system has different energies (aka masses) for different angular-momentum states. Similarly, a 2- or 3- quark system can have varying amounts of angular momentum corresponding to different mass multiplets. The angular momentum arises both from orbital motion and **also** from the intrinsic spins of the quarks. Even in the absence of rotational motion, the tensor relations eqs. (13) and (14) can be used because each quark has spin-$\frac{1}{2}$ and spin transformations follow exactly the same rules as isospin transformations.

In the end, we characterize the particles both with isospin and angular momentum.

3 SU(3)-flavor algebra

Reference: Thomson Chapter 9.6.

3.1 The action of SU(3)

The treatment of SU(3) symmetry closely follows the treatment of SU(2). SU(3) is the group of 3-dimensional unitary transformations with unit determinant. A representation of SU(3) is a set of unitary linear transformations in D dimensions (i.e. D x D matrices) whose multiplication rules are the same as the multiplication rules of SU(3) transformations.

SU(3) notation is a bit clearer than isospin notation. We denote the representations by their dimensionality so a 3D representation is simply written as the **3** representation. In fact, the 3D representation is used to define SU(3). An example of a 3D representation is the set of states formed by the up, down and strange quarks.

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = U \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{15}$$

where U is any 3x3 unitary matrix of determinant 1 and can be expressed as

$$U = e^{i\alpha\cdot\hat{\mathbf{T}}}, \tag{16}$$

where the 8 matrices $\hat{\mathbf{T}}_i = \frac{1}{2}\boldsymbol{\lambda}_i$ are the generators of SU(3). These form the basis of the SU(3) Lie Algebra.

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

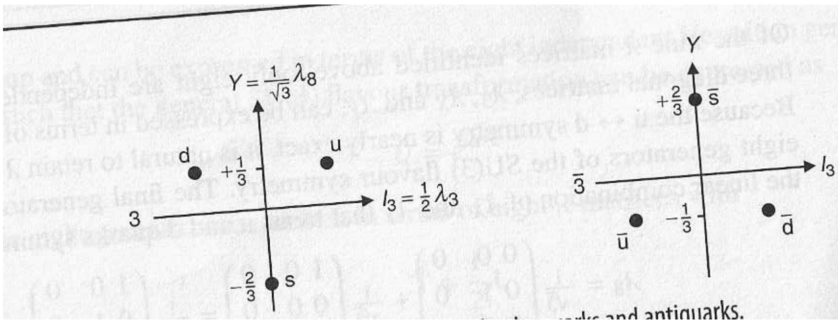
Note that $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3$, acting on vectors of the form $\begin{pmatrix} u \\ d \\ 0 \end{pmatrix}$ are just the Pauli spin matrices acting on the non-zero components. So the isospin group is a subgroup of SU(3).

Unlike the situation with isospin, there aren't any hadrons that transform in the **3** representation of SU(3) – only quarks.

3.2 Characterization of particles under SU(3)-flavor

The Casimir operator $\hat{T}^2 = \sum_{i=1}^{8} \hat{T}_i^2$ is proportional to the identity for all irreducible representations of SU(3) and its eigenvalue characterizes the representation. Unlike isospin, we don't use that eigenvalue to identify the representation. Instead, we use the dimension. Note that there is some ambiguity in this notation because two equal-dimension representations might be inequivalent. For example, there are two inequivalent dimension-3 representations, **3** and $\bar{\mathbf{3}}$.

In addition, particles are classified by their eigenvalues with respect to $\hat{T}_3 = \frac{1}{2}\lambda_3$ – the third component of isospin – and $\hat{Y} = \frac{1}{\sqrt{3}}\lambda_8$ – the *hypercharge*.



3.3 Tensor multiplication

The rules of SU(3) tensor multiplication can be used in several ways:

- Construction of effective SU(3)-flavor field theories for comparing decay amplitudes of various particle channels, similar to what we did in examining the Roper decays. The construction requires an invariant interaction-Lagrangian consisting of terms with 3 fields multiplied together. The systematic extraction of an invariant quantity of that sort, is done by extracting the scalar representation from the tensor product of the representations of the particles appearing in the decays.

- Construction of the particle zoo out of 2 and 3 quarks.

Note that antiquarks are in representation $\bar{\mathbf{3}}$. For mesons, the tensor rule is

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}, \tag{17}$$

and for baryons, the tensor rule is

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \tag{18}$$

Notice that both mesons and baryons appear in octets, and that baryons also appear in decuplets. Gell-Mann's key SU(3)-flavor discovery, was the prediction of the Ω as the tenth element of a decuplet.

4 The SU(3)-flavor zoo

4.1 The light mesons (Thomson 9.6.2)

Thomson uses commutation relations of the SU(3) generators ($\hat{\mathbf{T}}_i$) to explain how to create representations of SU(3). I'll skip those derivations and instead will assume the results. The light mesons are those built out of one quark (up, down or strange) and one anti-quark **and** furthermore are in the kinetic ground state[3]

Recall eq. (17)

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}. \tag{19}$$

This is illustrated diagramatically by

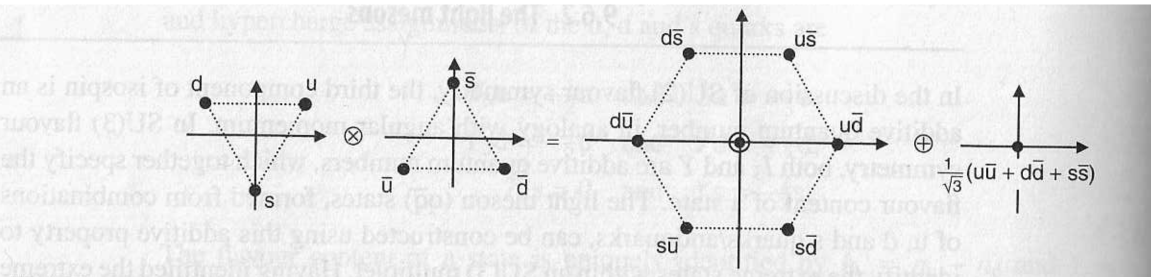


Fig. 9.13 SU(3) flavour $q\bar{q}$ multiplets. The two states at the centre of the octet are linear combinations of $|u\bar{u}\rangle$, $|d\bar{d}\rangle$ and $|s\bar{s}\rangle$ which are orthogonal to the singlet state.

The x-axis is the value of isospin-3 (the third component of isospin) and the y-axis is the value of the hypercharge. The three dots indicate that the dimensionality of the representation is 3.

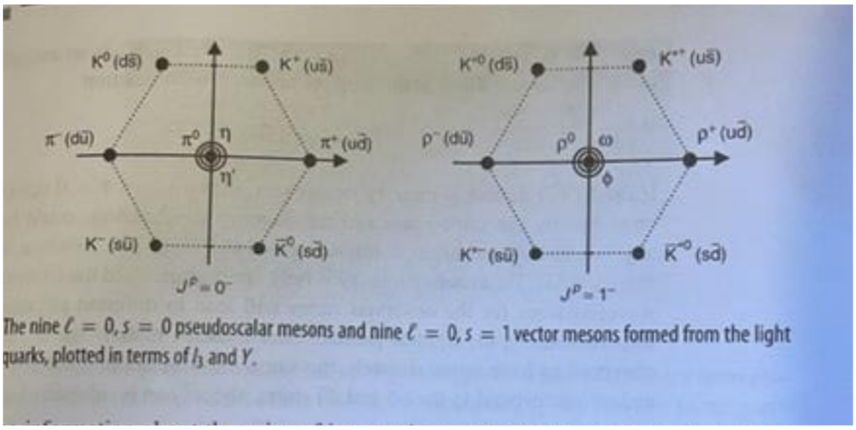
Recall that there are two inequivalent 3-d representations, labelled $\mathbf{3}$ and $\bar{\mathbf{3}}$. These correspond to the two orientations of triangle. The octet representation corresponds on the RHS of the diagram to the 8 dots (6 around the periphery and 2 in the center), whose total isospin-3 and hypercharge are plotted in the figure. The second diagram on the right is the singlet. Note that the scalar character of this singlet resembles the scalar character of the length-squared function $x^2 + y^2 + z^2$.

Now that we have shown the quark-antiquark constituents of the 9 mesons on the right or equivalently, the isospin-3/hypercharge combinations of the mesons on the right, we are ready to identify these with actual particles.

It turns out that there are several different sets of mesons corresponding to the above decompositions. These sets cluster around different masses and their differences correspond to other quantum numbers that we've ignored in the pure flavor analysis. Of particular interest are clusters characterized by bound-state orbital angular momentum $\mathbf{L} = 0$. These are like the s-states of the hydrogen atom, which correspond to the ground state of the non-relativistic Schrodinger equation.

[3]By 'kinetic bound state' I mean a bound state of quarks whose flavors and spins are specified, but not their positions or momenta.

There are two sets of states for **L** = 0. These correspond to spin-orientations. Since each quark has two possible spins, there are 4 possible spin-combinations which, according to the laws of spin tensor algebra, split into a singlet and a triplet. The singlet is S = 0 and the triplet is S = 1. The particle identifications are shown in the following figure.



The nine $\ell = 0, s = 0$ pseudoscalar mesons and nine $\ell = 0, s = 1$ vector mesons formed from the light quarks, plotted in terms of I_3 and Y.

4.2 Baryons

Similar analyses can be done for particles consisting of 3 quarks. These are covered in Thomson 9.6.4. In the relatively recent past, particles have also been discovered which have the characteristics of 4-quark bound states.