Thomson Chapter 10 SU(3)-color: Gauge Theory – Solutions to Exercises

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- EXERCISE: Assuming that $\mathcal{A} = e^{i\phi_1} + e^{i\phi_2}$,

  show that $|\mathcal{A}|^2 = 2\left[1 + \cos(\phi_1 - \phi_2)\right].$

  SOLUTION: Factor out $e^{i\phi_2}$.

$$\mathcal{A} = e^{-\phi_2}\left[1 + e^{i(\phi_1-\phi_2)}\right], \tag{1}$$

so

$$\begin{aligned} \mathcal{A}\mathcal{A}^\dagger &= e^{-i\phi_2}\left[1 + e^{i(\phi_1-\phi_2)}\right] e^{i\phi_2}\left[1 + e^{-i(\phi_1-\phi_2)}\right] \\ &= 1 + e^{i(\phi_1-\phi_2)}e^{-i(\phi_1-\phi_2)} + e^{i(\phi_1-\phi_2)} + e^{-i(\phi_1-\phi_2)} \\ &= 2 + 2\cos\left(\phi_1 - \phi_2\right) \end{aligned} \tag{2}$$

- EXERCISE: When performing a local charge-symmetry transformation of $\phi(x)$ you obtain

$$\begin{aligned} \partial_\mu\phi(x) &\to iq\left(\partial_\mu\chi(x)e^{iq\chi(x)}\right)\phi(x) + e^{iq\chi(x)}\partial_\mu\phi(x) \\ \partial^\mu\phi^*(x) &\to -iq\left(\partial^\mu\chi(x)e^{-iq\chi(x)}\right)\phi^*(x) + e^{-iq\chi(x)}\partial^\mu\phi^*(x). \end{aligned} \tag{3}$$

  When you take the product of the RHS's, you do **not** recover the product of the LHS's (although you do succeed in getting rid of the exponentials). Show this.

  SOLUTION: This is simply a matter of expanding out the RHS of the second equation times the RHS of the first equation.

$$\begin{aligned} &\left[-iq\left(\partial^\mu\chi(x)e^{-iq\chi(x)}\right)\phi^*(x) + e^{-iq\chi(x)}\partial^\mu\phi^*(x)\right]\left[iq\left(\partial_\mu\chi(x)e^{iq\chi(x)}\right)\phi(x) + e^{iq\chi(x)}\partial_\mu\phi(x)\right] \\ &= q^2\phi^*(x)\phi(x)\partial^\mu\alpha(x)\partial_\mu\alpha(x) + \partial^\mu\phi^*(x)\partial_\mu\phi^*(x). \end{aligned} \tag{4}$$

The cross terms, with a single derivative of $\alpha(x)$, cancel one another. The second term on the RHS is just the expression we started with before the symmetry transformation. However, the first term is new and therefore is responsible for breaking the symmetry.

- EXERCISE: we've shown the local invariance of the scalar Lagrangian,

$$\mathcal{L}'(x) = \frac{1}{2} D_\mu \phi^*(x) D^\mu \phi(x) - \frac{1}{2} m^2 \phi^*(x)\phi(x). \tag{5}$$

  Expand out the covariant derivatives in the new Lagrangian (compared to the old Lagrangian with ordinary derivatives) and identify the interaction terms (terms that aren't quadratic in the fields).

  SOLUTION: Recall that $D_\mu \phi(x) = (\partial_\mu + iqA_\mu)\phi(x)$. Then

$$\begin{aligned}
\frac{1}{2} D_\mu \phi^*(x) D^\mu \phi(x) &= \frac{1}{2} \left(\partial_\mu - iqA_\mu\right) \phi^*(x) \left(\partial^\mu + iqA^\mu\right) \phi(x) \\
&= \frac{1}{2} \partial_\mu \phi^*(x) \partial^\mu \phi(x) + \frac{1}{2} iqA^\mu \left[(\partial_\mu \phi^*(x))\phi(x) + \phi^*(x)(\partial_\mu \phi(x))\right] \\
&+ \frac{1}{2} q^2 A_\mu A^\mu \phi^*(x)\phi(x)
\end{aligned} \tag{6}$$

  Notice that the are cubic terms with one vector field, one derivative of a scalar field, and one undifferentiated scalar field. There are also quartic terms with two vector fields and two scalar fields and no derivatives.

- EXERCISE: Show that we can't have a mass term that is proportional to $A_\mu A^\mu$. Hint: show that such a term violates the local symmetry implied by the gauge transformation of A_μ.

  SOLUTION: If we had a term of the form $mA_\mu A^\mu$, then if it were locally symmetric, that would mean $mA'_\mu A'^\mu = mA_\mu A^\mu$ where $A'_\mu(x) = A_\mu(x) - \partial_\mu \chi(x)$. Try it.

$$\begin{aligned}
mA'_\mu A'^\mu &= m\left(A_\mu(x) - \partial_\mu \chi(x)\right)\left(A^\mu(x) - \partial^\mu \chi(x)\right) \\
&\neq mA_\mu(x) A^\mu(x).
\end{aligned} \tag{7}$$