Thomson Chapter 10 SU(3)-color: Gauge Theory

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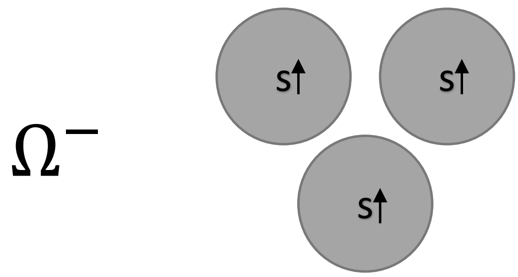
2 Preface: Color



So far, our quark model has been based on the idea that there are spin-$\frac{1}{2}$ quarks of various flavors, whose bound states are the observed hadrons. Qualitatively, this model was initially very successful at explaining particles that had been observed by the early 60's. But there was a serious glitch (quite apart from the fact that no-one has ever seen an isolated quark). If one assumed that the lowest-energy baryonic bound states (i.e., lowest-mass baryons such as the proton and neutron) have 0 angular momentum (this is the case for most potentials, such as the Coulomb potential), then the overall baryon wave-function (taking into account spin, flavor and orbital configuration) could **not** be antisymmetric. **But that contradicts the Pauli exclusion principle which requires fermionic wavefunctions to be antisymmetric.** A somewhat simplistic way of thinking about this, is to consider the Ω meson, which has a spin state where the 3 strange quarks all have spin-up. The Pauli exclusion principle would prohibit that, since it isn't possible to have two identical fermions (much less 3 particles) in the same state. You might ask why the quarks had to be fermions, but by the 1960's it was firmly established that a consistent field theory required that spin-$\frac{1}{2}$ should be fermions.

BUT NOT QUITE! In a very readable review by Wally Greenberg, The Origin of Quark Color, he explains how he'd been exploring the question of whether any other kind of statistics was possible besides fermions and bosons. He immediately applied his ideas to the mystery of baryon wave-function anti-symmetry and postulated that instead of fermions, the quarks were *para-fermions* of a certain type. Mathematically, his theory was equivalent to a theory in which each quark came in 3 types which he called "color". Once he'd introduced color, he was then able to use wave-function anti-symmetry (both for $L = 0$ bound states and higher-mass bound states) to predict mass patterns and other properties of many new baryons. Thomson, in section 9.4, goes into quite a bit of detail on how to construct baryon wave-functions that take into account color, flavor, spin and space (e.g. orbitals).

Greenberg describes an early conversation he had with Oppenheimer, in which he explained his idea of color.

> When I asked him if he had read my paper, he said, "It's beautiful." I was elated. My elation was, however, short-lived, because Oppenheimer's next statement was, "but I don't believe a word of it."

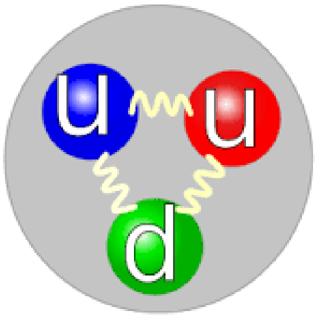
Notwithstanding Oppenheimer, some physicists were drawn to Greenberg's idea and began to expand on it. Earlier, in 1954, Yang and Mills had explored the idea that strong interactions had an origin analogous to that of the electromagnetic field. In the abstract to their paper, they said

> The electric charge serves as a source of electromagnetic field; an important concept in this case is gauge invariance which is closely connected with (1) the equation of motion of the electromagnetic field, (2) the existence of a current density, and (3) the possible interactions between a charged field and the electromagnetic field. We have tried to generalize this concept of gauge invariance to apply to isotopic spin conservation.

However, the work of Yang and Mills on isotopic spin wasn't able to be reconciled with experiment, partly because they couldn't propose a mechanism that broke the symmetry (the proton and neutron have different masses). The color property, as explained by Greenberg, was a complete (and unbroken) symmetry of the theory. So after Greenberg's introduction of color, the idea of a color-generalization to gauge symmetry was revived by Han and Nambu (in 1965). This theory, over time, became wildly successful.[1] We therefore begin with a thorough discussion of gauge symmetry as it appears

[1] There remains a question which puzzles me as well as others. Namely, why didn't the original Yang-Mills theory work, especially many decades later after a better understanding of how symmetry-breaking can be accommodated in a gauge theory? In other words, why does SU(3)-flavor (or more appropriately SU(6)-flavor) not generalize to a gauge theory in the same way that SU(3)-color does? As far as I know, this question has been explored but without any kind of convincing resolution. It's all part of the more general exploration of supersymmetrical and grand-unified theories. In my opinion, we need some convincing reason why there are only 3 generations of particles, which appear to behave more or less the same as one another.

in classical and quantum electrodynamics, and then show how these ideas become generalized following Yang and Mills.



3 The electromagnetic vector potential and gauge invariance

We start with the classical theory. Maxwell's equations:

$$\begin{aligned} \nabla \times \mathbf{B} - \partial_t \mathbf{E} &= \mathbf{J} \\ \nabla \cdot \mathbf{E} &= \rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0 \end{aligned} \tag{1}$$

In this form, Lorentz invariance isn't obvious. Instead, use an alternative form. Define the electromagnetic field strength tensor $F_{\mu\nu}$ as

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{2}$$

and the current 4-vector $J^\mu = (\rho, J^x, J^y, J^z)$.

Then Maxwell's equations can be rewritten using 'Lorentz covariant' tensors – i.e., tensors which, under a change of inertial frame undergo Lorentz transformations.

$$\begin{aligned} \partial_\mu F^{\nu\mu} &= J^\mu \\ \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} &= 0 \end{aligned} \tag{3}$$

These equations don't have any different content than the original Maxwell equations, but they are more apparently Lorentz invariant. (Note that we obtained this manifest covariance at the expense of adding some redundancy into the definition of $F_{\mu\nu}$. That often happens when exposing a hitherto hidden symmetry.)

Early on (19th century) mathematicians and physicists introduced the scalar potential ϕ and vector potential $\mathbf{A}$ as mathematical devices for simplifying the solutions of electromagnetic equations. These can be combined into a single 4-vector A^μ with the property that the electromagnetic field strength tensor can be written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4}$$

Since the electric and magnetic fields are components of $F_{\mu\nu}$, this means that the 4 components of the vector potential are sufficient to determine the 6 components of the electric and magnetic field.

Notice that there is a many-to-one correspondence between the EM fields and the vector potentials. If two vector potentials are related by

$$A'_\mu = A_\mu - \partial_\mu \chi, \tag{5}$$

for **any** scalar function $\chi(t, \mathbf{x})$, then it's easy to see that the LHS of eq. (4) is the same for both A and A'. That is, the $\mathbf{E}$ and $\mathbf{B}$ fields don't depend on χ. **We call eq. (5) a gauge transformation. In classical EM, all electromagnetic effects are given by the electric and magnetic fields, so physics is independent of χ.**

4 The vector potential in classical and quantum theories

Classically, the electric and magnetic fields act on a particle by exerting a force on the particle

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{6}$$

What about the vector potential? Feynman addresses this in a beautiful discussion in chapter 15 of Volume II of The Feynman Lectures. In chapter 15.4, he asks

"**Is the vector potential merely a device which is useful in making calculations .. or is the vector potential a "real" field?** Isn't the magnetic field the "real" field, because it is responsible for the force on a moving particle? First we should say that the phrase "a real field" is not very meaningful. ... What we mean here by a "real" field is this: a real field is a mathematical function we use for avoiding the idea of action at a distance ...A "real" field is then a set of numbers we specify in such a way about that what happens *at a point* depends only on the numbers *at that point*."

Feynman then goes on to explain that in quantum mechanics, the idea of force loses the central role it plays in classical physics. Instead, energy and momentum and ultimately the vector potential, are more important.

This point was clarified in a surprising paper written in 1956 by Aharonov and Bohm and explained by Feynman. They considered a solenoid.

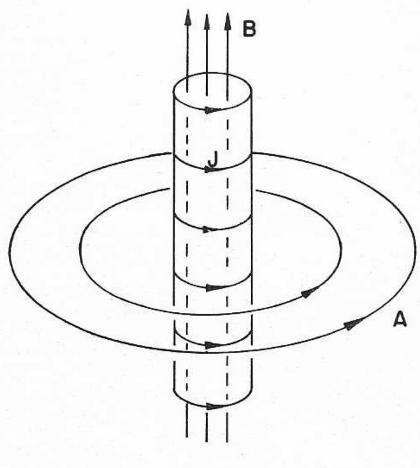


Fig. 15–6. The magnetic field and vector potential of a long solenoid.

The interior of the solenoid contains a magnetic field, but the exterior has $\mathbf{B} = 0$. Classically, a particle travelling on the exterior of the solenoid experiences no effect since the magnetic force is 0. However, the vector potential outside the solenoid can't be 0. This follows from the fact that by Stoke's theorem

$$\oint \mathbf{A} \cdot d\mathbf{s} = \int\int (\nabla \times \mathbf{A}) \cdot \hat{\mathbf{n}} dS = \int\int \mathbf{B} \cdot \hat{\mathbf{n}} dS. \tag{7}$$

The LHS is the integral over a contour that surrounds the solenoid, whereas the RHS is an integral over the surface enclosed by the contour. That surface also encloses the solenoid and therefore the magnetic field inside the solenoid. Since that magnetic field is non-zero, the LHS is non-zero and thus the vector potential cannot be 0 everywhere outside the solenoid.

In quantum mechanics, the wave equation depends on the vector potential and thus the particle evolution changes outside the solenoid even though no magnetic field is present there.

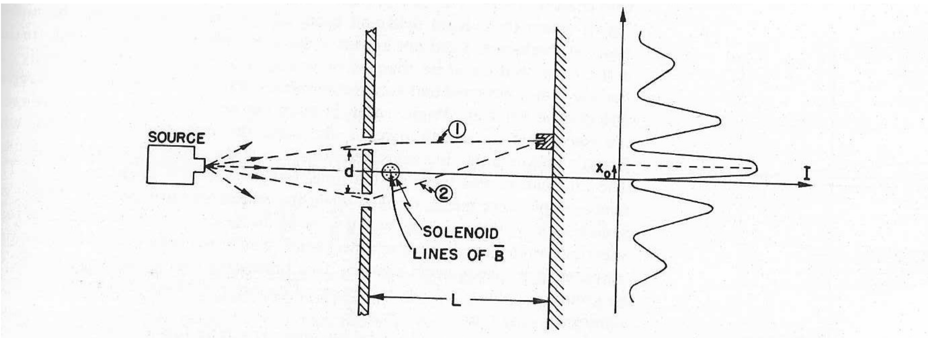


Fig. 15–7. A magnetic field can influence the motion of electrons even though it exists only in regions where there is an arbitrarily small probability of finding the electrons.

Experiments ultimately confirmed the Aharonov-Bohm effect. Therefore, in quantum mechanics, the magnetic field is **not** a "real" field in the sense defined above. (Strictly speaking, using the definition of "real" field, the vector potential also isn't a "real" field since its local effects turn out to depend on contour integrals.)

There are several ways of deriving the Aharonov-Bohm effect. One can start with the modified Schrodinger equation.

$$i\partial_t \psi = \frac{1}{2m} \left(-i\nabla - q\mathbf{A}\right)^2 \psi. \tag{8}$$

We see that this equation involves the vector potential, rather than the magnetic field. About this, Feynman says

> "This subject has an interesting history. The theory we have described was known from the beginning of quantum mechanics

in 1926. The fact that the vector potential appears in the wave equation of quantum mechanics ... was obvious from the day it was written. That it cannot be replaced by the the magnetic field in any easy way was observed by one man after the other who tried to do so."

Another way of deriving the Aharonov-Bohm effect comes from the path integral approach. This turns out to be the clearest derivation. Of course, since the path integral is ultimately equivalent to the Schrodinger equation, the derivations are equivalent to one another. Consider a particle going from a point $x = A$ to a point $x = B$ over a period of time T. In the path integral formulation, the probability amplitude for this process is given as

$$\mathcal{A} = \int \mathcal{D}x(t) e^{iS(x(t))}, \tag{9}$$

where the integral is taken over all paths from A to B, and the function S is the action. For a non-relativistic particle under the influence of an electromagnetic field, the action is

$$S = \int dt \left[\frac{m}{2} (\frac{d\mathbf{x}}{dt} \cdot \frac{d\mathbf{x}}{dt}) + q\mathbf{A} \cdot \frac{d\mathbf{x}}{dt} \right] \equiv S_0 + q \int dt \left(\mathbf{A} \cdot \frac{d\mathbf{x}}{dt} \right) = S_0 + q \int_A^B \mathbf{A} \cdot d\mathbf{s}. \tag{10}$$

S_0 is the free action. The path integral (probability amplitude) becomes

$$\mathcal{A} = \int \mathcal{D}x(t) e^{iS_0(x(t))} e^{q \int_A^B \mathbf{A} \cdot d\mathbf{s}}. \tag{11}$$

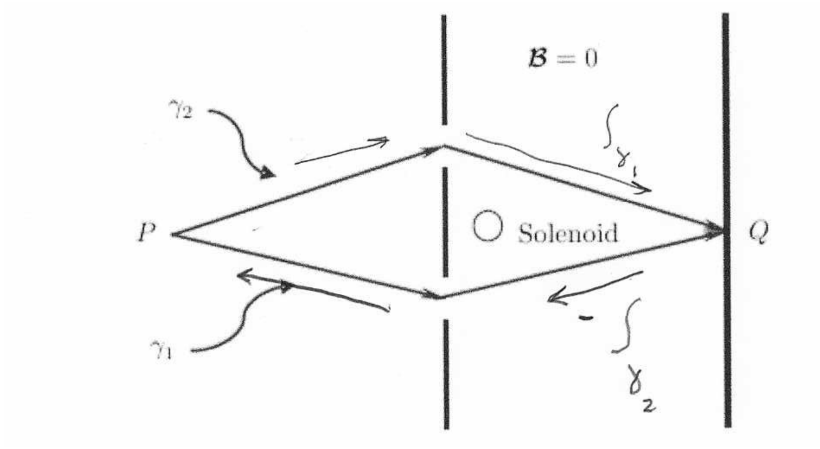
Now we wave our arms a bit. You may recall that one of the virtues of the path integral, is that it bridges classical and quantum mechanics. The dominant contribution to the path integral arises from the *stationary* paths – i.e. those that satisfy the Euler-Lagrange equations. Other paths contribute but are suppressed by factors of $\hbar$. In the double slit experiment illustrated above, the dotted lines represent the dominant paths which can pass through the two slits. Assume these are the only paths that matter so that

$$\mathcal{A} \approx e^{iS_0^1(x(t))} e^{q \int_1 \mathbf{A} \cdot d\mathbf{s}} + e^{iS_0^2(x(t))} e^{q \int_2 \mathbf{A} \cdot d\mathbf{s}} \tag{12}$$

where the indices 1 and 2 refer to the upper and lower path respectively. Generically this looks like $\mathcal{A} = e^{i\phi_1} + e^{i\phi_2}$.

EXERCISE: Show that $|\mathcal{A}|^2 = 2\left[1 + \cos(\phi_1 - \phi_2)\right]$.

From this it follows that the probability depends on the vector potential **only** as $\int_2 \mathbf{A} \cdot d\mathbf{s} - \int_1 \mathbf{A} \cdot d\mathbf{s}$.



We see that

$$\int_2 \mathbf{A} \cdot d\mathbf{x} - \int_1 \mathbf{A} \cdot d\mathbf{x} = \oint \mathbf{A} \cdot d\mathbf{s} \tag{13}$$

where the contour surrounds the solenoid. But this is the same contour integral we encountered in eq. (7) and it is non-zero because it surrounds the solenoid's magnetic field. Hence the process-probability depends on the magnetic field even though the particle appears to travel only in a region with no magnetic field (mind you, some of the paths in the path integral indeed intersect the solenoid, so I'm not sure this is very mysterious).

One more thing. Now that we've seen that the vector potential is what matters in quantum mechanics, what if we do a gauge transformation to

$$A_\mu^{'} = A_\mu - \partial_\mu \chi? \tag{14}$$

Does quantum mechanics give us a different result for A than for A'? The answer is no. To see this, consider again eq. (7).

$$\oint \mathbf{A'} \cdot d\mathbf{s} = \int\int \mathbf{B} \cdot \hat{\mathbf{n}} dS + \int\int (\nabla \times \nabla\chi) \cdot \hat{\mathbf{n}} dS. \tag{15}$$

But the last term is 0 since $(\nabla \times \nabla)\,\chi = 0$. So the contribution is gauge invariant as it should be.

5 Local charge symmetry and gauge invariance

5.1 Derivation of the QED Lagrangian

Consider a simple complex scalar field theory whose Lagrangian consists only of kinetic and mass terms.

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu \phi^*(x)\partial^\mu \phi(x) - \frac{1}{2}m^2\phi^*(x)\phi(x). \tag{16}$$

This theory is invariant under the symmetry transformation

$$\phi(x) \to e^{i\alpha}\phi(x). \tag{17}$$

To see this, note that $\phi^*(x) \to e^{-i\alpha}\phi^*(x)$ so that

$$\begin{aligned}\frac{1}{2}\partial_\mu \phi^*(x)\partial^\mu \phi(x) - \frac{1}{2}m^2\phi^*(x)\phi(x) &\to \frac{1}{2}\partial_\mu \phi^*(x)e^{-i\alpha}e^{i\alpha}\partial^\mu \phi(x) - \frac{1}{2}m^2 e^{-i\alpha}e^{i\alpha}\phi^*(x)\phi(x)\\ &= \frac{1}{2}\partial_\mu \phi^*(x)\partial^\mu \phi(x) - \frac{1}{2}m^2\phi^*(x)\phi(x).\end{aligned} \tag{18}$$

This symmetry is true for any choice of α. We call this symmetry a **global charge symmetry**.

IMPORTANT: To prove this, we relied on the fact that

$$e^{-i\alpha}\partial_\mu \left(e^{i\alpha}\phi(x)\right) = e^{-i\alpha}e^{i\alpha}\partial_\mu \phi(x). \tag{19}$$

This is true only because $e^{i\alpha}$ is a constant.

Now imagine that we want to extend this symmetry so that a different α could be chosen for each point in spacetime. That is,

$$\phi(x) \to e^{iq\chi(x)}\phi(x) \tag{20}$$

where we've replace the constant α with a function $q\chi(x)$ (we could have written a function $\beta(x)$ but by convention, we always factor out an arbitrary constant q). This is known as a **local transformation** and what we are looking for, is a way for this local transformation to be a **local symmetry**. The mass term continues to be invariant.

$$\frac{1}{2}m^2\phi^*(x)\phi(x) \to \frac{1}{2}m^2 e^{-iq\chi(x)}e^{iq\chi(x)}\phi^*(x)\phi(x) = \frac{1}{2}m^2\phi^*(x)\phi(x) \tag{21}$$

However, the kinetic term is **not** invariant, because the derivative acts on the phase factor (unlike the case when the phase factor is constant). Specifically, notice that

$$\begin{aligned} \partial_\mu \phi(x) &\to iq \left(\partial_\mu \chi(x) e^{iq\chi(x)}\right) \phi(x) + e^{iq\chi(x)} \partial_\mu \phi(x) \\ \partial^\mu \phi^*(x) &\to -iq \left(\partial^\mu \chi(x) e^{-iq\chi(x)}\right) \phi^*(x) + e^{-iq\chi(x)} \partial^\mu \phi^*(x). \end{aligned} \tag{22}$$

When you take the product of the RHS's, you do **not** recover the product of the LHS's (although you do succeed in getting rid of the exponentials).

EXERCISE: Show this.

Here's the grand trick that lets you turn the global symmetry into a local symmetry.. Introduce a new field. For grins, we'll call it A_μ. (For a moment, pretend we never heard of electromagnetism.) Using suggestive notation, we'll define

$$D_\mu \phi(x) = (\partial_\mu + iqA_\mu)\phi(x). \tag{23}$$

This resembles the definition, in geometry, of a covariant derivative and indeed we will call it a covariant derivative. Let's consider the transformations of both the ϕ and A_μ fields:

$$\begin{aligned} \phi(x) &\to e^{iq\chi(x)} \phi(x) \\ A_\mu(x) &\to A_\mu(x) - \partial_\mu \chi(x). \end{aligned} \tag{24}$$

This is a local transformation. As before, the mass term (which doesn't depend on derivatives or on A_μ) is invariant under the local transformation. What happens to $D_\mu \phi(x)$ under this local transformation?

$$\begin{aligned} D_\mu \phi(x) &= \partial_\mu \phi(x) + iqA_\mu(x)\phi(x) \\ &\to \partial_\mu \left(e^{iq\chi(x)}\phi(x)\right) + iq(A_\mu - \partial_\mu \chi(x)) e^{iq\chi(x)} \phi(x) \\ &= \left(iq\partial_\mu \chi(x) - iq\partial_\mu \chi(x)\right) e^{iq\chi(x)} \phi(x) + e^{iq\chi(x)} \left(\partial_\mu \phi(x) + iqA_\mu(x)\phi(x)\right) \\ &= e^{iq\chi(x)} \left(\partial_\mu \phi(x) + iqA_\mu(x)\phi(x)\right) \\ &= e^{iq\chi(x)} D_\mu \phi(x). \end{aligned} \tag{25}$$

Similarly, $D^\mu \phi^*(x) \to e^{-iq\chi(x)} D^\mu \phi^*(x)$. Together, we obtain

$$D_\mu \phi^*(x) D^\mu \phi(x) \to D_\mu \phi^*(x) e^{-iq\chi(x)} e^{iq\chi(x)} D^\mu \phi(x) = D_\mu \phi^*(x) D^\mu \phi(x). \tag{26}$$

This modified kinetic term is therefore invariant under the local transformation. Based on all this, we've shown the local invariance of the new

Lagrangian,

$$\mathcal{L}'(x) = \frac{1}{2} D_\mu \phi^*(x) D^\mu \phi(x) - \frac{1}{2} m^2 \phi^*(x)\phi(x). \tag{27}$$

Now you might object that we cheated by introducing, in eq. (30), a completely arbitrary transformation for the A_μ field. And if this were the only term in the Lagrangian with the A_μ field, then indeed there might not be much content in this transformation. However, in order for the A_μ field to actually **do** anything, it needs to be involved elsewhere in the Lagrangian. In particular, it needs a kinetic (quadratic, with a derivative or two) term so that it can propagate. **So we need to add to the Lagrangian a kinetic term which is invariant under the transformation of A_μ.** Here's the magic. The A_μ transformation is one we've encountered before. **It's a gauge transformation!** And we know that $F_{\mu\nu}$ is invariant under gauge transformations. That means we can add a kinetic term proportional to $F_{\mu\nu}F^{\mu\nu}$, and that term will be automatically invariant under the local transformation. And now we're done! We have a theory which is locally invariant and which includes kinetic terms for both the scalar and A_μ field. Not only that, but the theory uniquely determines the interaction between the scalar and vector fields.

EXERCISE: Expand out the covariant derivatives in the new Lagrangian and identify the interaction terms (terms that aren't quadratic in the fields).

EXERCISE: Show that we can't have a mass term that is proportional to $A_\mu A^\mu$. Hint: show that such a term violates the local symmetry implied by the gauge transformation of A_μ.

So far, this discussion has been about scalar fields. However, the electron and other fermions are described by Dirac fields. Recall that the Dirac Lagrangian for a non-interacting fermion is

$$\mathcal{L}_D = \bar{\psi}(x) \left(i\partial\!\!\!/ - m \right) \psi(x) \tag{28}$$

As before, this Lagrangian has a global symmetry $\psi(x) \to e^{i\alpha}\psi(x)$. And as before, if we wish to promote this global symmetry to a local symmetry, we have to introduce a new field A_μ. Then, again as before, the Lagrangian

$$\mathcal{L}'_D = \bar{\psi}(x) \left(iD\!\!\!\!/ - m \right) \psi(x) \tag{29}$$

is invariant under the transformations

$$\begin{aligned} \psi(x) &\to e^{iq\chi(x)}\psi(x) \\ A_\mu(x) &\to A_\mu(x) - \partial_\mu \chi(x). \end{aligned} \tag{30}$$

If we expand out the Lagrangian of eq. (29), and then add the kinetic term for the A_μ field, we get the familiar QED Lagrangian

$$\mathcal{L}'_D = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(x)\left(i\partial\!\!\!/ - m\right)\psi(x) + iq\bar{\psi(x)}A\!\!\!/\psi(x). \tag{31}$$

5.2 U(1)

The analysis of electrodynamic gauge symmetry does not require any reference to group symmetries. However, when we generalize to SU(3), it's useful to make a connection between the above discussion, and group theory. Let's go back to the global transformation law $\phi(x) \to e^{i\alpha}\phi(x)$. This is actually a 1D representation of the group U(1). To see this, consider a 1D representation of U(1) with representations of g_1 and g_2 as

$$\begin{aligned} g_1 &\to U_1 = e^{\alpha_1} \\ g_2 &\to U_2 = e^{\alpha_2} \end{aligned} \tag{32}$$

Then if $g_1 \circ g_2 = g_3$ and $g_3 \to U_3 = e^{\alpha_3}$, it turns out that $U_1 U_2 = U_3$.

5.3 Historical notes

For historical reasons, the A_μ field is called a **gauge field**. It plays somewhat the same role as the metric in general relativity, which permits the laws of nature to be generally covariant under coordinate transformations. In fact, some of the first developments of gauge theory were due to Weyl who, as early as 1918, sought to unify gravity and electromagnetism. He was perturbed by the notion that on a Riemann manifold, the directions of vectors at different points could be compared to one another by using a connection, but that the lengths of the vectors were assumed to be the same when transported to a different point. He thus generalized Riemann geometry to allow a length transformation, and he then proposed that this had to do with electromagnetism. Einstein responded, as reported in Early History of Gauge Theory and Weak Interactions.

> Einstein admired Weyl's theory as "a coup of genius of the first rate ...", but immediately realized that it was physically untenable: "Although your idea is so beautiful, I have to declare frankly that, in my opinion, it is impossible that the theory corresponds to nature."

Later, in 1929, Weyl adapted his original idea to both correct the issue discovered by Einstein, and to accommodate quantum mechanics.[2] Instead of a length transformation, he considered a phase transformation (which turns out to be the phase factor used in the Aharanov-Bohm discussion above). This early history, as well as subsequent generalizations to more complex groups such as SU(3), is covered in Straumann's interesting review above. I especially enjoyed, in that paper, numerous quotes and pieces of correspondence between luminaries such as Weyl, Pauli and Einstein. For example, here's a biting letter from Pauli to Weyl in response to an article written by Weyl in 1929.

> "Before me lies the April edition of the Proc.Nat.Acad. (US). Not only does it contain an article from you under "Physics" but shows that you are now in a 'Physical Laboratory': from what I hear you have even been given a chair in 'Physics' in America. I admire your courage; since the conclusion is inevitable that you wish to be judged, not for success in pure mathematics, but for your true but unhappy love for physics [5]"

6 Summary up to now

- The vector potential A_μ was introduced in classical mechanics as an elegant mathematical quantity which was also sometimes useful for more easily solving Maxwell's equations.
- In classical mechanics, the forces on particles are proportional to the **E** and **B** fields.
- Vector potentials differing by gauge transformations (an additional term of the form $\partial_\mu \chi(x)$), correspond to the same **E** and **B** fields so we might suspect that the A_μ fields are less "real" than the electric or magnetic fields.

[2] An unusually thorough analysis of the relationship between geometric connections and gauge connections, can be found in chapter 25.2 of Schwartz's text.

- This suspicion is wrong. In fact, the quantum mechanical equations of motion only involve the A_μ fields. In the Aharanov-Bohm experiment, particles appear to move in a region of space where $\mathbf{B} = 0$ but nevertheless have their wavefunctions altered by the fact that in that region $\mathbf{A} \neq 0$.

- The origin of the A field can be inferred from a local-symmetry hypothesis. Namely, we require that the theory of fermions not only be invariant under the global U(1) group known as *charge symmetry*, i.e. global phase transformations of the form $\psi(x) \to e^{iq\alpha}\psi$, but it should also be invariant under point-by-point transformations $\psi(x) \to e^{iq\chi(x)}\psi$.

- This requirement can only be accomplished by introducing a new field that plays a similar role to the metric in general relativity. It 'connects' vectors (such as derivatives of a field) from one point to another in spacetime.

- That new field is then identified as the electromagnetic vector potential A_μ, and its transformation law turns out to be the gauge transformation. A kinetic term must be added to the Lagrangian, and that term must be invariant under that gauge transformation. The appropriate kinetic term is proportional to $F_{\mu\nu}F^{\mu\nu}$.

- The resulting exact form of a locally charge-invariant Lagrangian is then QED.

- In the meantime, the theory of quarks emerged. Three quarks make up a baryon. For example, 3 strange quarks make up an ω^- baryon. In order to avoid conflicts with Pauli's exclusion principle, we need to invent a new quantum number so that all three strange quarks are actually different than one another. The new quantum number is called color and (see below) represents an exact global symmetry.

- Just as we did for the global charge symmetry of fermions, we can explore what happens if we promote the global color symmetry of quarks to a local theory. That's what we'll do next.

7 Color symmetry – SU(3)

We have previously encountered the group SU(3) as way of explaining patterns of masses among baryons. This is called SU(3)-flavor, and it can be simply understood as the theory of 3 different quarks, which are labeled by a quantity we call **flavor** and whose interactions between one another are flavor-independent.

SU(3)-color is an entirely different thing. For **each** type of quark (for example, the strange quark), there are 3 distinct particles of identical mass and these are labeled by a quantity we call color. For example – red, green and blue. Whereas different-flavored quarks have different masses, different-colored quarks have the same mass. Thus 'color' is a true symmetry **just like charge symmetry is a true symmetry of electrodynamics.** Because there are 3 colors, the color symmetry is SU(3)

7.1 Refresher: The action of SU(3)

This material was covered in notes for section 9.6 of Thomson.

SU(3) is the group of 3-dimensional unitary transformations with unit determinant. A representation of SU(3) is a set of unitary linear transformations in D dimensions (i.e. D x D matrices) whose multiplication rules are the same as the multiplication rules of SU(3) transformations.

An example of a 3D representation is the set of states formed by the blue, green and red strange-quarks. Or if you prefer a different example, consider the blue, green and red up-quarks. Think of these designators as vector components. Then

$$\begin{pmatrix} b' \\ g' \\ r' \end{pmatrix} = U \begin{pmatrix} b \\ g \\ r \end{pmatrix} \tag{33}$$

where U is any 3x3 unitary matrix of determinant 1 and can be expressed as

$$U = e^{i\alpha\cdot\hat{\mathbf{T}}}, \tag{34}$$

where the 8 matrices $\hat{\mathbf{T}}_i = \frac{1}{2}\boldsymbol{\lambda}_i$ are the generators of SU(3). These form the basis of the SU(3) Lie Algebra.

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Now, we'll change notation slightly. Instead of writing

$$\begin{pmatrix} b \\ g \\ r \end{pmatrix}, \tag{35}$$

we'll write ψ where we define ψ to be the 3-component object

$$\psi = \begin{pmatrix} \psi_b \\ \psi_g \\ \psi_r \end{pmatrix}. \tag{36}$$

Then the SU(3)-color transformations act on ψ.

7.2 Making SU(3)-color into a local symmetry

This section follows 10.1.1 in Thomson.

We're now ready to make the connection with electrodynamics. Recall that the charge symmetry is the U(1)-group acting on, for example, the electron as $\psi_e \to e^{i\alpha}\psi_e$.

Now we have the color symmetry as the SU(3)-group acting on, for example, the strange quarks, as $\psi \to e^{i\boldsymbol{\alpha}\cdot\hat{\mathbf{T}}}\psi$. We can confirm, as we did for the charge (phase) transformations, that the free quark Lagrangian is indeed invariant under color transformations.

$$\bar{\psi}(x)\left(i\partial\!\!\!/ - m\right)\psi(x) = \bar{\psi}(x)e^{-i\boldsymbol{\alpha}\cdot\hat{\mathbf{T}}}\left(i\partial\!\!\!/ - m\right)e^{i\boldsymbol{\alpha}\cdot\hat{\mathbf{T}}}\psi(x), \tag{37}$$

since $e^{-i\boldsymbol{\alpha}\cdot\hat{\mathbf{T}}}e^{i\boldsymbol{\alpha}\cdot\hat{\mathbf{T}}} = 1$. (Recall that $\not{\partial}\psi(x)$ just means $\gamma_\mu\partial^\mu\psi(x)$). Just as we saw with global charge transformations, the derivative term 'commutes' with the constant (in spacetime) SU(3) transformation.

Now, following Weyl's reasoning used for charge symmetry, let's propose that the SU(3) symmetry should be local. In other words, the theory of color should become a theory where the equations of motion are invariant under transformations of the form

$$\psi(x) \to e^{ig_s\boldsymbol{\alpha}(x)\cdot\hat{\mathbf{T}}}\psi(x) \tag{38}$$

where we've replaced the constant $\boldsymbol{\alpha}$ with a function $g_s\boldsymbol{\alpha}(x)$ (we could have written a function $\boldsymbol{\beta}(x)$ but by convention, we always factor out an arbitrary constant g_s known as the strong coupling constant).

Now it's no longer true that $\not{\partial}$ 'commutes with the transformation factor. Rather,

$$\not{\partial}\left(e^{ig_s\boldsymbol{\alpha}(x)\cdot\hat{\mathbf{T}}}\psi(x)\right) = e^{ig_s\boldsymbol{\alpha}(x)\cdot\hat{\mathbf{T}}}\left[\not{\partial}\psi(x) + \left(ig_s\not{\partial}\boldsymbol{\alpha}(x)\cdot\hat{\mathbf{T}}\right)\psi(x)\right]. \tag{39}$$

Consequently, the free quark theory is not invariant under the local transformation (the transformed derivative term will have an extra component). This is similar to what we encountered with the local U(1) symmetry (charge) and we fixed the problem by introducing a new vector field which transformed appropriately when acted on by the charge symmetry.

Although it may not be obvious, this situation is considerably more complicated than the case with charge. The extra term involves the $\hat{\mathbf{T}}$ matrices and when we add a term to the Lagrangian which can effectively restore the symmetry, it turns out that term needs to involve $\hat{\mathbf{T}}$ matrices. Furthermore, since those matrices don't commute, we'll need to take advantage of the algebraic identity (the definition of the SU(3) Lie Algebra)

$$[\hat{T}_i, \hat{T}_j] = if_{ijk}\hat{T}_k. \tag{40}$$

The f_{ijk} are called the *structure constants* of SU(3).

Anyway, one can follow the argument used for local charge symmetry, and after taking care of the complexities involving the non-commuting matrices, we obtain the following. Define the 'covariant derivative' D_μ by

$$D_\mu = \partial_\mu + ig_sG^a_\mu\hat{T}^a \tag{41}$$

where we use the convention that the indices a are summed over because they are repeated. The G_μ^a are a collection of vector fields where a ranges from 1 to 8 (corresponding to the 8 SU(3) generators) and are known as **strong** fields (analogous to the A_μ field). Then under a local SU(3) transformation, let

$$G_\mu^k \to G_\mu^{k'} = G_\mu^k - \partial_\mu \alpha_k - g_s f_{ijk} \alpha_k G_\mu^j. \tag{42}$$

This is called the SU(3) gauge-transformation of the strong fields. Notice that the first two terms on the right are similar to those of the gauge transformation of the A_μ field. The term proportional to f_{ijk} is new. This terms arises when the Lie Algebra generators don't commute. The mathematical term for such Lie Algebras, is **nonabelian** and leads to the expression for such theories as "non-abelian gauge theories".

From all this, we can finally show that under SU(3)-local transformations, there is an invariance of the expression $\bar{\psi}(x)\left(i\not{D} - m\right)\psi(x) = \bar{\psi}(x)\left(i\not{\partial} + i g_s \not{G}^a \hat{T}^a - m\right)\psi(x)$.

To summarize: We started with a free theory of colored quark fields. This theory is invariant under global (i.e., spacetime independent) SU(3) transformations of the colored quark fields. We then examined how that symmetry could become local. Following the model of gravity, where Lorentz invariance becomes a local symmetry with spacetime points connected by the metric tensor, and following the example of how charge invariance becomes a local symmetry with spacetime points connected by the electromagnetic potential field, we introduced strong fields with specific SU(3) transformations that we call SU(3)-gauge transformations. Just as happens in gravity and charge-symmetry, the symmetry-requirement forces a specific form of interaction between the strong field and the quark fields.

We're not done. Although we've introduced the strong fields and shown how they interact with quark fields, we haven't yet said how the strong fields evolve – i.e. how they become *dynamical*. To do that, we must introduce kinetic terms for the strong fields, i.e., quadratic terms involving one or two derivatives. Those terms would be analogous to $F_{\mu\nu}F^{\mu\nu}$ for the A_μ fields in electromagnetism.

We'll do this later. Then, the complete theory of quarks and strong fields will be done. Just as electromagnetic fields are associated with photons, the strong fields will be associated with particles we call gluons.