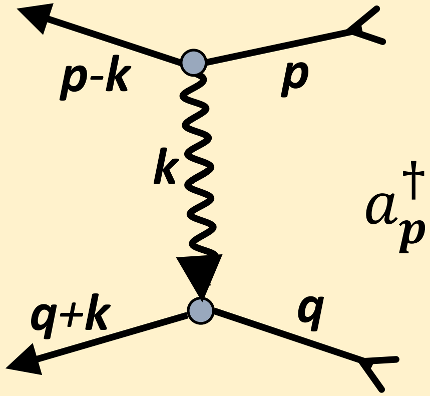
Making sense of QFT

Lecture 6: Feynman's Renormalization

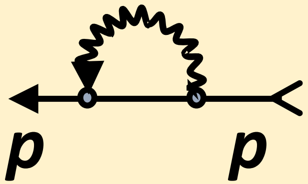
by Eugene Stefanovich

Mixed bag results from last lecture about QED in the 2nd order



$$a^{\dagger}_{\mathbf{p}-\mathbf{k}}a^{\dagger}_{\mathbf{q}+\mathbf{k}}a_{\mathbf{p}}a_{\mathbf{q}}$$

Rather accurate result for electron-electron scattering

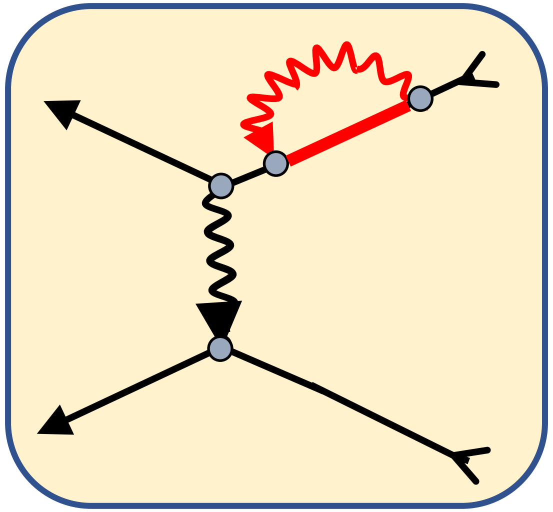


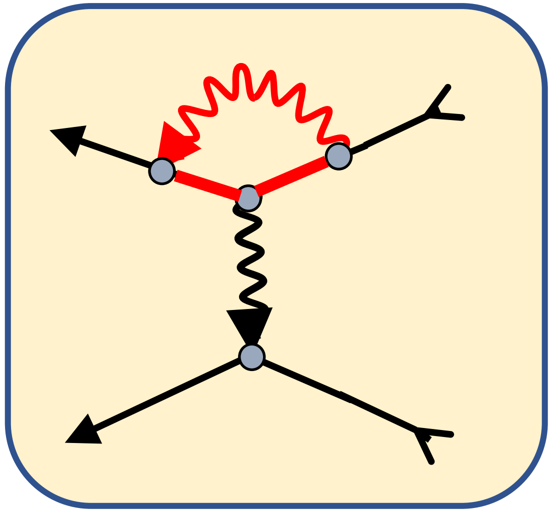
$$a^{\dagger}_{\mathbf{p}}a_{\mathbf{p}}$$

Divergent S-matrix contribution for electron self-scattering

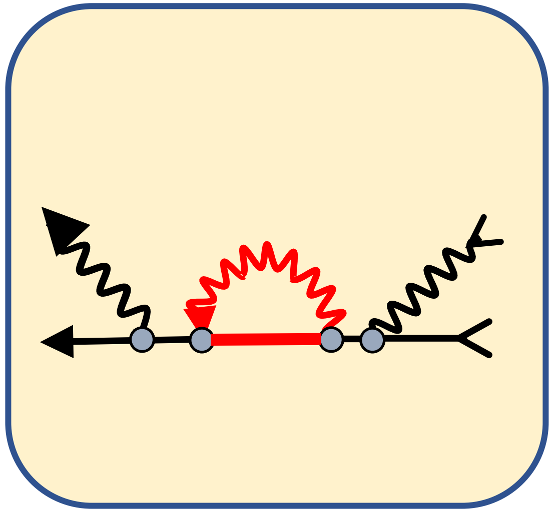
Can we simply ignore unrealistic self-scattering terms?

No. Here are examples of physical 4th order terms
with divergent loop integrals:





Electron-electron scattering



Electron-photon scattering

The idea of renormalization

- Our QED Hamiltonian H is incorrect, because it results in a divergent S-matrix.
- We need to modify the Hamiltonian to fix these problems.
- We will modify H by adding *counterterms* in such a way that all S-matrix divergences are canceled out.
- We will develop special *renormalization conditions* for selecting counterterms.

Three renormalization conditions

Mass renormalization condition: There should be no renorm terms in the S-matrix.

Charge renormalization condition: Scattering of particles at low collision energies should be no different from classic results (Coulomb-Möller formula for electron-electron scattering and Thomson-Compton formula for electron-photon scattering).

Relativistic invariance condition: Addition of counterterms should not destroy the relativistic invariance of QED. So, the counterterms should be constructed by the same Weinberg's rules as were used for V_1.

Selection of counterterms

- Original interaction density

$$V_1(\tilde{x}) = -e\overline{\psi}(\tilde{x})\gamma^\mu\psi(\tilde{x})A_\mu(\tilde{x})$$

- Mass renormalization counterterm density

$$R(\tilde{x}) = \delta\overline{\psi}(\tilde{x})\psi(\tilde{x}) + \varepsilon\overline{\psi}(\tilde{x})\left(-i\hbar c\gamma^\mu\partial_\mu + mc^2\right)\psi(\tilde{x})$$

- Charge renormalization counterterm density

$$W(\tilde{x}) = \sigma\overline{\psi}(\tilde{x})\gamma^\mu\psi(\tilde{x})A_\mu(\tilde{x})$$

- New Hamiltonian with counterterms

$$H^c = H_0 + \int d\boldsymbol{x}\left(V_1(0,\boldsymbol{x}) + R(0,\boldsymbol{x}) + W(0,\boldsymbol{x})\right)$$

$$= H_0 + V_1 + R + W$$

Selection of counterterms

- We will now treat δ, ε and σ as fitting parameters that should be adjusted to satisfy renormalization conditions.
- We will do this perturbatively, so renormalization constants are given by series in the powers of electron's charge:

$$\delta = \boldsymbol{\delta_2} + \delta_4 + \delta_6 + \cdots$$
$$\varepsilon = \boldsymbol{\varepsilon_2} + \varepsilon_4 + \varepsilon_6 + \cdots$$
$$\sigma = \boldsymbol{\sigma_3} + \sigma_5 + \sigma_7 + \cdots$$

- In our lectures we will not go beyond the 4th order S-matrix. So, renormalization constants in the lowest orders $\boldsymbol{\delta_2}$, $\boldsymbol{\varepsilon_2}$, $\boldsymbol{\sigma_3}$ will be sufficient for our purposes.
- In other words, our approximate renormalized Hamiltonian is

$$H^c = H_0 + V_1 + R_2 + W_3$$

Calculation of renormalized S-matrix

- Next we substitute the new interaction in the usual Dyson's formula for the S-matrix

$$
\begin{aligned}
S^c = 1 + \int_{-\infty}^{\infty} dt[&(V_1 + R_2 + W_3) \\
&+(V_1 + R_2 + W_3)\,(V_1 + R_2 + W_3) \\
&+(V_1 + R_2 + W_3)\,(V_1 + R_2 + W_3)\,(V_1 + R_2 + W_3) \\
&+(V_1 + R_2 + W_3)\,(V_1 + R_2 + W_3)\,(V_1 + R_2 + W_3)\,(V_1 + R_2 + W_3) + \cdots]
\end{aligned}
$$

Calculation of renormalized S-matrix

- and obtain a perturbative expansion

$$S^c = 1 + S_1^c + S_2^c + S_3^c + S_4^c + \ldots$$

- with the following terms

$$S_1^c = \int_{-\infty}^{\infty} dt V_1$$

no effect from renormalization

$$S_2^c = \int_{-\infty}^{\infty} dt (V_1 V_1 + R_2)$$

from the mass renormalization condition it follows $R_2 = -(V_1 V_1)^{ren}$

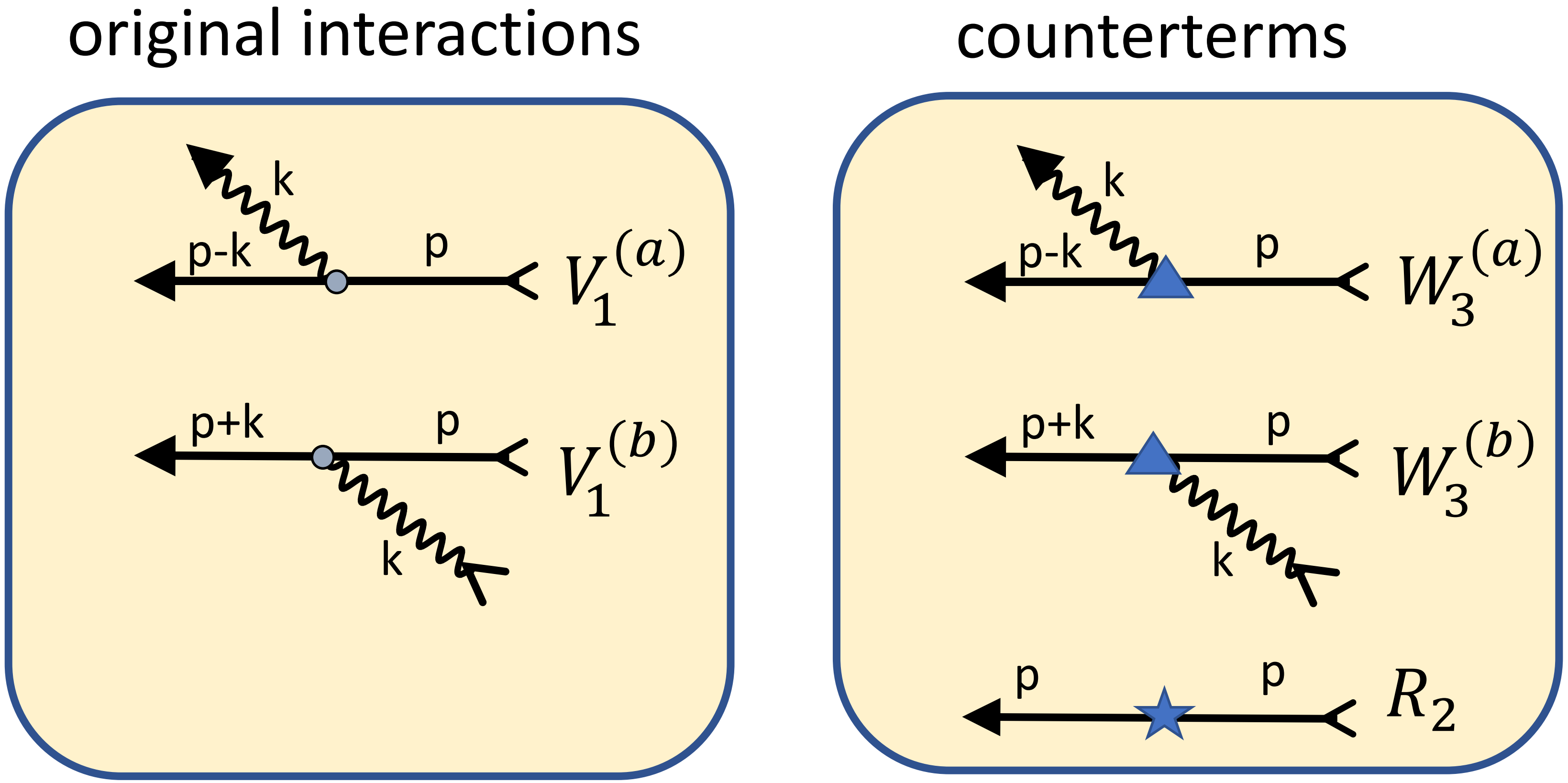
$$S_3^c = \int_{-\infty}^{\infty} dt (V_1 V_1 V_1 + V_1 R_2 + R_2 V_1 + W_3)$$

all extra terms are virtual and do not contribute to S_3^c

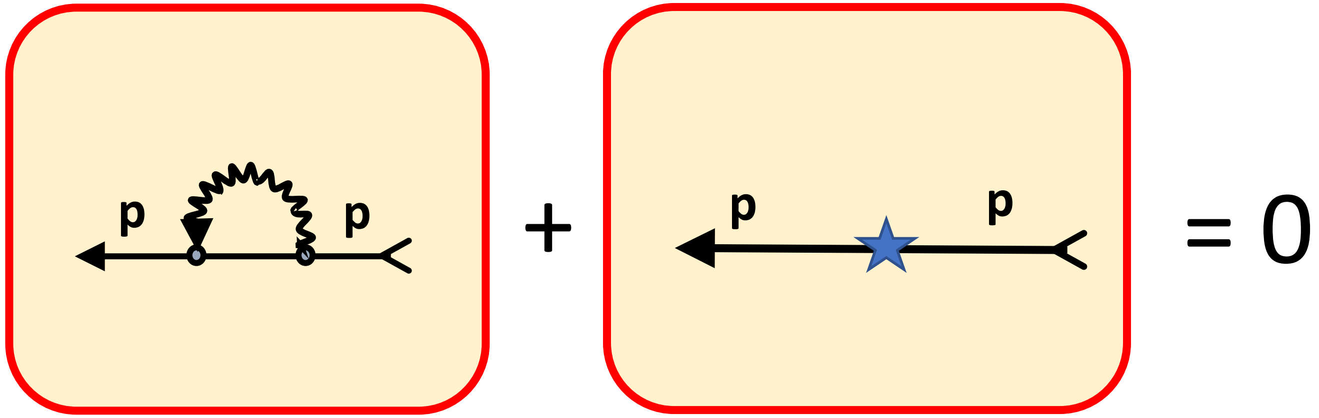
$$S_4^c = \int_{-\infty}^{\infty} dt (V_1 V_1 V_1 V_1 + R_2 V_1 V_1 + V_1 R_2 V_1 + V_1 V_1 R_2 + V_1 W_3 + W_3 V_1 + R_2 R_2)$$

- Renorm term $R_2 R_2$ will be cancelled by R_4
- Charge renormalization condition should be used to select W_3

Feynman graphs for new interaction vertices



Cancellation of divergences in the 2nd order renormalized S-matrix



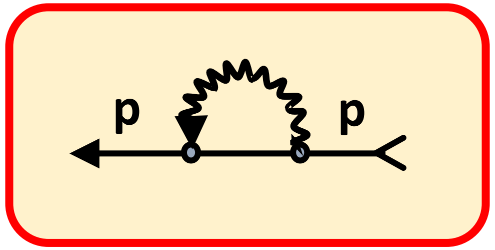
$$(V_1V_1)^{ren} \quad + \quad R_2 \quad = 0$$

- $(V_1V_1)^{ren}$ is infinite, so how are we going to use infinite R_2 in calculations?
- The answer is given by *regularization*.

Regularization

- In order to avoid manipulations with infinite quantities, we introduce *regularization,* e.g. perform loop integrals only within large radius Λ in the momentum space.
- Example: loop integral in the electron self-scattering diagram

$$C(p) \propto \int \frac{d\boldsymbol{k}}{k(\omega_{\boldsymbol{p}-\boldsymbol{k}} - \omega_{\boldsymbol{p}} + k)} \propto \int_0^{\Lambda} \frac{k^2}{k^2} dk \propto \Lambda.$$



- This is *ultraviolet divergence*, because the integral diverges in the high integration momentum limit $k \to \infty$.

Regularization

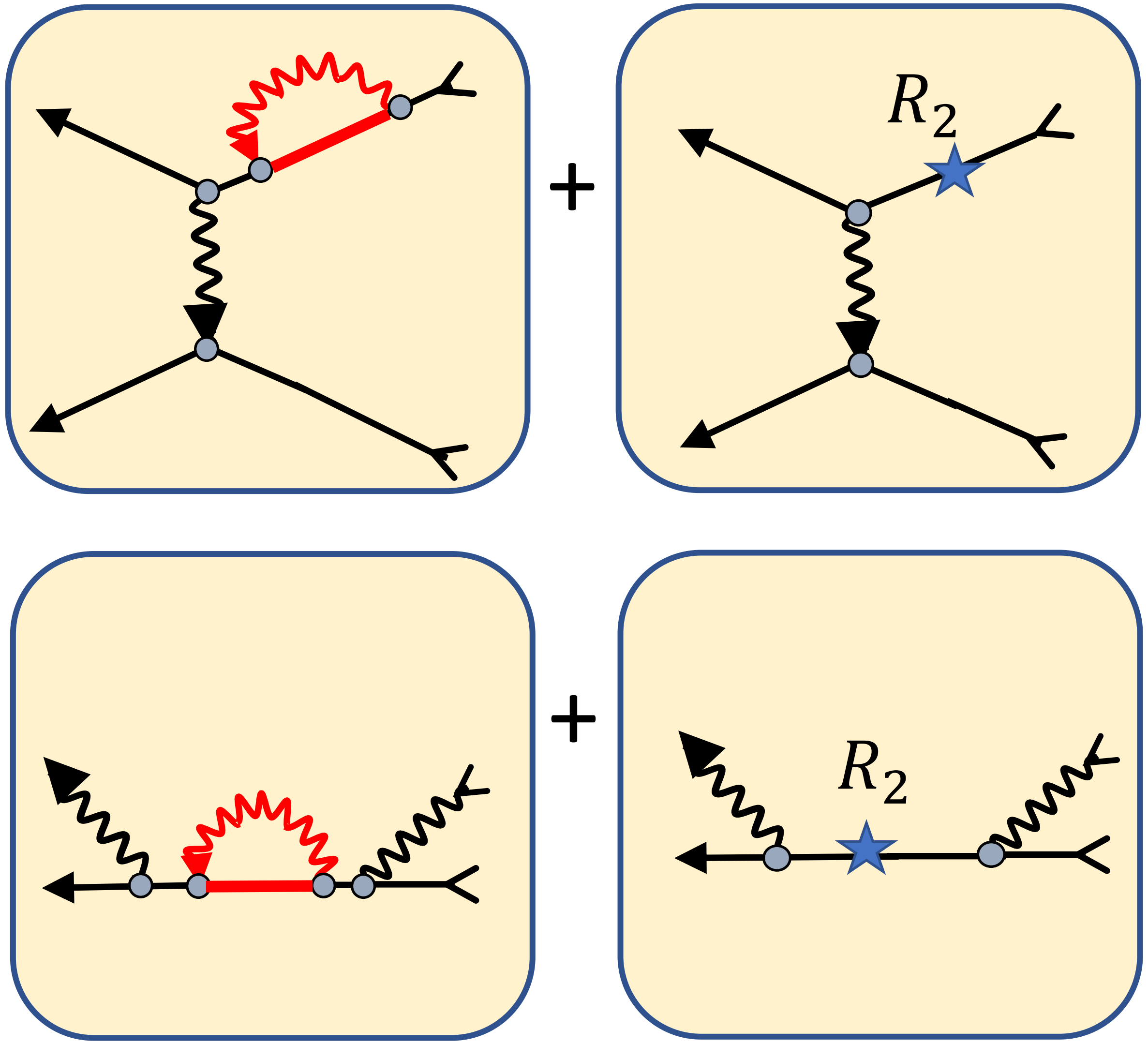
- After introducing the artificial integration cutoff Λ, all renormalization constants and counterterms become functions of Λ that tend to infinity as $\Lambda \to \infty$

$$\delta_2(\Lambda) \to \infty, \varepsilon_2(\Lambda) \to \infty, \sigma_3(\Lambda) \to \infty, R_2(\Lambda) \to \infty$$

- While Λ is finite, all intermediate quantities remain finite and we can do mathematical manipulations with them.
- At the end of calculations, we will take the physical limit $\Lambda \to \infty$. All quantities that have good physical meaning (e.g., scattering amplitudes, energies) must tend to finite values in this limit.

Cancelation of divergences in the 4th order renormalized S-matrix

original S-matrix $V_1 V_1 V_1 V_1$ with counterterms



= 0

= radiative correction to electron-photon scattering. Its value is finite and small in the limit $\Lambda \to \infty$

Cancelation of divergences in the 4th order renormalized S-matrix

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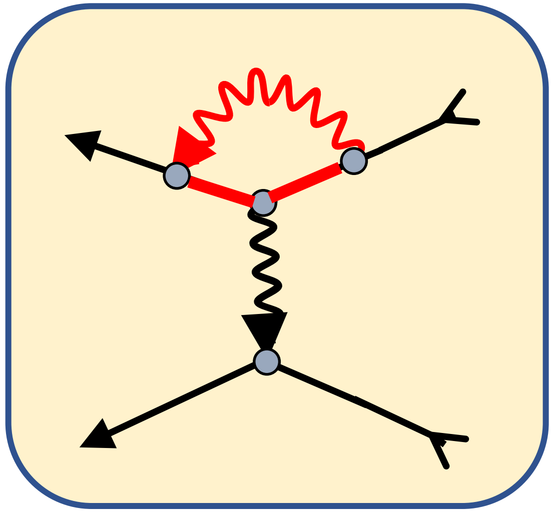

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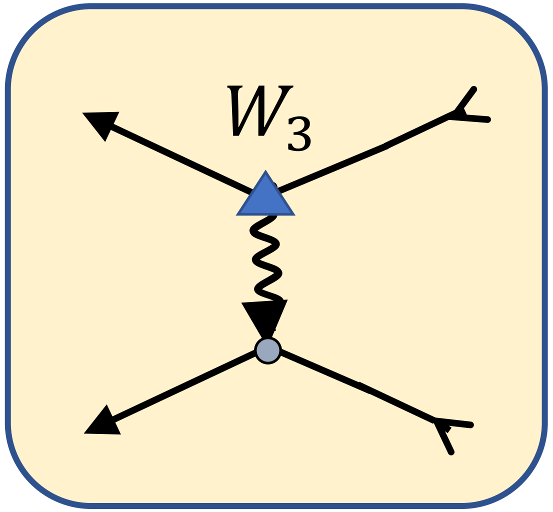


= radiative correction to electron-electron scattering

- In the expansion for the renormalized S-matrix, for each original divergent diagram there exists a divergent diagram with conterterm vertex, such that the sum of the two diagrams is either zero or a small radiative correction.
- Therefore, all ultraviolet divergences are canceled out.

Renormalization in higher perturbation orders

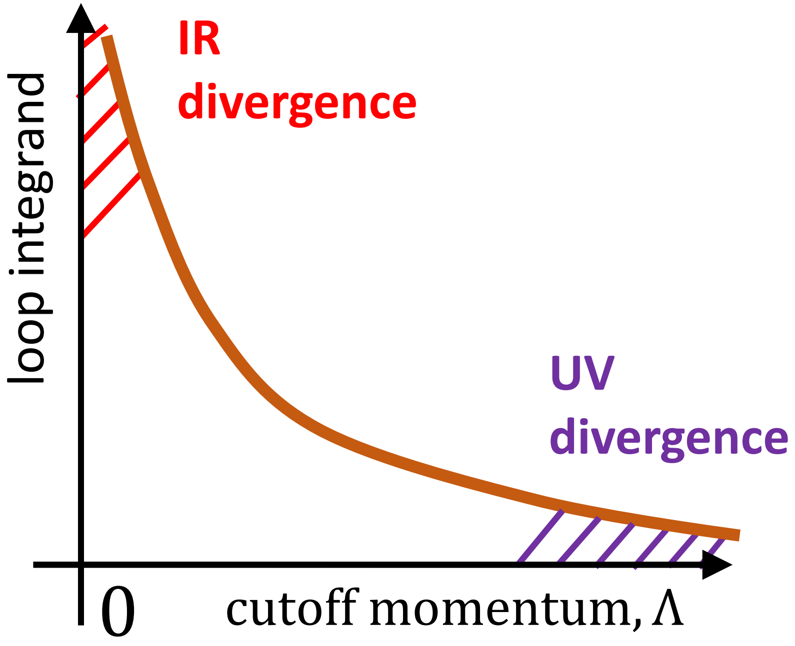
- Renormalization can be continued in higher perturbation orders.
- Renormalization conditions remain the same.
- Operator types of counterterms remain the same.
- Only Λ-dependencies of renormalization constants have to be re-calculated in each order

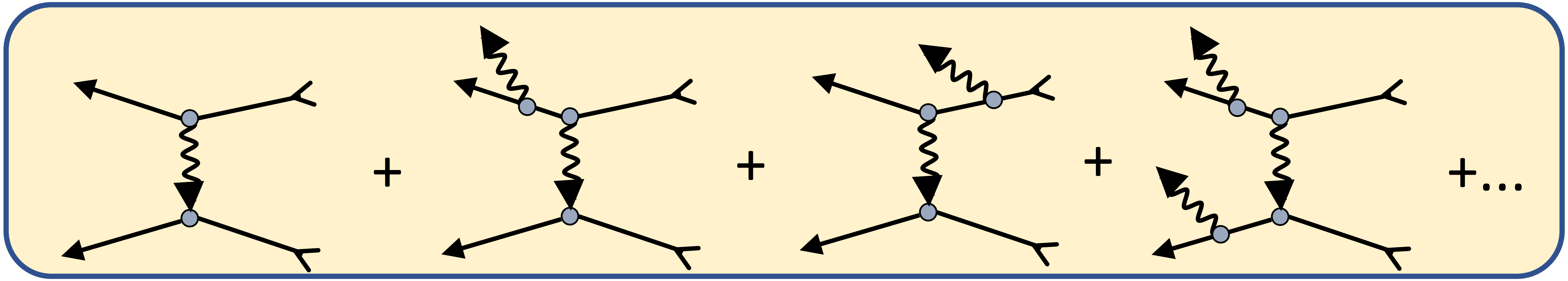
$$\delta(\Lambda) = \delta_2(\Lambda) + \delta_4(\Lambda) + \delta_6(\Lambda) + \cdots$$
$$\varepsilon(\Lambda) = \varepsilon_2(\Lambda) + \varepsilon_4(\Lambda) + \varepsilon_6(\Lambda) + \cdots$$
$$\sigma(\Lambda) = \sigma_3(\Lambda) + \sigma_5(\Lambda) + \sigma_7(\Lambda) + \cdots$$

- This is a good feature of *renormalizable* theories, such as QED.
- In *non-renormalizable* theories, the number of operator types of counterterms grows with the perturbation order.

Infrared divergences

- *Ultraviolet divergences* in loop integrals result from slow decay of integrands at large momenta.
- Ultraviolet divergences are fixed by renormalization.
- *Infrared divergences* show up as singularities of integrands at low momenta.
- Infrared divergences result from
  - zero photon mass
  - the possibility of multiple "soft photon" creation in QED process.
- They can be "fixed" by summing diagrams from higher perturbation orders.





Common interpretation of renormalization

- Masses m and charges e of particles present in the theory diverge in the physical limit $\Lambda \to \infty$.
- This is OK as these masses and charges belong to non-interacting (*bare*) particles, which cannot be observed.
- Self-interaction adds infinite corrections to m and e, so that *physical particles* have finite masses and charges measured in experiments.

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"Today we have no paradoxes - maybe. We have this infinity that comes in when we put all the laws together, but the people sweeping the dirt under the rug are so clever that one sometimes thinks this is not a serious paradox." Richard Feynman

Richard Feynman

Achievements and failures of the renormalized QED

- Good news: Very accurate S-matrix S^c whose predictions agree with experiment:
  - Scattering cross-sections.
  - Anomalous electron magnetic moment $(g-2)/2$:

experiment:	0.00115965218073
QFT:	0.001159652181643

  - Energies of atomic levels (Lamb shifts).
  - Taking into account that original theory was mutilated by renormalization, this agreement is nothing short of a miracle.
- Bad news: Ill-defined Hamiltonian H^c (with infinite counterterms):
  - We cannot analyze bound states by diagonalization of H^c.
  - We cannot study time evolution of states and observables by applying the operator $e^{iH^c t/\hbar}$.
  - This is a serious drawback, which does not allow us to claim a complete, self-consistent theory.
  - In the next lecture we will consider the Greenberg-Schweber dressing, which removes all inconsistencies and divergences from QED.

Thank you!