Making sense of QFT

Lecture 5: Quantum Electrodynamics

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Outline

1. Simplified QED
2. S-matrix
3. Dyson's perturbation theory
4. Feynman diagrams
5. Electron-electron scattering
6. Electron self-scattering

Fock space

- Fock space is built as a direct sum of N-particle spaces:

$$\mathcal{H}_{Fock} = |vac\rangle \oplus \mathcal{H}_e \oplus \mathcal{H}_\gamma \oplus \mathcal{H}_{ee} \oplus \mathcal{H}_{e\gamma} \oplus \mathcal{H}_{\gamma\gamma} \oplus \cdots$$

- It contains states of any system with any number of particles.
- A general state $|\Psi\rangle \in \mathcal{H}_{Fock}$ is a superposition of states with different numbers of particles, e.g.,
  (atom in an excited state) + (atom in the ground state and a photon)

Fock space (creation and annihilation operators)

- *Creation operator* $a_{\boldsymbol{p}}^{\dagger}$ acts on $|\Psi\rangle$ and adds to this state one particle with momentum $\boldsymbol{p}$ and spin projection/helicity σ. (If the particle is a fermion and the state $|\boldsymbol{p}\sigma\rangle$ was already present in $|\Psi\rangle$, then $a_{\boldsymbol{p}\sigma}^{\dagger}|\Psi\rangle = 0$).
- *Annihilation operator* $a_{\boldsymbol{p}\sigma}$ acts on $|\Psi\rangle$ and removes from this state one particle with momentum $\boldsymbol{p}$ and spin projection/helicity σ.
  (If the state $|\boldsymbol{p}\sigma\rangle$ was not present in $|\Psi\rangle$, then $a_{\boldsymbol{p}\sigma}|\Psi\rangle = 0$).

Some physical operators in the Fock space

- Total number of electrons

$$N_e = \sum_\sigma \int d\boldsymbol{p}\, a^\dagger_{\boldsymbol{p}\sigma} a_{\boldsymbol{p}\sigma}$$

- Total energy of non-interacting electrons (Poincaré generator)

$$H_0 = \sum_\sigma \int d\boldsymbol{p} \sqrt{p^2c^2 + m^2c^4}\, a^\dagger_{\boldsymbol{p}\sigma} a_{\boldsymbol{p}\sigma}$$

Example of Weinberg's method: Quantum electrodynamics

- **electron-positron quantum field**:

$$\psi_a(\tilde{x}) \equiv \psi_a(t, \boldsymbol{x})$$
$$= \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^{3/2}} \sqrt{\frac{m_e c^2}{\omega_{\boldsymbol{p}}}} \sum_{s_z} (e^{-\frac{i}{\hbar}\tilde{p}\cdot\tilde{x}} u_a(\boldsymbol{p}, s_z) a_{\boldsymbol{p}s_z} + e^{\frac{i}{\hbar}\tilde{p}\cdot\tilde{x}} v_a(\boldsymbol{p}, s_z) b^{\dagger}_{\boldsymbol{p}s_z}). \quad \text{(B.34)}$$

u_a and v_a are 4-component bispinors, which are carefully selected to satisfy all requirements for the fields.

- **photon quantum field:**

$$\mathcal{A}_\mu(\tilde{x}) \equiv \mathcal{A}_\mu(t, \boldsymbol{x})$$
$$= \frac{\hbar c}{(2\pi\hbar)^{3/2}} \int \frac{d\boldsymbol{p}}{\sqrt{2pc}} \sum_{\tau} [e^{-\frac{i}{\hbar}\tilde{p}\cdot\tilde{x}} e_\mu(\boldsymbol{p}, \tau) c_{\boldsymbol{p}\tau} + e^{\frac{i}{\hbar}\tilde{p}\cdot\tilde{x}} e^*_\mu(\boldsymbol{p}, \tau) c^{\dagger}_{\boldsymbol{p}\tau}], \quad \text{(C.2)}$$

e_μ is a 4-component polarization function,

- **Potential energy density of QED**

$$V_1(t, \boldsymbol{x}) = -e\overline{\psi}(t, \boldsymbol{x})\gamma^\mu \psi(t, \boldsymbol{x}) A_\mu(t, \boldsymbol{x})$$

is relativistically invariant in the Wigner-Dirac-Weinberg sense.

- **Full interacting Hamiltonian of QED**

$$H = H_0 + V_1 = H_0 - e\int \overline{\psi}(0, \boldsymbol{x})\gamma^\mu \psi(0, \boldsymbol{x}) A_\mu(0, \boldsymbol{x}) d\boldsymbol{x}$$

Simplified QED

- Simplify notation by dropping intergations signs, indices, numerical factors, etc.

$$V_1 = -e \int \overline{\psi}(0,\boldsymbol{x})\gamma^{\mu}\psi(0,\boldsymbol{x})A_{\mu}(0,\boldsymbol{x})d\boldsymbol{x}$$

$$\sim e\overline{\psi}\psi A$$
$$= e(a^{\dagger} + b)(a + b^{\dagger})(c + c^{\dagger})$$

- Ignore terms involving operators of positrons b and $b^{\dagger}$ and spin components

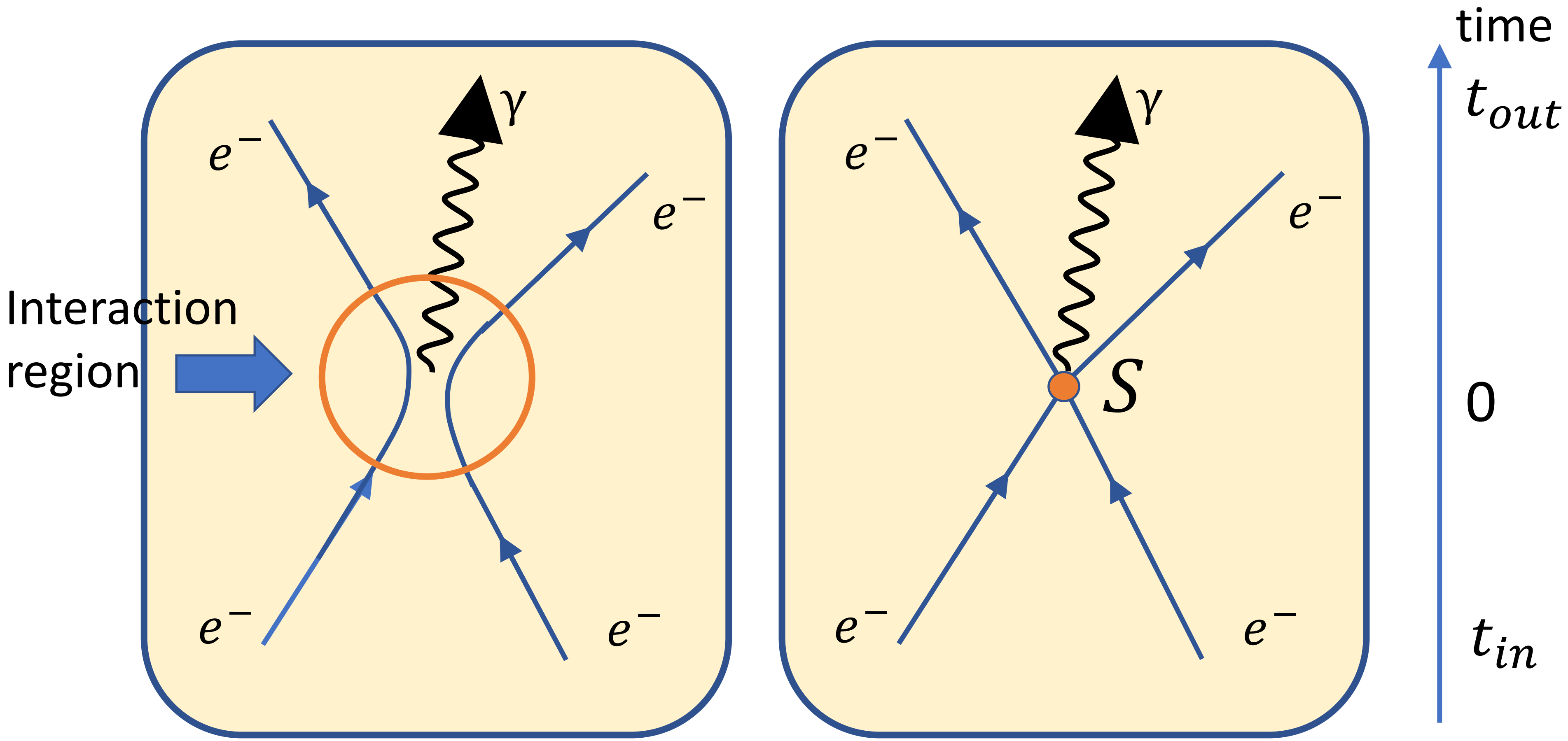
$$H = H_0 + V_1$$

$$H_0 = \int d\boldsymbol{p}\,\omega_{\boldsymbol{p}} a_{\boldsymbol{p}}^{\dagger} a_{\boldsymbol{p}} + \int d\boldsymbol{k}\,ck c_{\boldsymbol{k}}^{\dagger} c_{\boldsymbol{k}}$$

$$V_1 \approx e(a^{\dagger}ac + a^{\dagger}c^{\dagger}a) = V_1^{(a)} + V_1^{(b)}$$

S-matrix

Exact description of scattering | S-matrix description of scattering

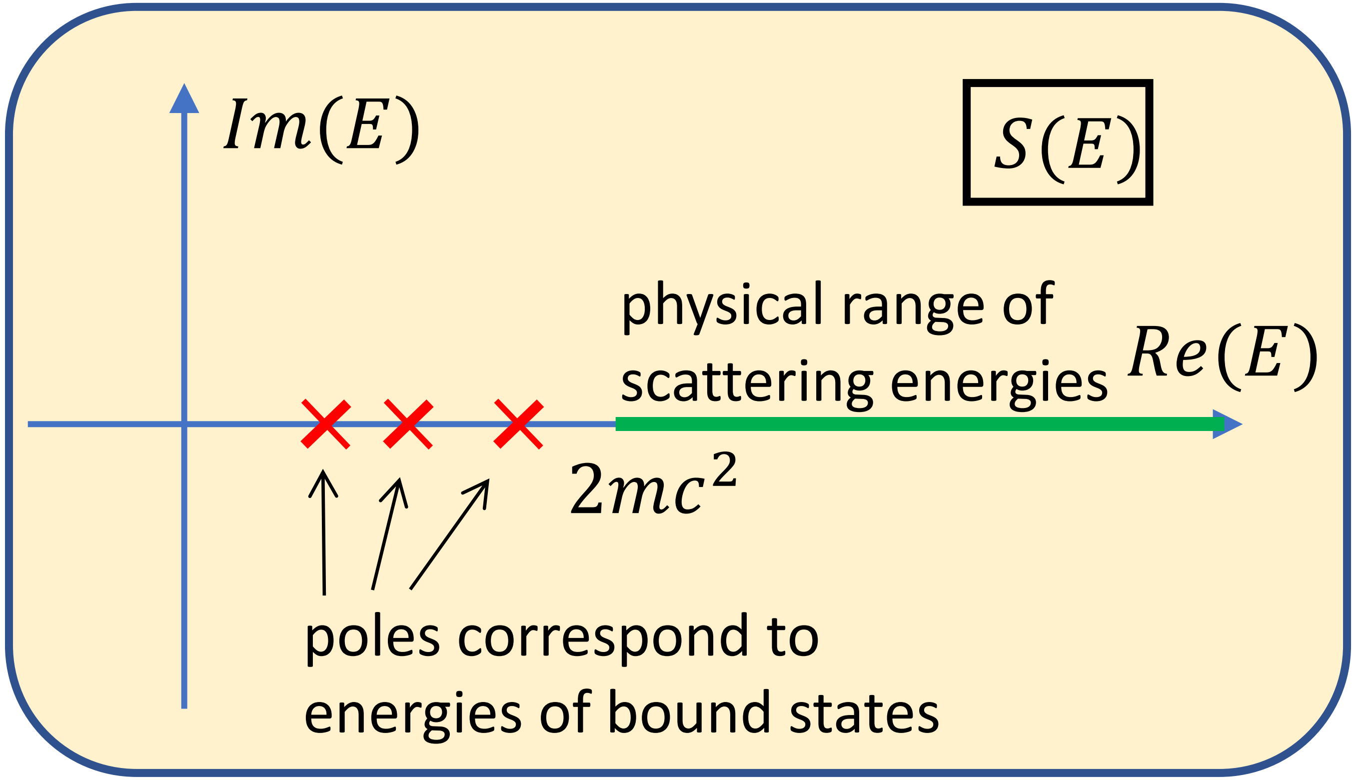


$$|\Psi(t_{out})\rangle = e^{-iH(t_{out}-t_{in})/\hbar}|\Psi(t_{in})\rangle.$$

$$|\Psi(t_{out})\rangle = e^{-iH_0(t_{out}-0)/\hbar} S e^{-iH_0(0-t_{in})/\hbar}|\Psi(t_{in})\rangle.$$

S-matrix, experimental predictions

- From S-matrix coefficients one can calculate scattering cross-sections.
- Positions of poles of the function $S(E)$ on the complex energy plane coincide with energies of bound states.



Schematic pole structure for the electron-positron S-matrix

S-matrix in perturbation theory

- Dyson's formula

$$S = 1 - \frac{i}{\hbar}\int_{-\infty}^{+\infty} V(t')dt' - \frac{1}{\hbar^2}\int_{-\infty}^{+\infty} V(t')dt' \int_{-\infty}^{t'} V(t'')dt'' + \cdots . \qquad (7.11)$$

- where "t-dependent interaction Hamiltonian" is

$$V(t) = e^{\frac{i}{\hbar}H_0 t} V e^{-\frac{i}{\hbar}H_0 t}. \qquad (7.10)$$

S-matrix, further simplifications

$$S = 1 + S_1 + S_2 + S_3 + S_4 + \dots$$

$$S_1 \propto \int_{-\infty}^{+\infty} V_1 dt,$$

$$S_2 \propto \int_{-\infty}^{+\infty} V_1 V_1 dt,$$

$$S_3 \propto \int_{-\infty}^{+\infty} V_1 V_1 V_1 dt,$$

$$S_4 \propto \int_{-\infty}^{+\infty} V_1 V_1 V_1 V_1 dt.$$

$$V_1(t) \equiv e^{iH_0 t/\hbar} V_1 e^{-iH_0 t/\hbar} = V_1^{(a)}(t) + V_1^{(b)}(t).$$

Useful formulas:

$$e^{iH_0 t/\hbar} a_{\boldsymbol{p}}^{\dagger} e^{-iH_0 t/\hbar} = e^{i\omega_{\boldsymbol{p}} t/\hbar} a_{\boldsymbol{p}}^{\dagger}$$

$$e^{iH_0 t/\hbar} a_{\boldsymbol{p}} \, e^{-iH_0 t/\hbar} = e^{-i\omega_{\boldsymbol{p}} t/\hbar} a_{\boldsymbol{p}}$$

$$e^{iH_0 t/\hbar} c_{\boldsymbol{k}}^{\dagger} e^{-iH_0 t/\hbar} = e^{ickt/\hbar} c_{\boldsymbol{k}}^{\dagger}$$

$$e^{iH_0 t/\hbar} c_{\boldsymbol{k}} \, e^{-iH_0 t/\hbar} = e^{-ickt/\hbar} c_{\boldsymbol{k}}$$

t-dependence of operators under integrals $\int_{-\infty}^{\infty} \dots dt$

$$e^{iH_0 t/\hbar}[operator]e^{-iH_0 t/\hbar} = \exp(iEt/\hbar)[operator]$$

where *E* is "energy function" = (sum of energies of created particles)
-(sum of energies of annihilated particles)

general form of terms in the S-matrix

$$S_n = \int_{-\infty}^{\infty} e^{iEt/\hbar}[operator]dt = 2\pi\hbar\delta(E)[operator]$$

S-matrix, 1st perturbation order

$$S_1 \propto \int_{-\infty}^{+\infty} V_1(t)dt = \int_{-\infty}^{+\infty} dt \ \left(V_1^{(a)}(t) + V_1^{(b)}(t)\right)$$

$$= \left(\int_{-\infty}^{+\infty} dt \ e^{i(\omega_{\boldsymbol{p}-\boldsymbol{k}}+ck-\omega_{\boldsymbol{p}})t/\hbar}\right) a^\dagger_{\boldsymbol{p}-\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} a_{\boldsymbol{p}}$$

$$+ \left(\int_{-\infty}^{+\infty} dt \ e^{i(\omega_{\boldsymbol{p}+\boldsymbol{k}}-ck-\omega_{\boldsymbol{p}})t/\hbar}\right) a^\dagger_{\boldsymbol{p}+\boldsymbol{k}} a_{\boldsymbol{p}} c_{\boldsymbol{k}},$$

$$= 2\pi\hbar\delta(\omega_{\boldsymbol{p}-\boldsymbol{k}} + ck - \omega_{\boldsymbol{p}}) a^\dagger_{\boldsymbol{p}-\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} a_{\boldsymbol{p}}$$

$$+2\pi\hbar\delta(\omega_{\boldsymbol{p}+\boldsymbol{k}} - ck - \omega_{\boldsymbol{p}}) \ a^\dagger_{\boldsymbol{p}+\boldsymbol{k}} a_{\boldsymbol{p}} \ c_{\boldsymbol{k}}$$

- the arguments of delta functions never turn to zero, because a free electron cannot emit/absorb a photon without violating energy-momentum conservation laws.
- **Conclusion:** 1st order S-matrix vanishes $S_1 = 0$

S-matrix, 2nd perturbation order

$$S_2 \propto \left(V_1^{(a)} + V_1^{(b)}\right)\left(V_1^{(a)} + V_1^{(b)}\right).$$

$$S_2 \propto V_1^{(b)}V_1^{(b)} + V_1^{(b)}V_1^{(a)} + V_1^{(a)}V_1^{(b)} + V_1^{(a)}V_1^{(a)}$$

$$\propto \left(a^\dagger_{\mathbf{p}'+\mathbf{k}'} a_{\mathbf{p}'} c_{\mathbf{k}'}\right)\left(a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}} c_{\mathbf{k}}\right) + \boxed{\left(a^\dagger_{\mathbf{p}'+\mathbf{k}'} a_{\mathbf{p}'} c_{\mathbf{k}'}\right)\left(a^\dagger_{\mathbf{p}-\mathbf{k}} c^\dagger_{\mathbf{k}} a_{\mathbf{p}}\right)}$$

$$+ \left(a^\dagger_{\mathbf{p}'-\mathbf{k}'} c^\dagger_{\mathbf{k}'} a_{\mathbf{p}'}\right)\left(a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}} c_{\mathbf{k}}\right) + \left(a^\dagger_{\mathbf{p}'-\mathbf{k}'} c^\dagger_{\mathbf{k}'} a_{\mathbf{p}'}\right)\left(a^\dagger_{\mathbf{p}-\mathbf{k}} c^\dagger_{\mathbf{k}} a_{\mathbf{p}}\right).$$

- Let us focus on the 2nd term

$$V_1^{(b)} \times V_1^{(a)} = \boxed{\left(a^\dagger_{\mathbf{p}'+\mathbf{k}'} a_{\mathbf{p}'} c_{\mathbf{k}'}\right)\left(a^\dagger_{\mathbf{p}-\mathbf{k}} c^\dagger_{\mathbf{k}} a_{\mathbf{p}}\right)}$$

- For analysis, it is more convenient to rewrite this term in the "normal order", where all creation operators are on the left and all annihilation operators are on the right. Calculations are non-trivial due to the non-commutativity of the operators.

S-matrix, 2nd perturbation order

- Permutations of creation and annihilation operators produce extra terms

$$a_{\boldsymbol{p}'}a_{\boldsymbol{p}}^{\dagger} = -a_{\boldsymbol{p}}^{\dagger}a_{\boldsymbol{p}'} + \delta(\boldsymbol{p} - \boldsymbol{p}')$$

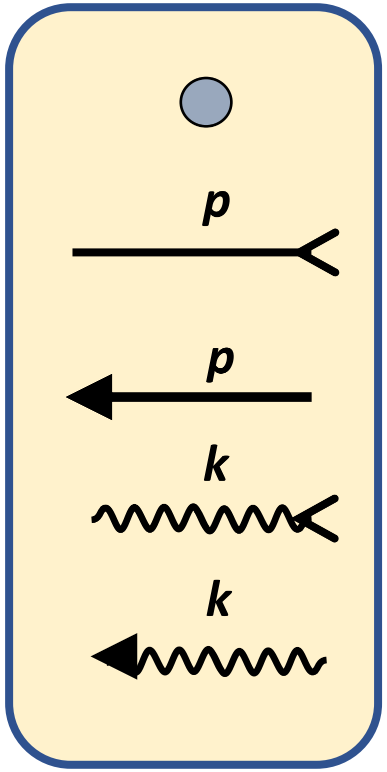
$$c_{\boldsymbol{k}'}c_{\boldsymbol{k}}^{\dagger} = c_{\boldsymbol{k}}^{\dagger}c_{\boldsymbol{k}'} + \delta(\boldsymbol{k} - \boldsymbol{k}')$$

- Then our product in the normal order is

$$\begin{aligned}(a^{\dagger}ac)(a^{\dagger}c^{\dagger}a) = a^{\dagger}\, a^{\dagger}\, c^{\dagger}aac \\ +a^{\dagger}\, a^{\dagger}\, c^{\dagger}aa \\ +a^{\dagger}\, c^{\dagger}ac \\ +a^{\dagger}a\end{aligned}$$

- These calculations are boring. Elegant solution -- Feynman diagrams.

S-matrix, Feynman diagrams



Vertex (numerical factor)

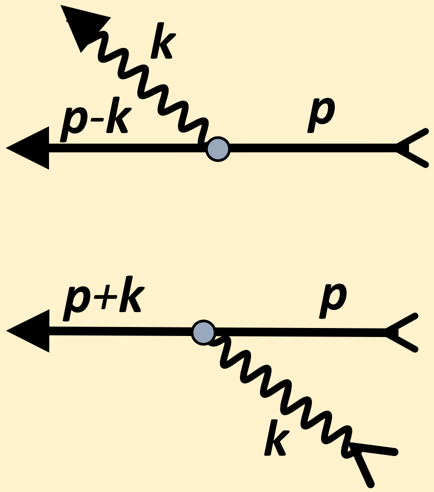
Electron annihilation operator $a_{\boldsymbol{p}}$

Electron creation operator $a_{\boldsymbol{p}}^{\dagger}$

Photon annihilation operator $c_{\boldsymbol{k}}$

Photon creation operator $c_{\boldsymbol{k}}^{\dagger}$

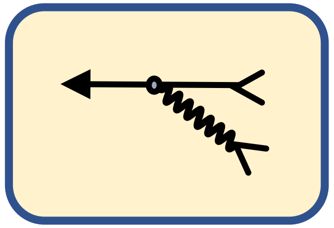
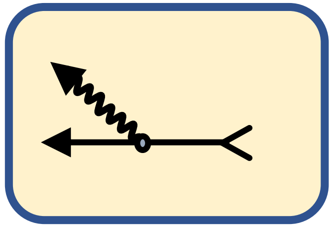
S-matrix, Feynman diagrams

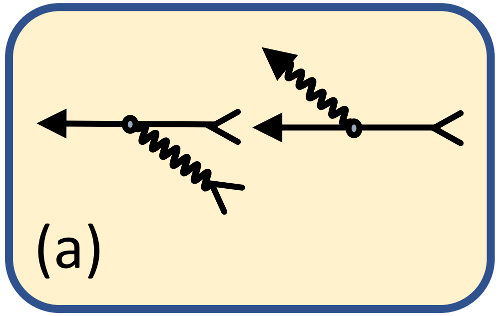


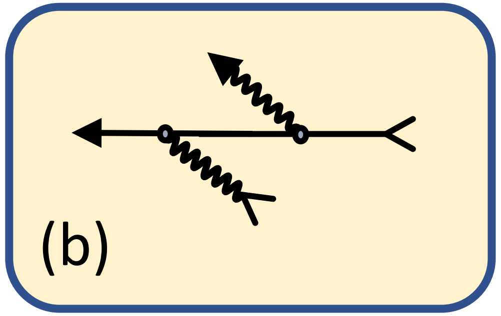
$$a^{\dagger}_{\boldsymbol{p}-\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}} a_{\boldsymbol{p}} = V_1^{(a)}$$

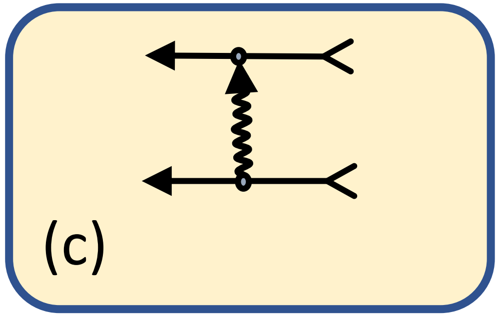
$$a^{\dagger}_{\boldsymbol{p}+\boldsymbol{k}} a_{\boldsymbol{p}} c_{\boldsymbol{k}} = V_1^{(b)}$$

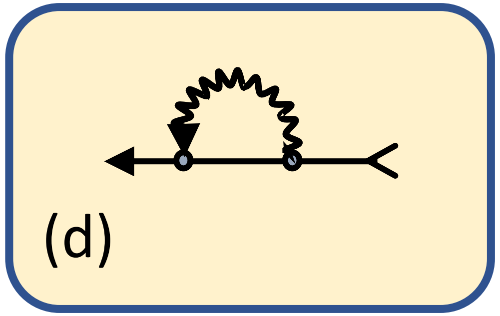
S-matrix, Feynman diagrams, Wick's theorem

$V_1^{(b)} \times V_1^{(a)} =$  $\times$  $=$ sum of diagrams with all possible arrow-to-tail couplings

$=$ 

$+$ 

$+$ 

$+$ 

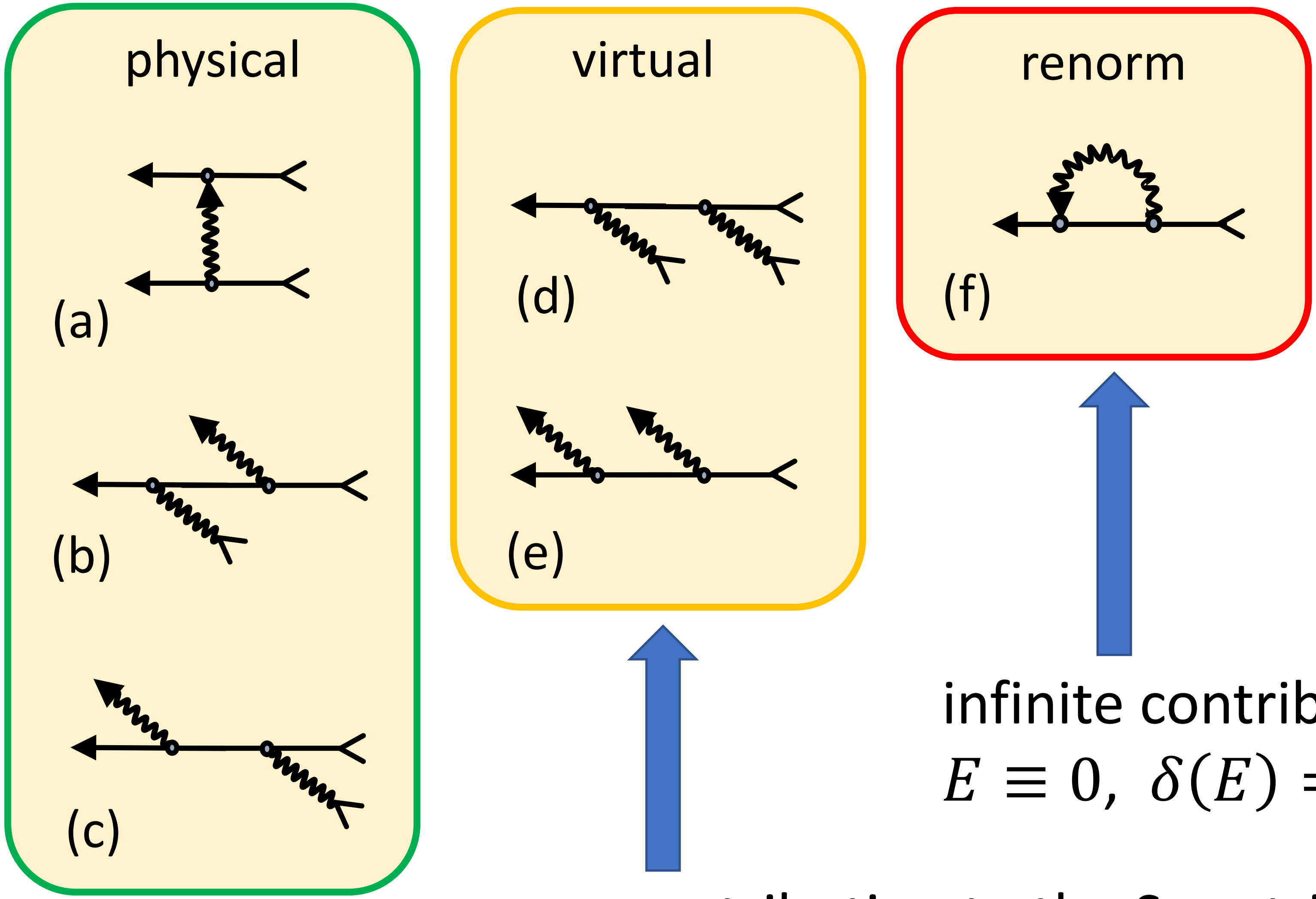
self-scattering of an electron on itself

electron-electron scattering

electron-photon scattering

unconnected diagram without physical meaning

All terms in the 2nd order S-matrix S_2



infinite contribution to the S-matrix

$E \equiv 0, \ \delta(E) = \infty$

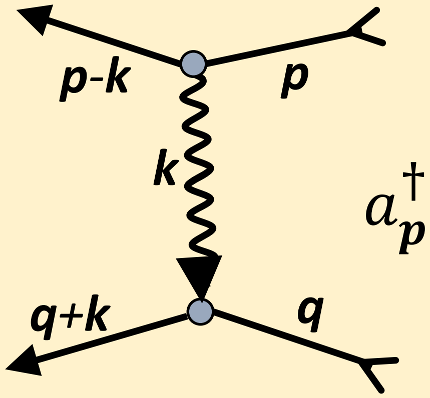
zero contribution to the S-matrix

$E \neq 0, \ \delta(E) = 0$

finite contribution to the S-matrix

$E = 0$ and $\delta(E) \neq 0$ only on the "energy shell" (energy is conserved)

Electron-electron scattering diagram

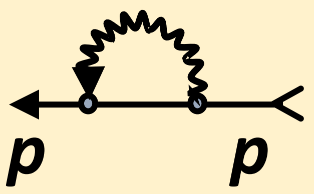


$$a^{\dagger}_{\boldsymbol{p}-\boldsymbol{k}}a^{\dagger}_{\boldsymbol{q}+\boldsymbol{k}}a_{\boldsymbol{p}}a_{\boldsymbol{q}}$$

$$S_2 \approx N \int d\boldsymbol{p}d\boldsymbol{q}d\boldsymbol{k}\,\frac{\delta\left(\omega_{\boldsymbol{p}-\boldsymbol{k}} + \omega_{\boldsymbol{q}+\boldsymbol{k}} - \omega_{\boldsymbol{p}} - \omega_{\boldsymbol{q}}\right)}{k^2}\,a^{\dagger}_{\boldsymbol{p}-\boldsymbol{k}}a^{\dagger}_{\boldsymbol{q}+\boldsymbol{k}}a_{\boldsymbol{p}}a_{\boldsymbol{q}}$$

- From the numerical coefficient (Nk^{-2}) one can calculate electron-electron scattering cross-section, which agrees pretty well with experimental data.

Diagram of electron self-scattering



$a_{\boldsymbol{p}}^{\dagger} a_{\boldsymbol{p}}$

- This looks like permanent self-scattering due spontaneous emission/absorption of photons by a free electron.
- The presence of a loop in the diagram results in an infinite numerical factor (loop integral).
- However, even if the loop integral were finite, the presence of such a t-independent renorm term in the S-matrix would be unacceptable, because its t-integral diverges:

$$S_2 = \int_{-\infty}^{\infty} \left(\int d\boldsymbol{p} f(\boldsymbol{p})\, a_{\boldsymbol{p}}^{\dagger} a_{\boldsymbol{p}} \right) dt = \infty.$$

- Our theory has a serious defect. It has to be fixed by **renormalization**.

Thank you!