Making sense of QFT

Lecture 7: Greenberg-Schweber dressing

by Eugene Stefanovich

Renormalized QED

- From last lecture, the Hamiltonian of renormalized QED is

$$H^c = H_0 + V_1 + R_2 + W_3$$

where V_1 is original interaction, R_2 and W_3 are counterterms.

- We will add to our theory also protons and antiprotons described by Dirac's quantum field $\Psi(\tilde{x})$

$$V_1(\tilde{x}) = -e\overline{\psi}(\tilde{x})\gamma^\mu\psi(\tilde{x})A_\mu(\tilde{x}) + e\overline{\Psi}(\tilde{x})\gamma^\mu\Psi(\tilde{x})A_\mu(\tilde{x})$$

- R_2 and W_3 should be modified correspondingly.

Renormalized QED

- All relevant operators in QED have form $X = X^{physical} + X^{virtual} + X^{renorm}$

Type of operator	Example	Acceptable in S-matrix?
virtual	$a^\dagger ac, a^\dagger c^\dagger a, a^\dagger b^\dagger c^\dagger$	yes, zero contribution
physical	$a^\dagger a^\dagger aa, a^\dagger a^\dagger aac$	yes
renorm	$a^\dagger a$	no

- Properties of interaction operators in $H^c = H_0 + V_1 + R_2 + W_3$:
  - V_1 is finite and virtual
  - R_2 is infinite and renorm
  - W_3 is infinite and virtual

- After substituting the renormalized interaction $V_1 + R_2 + W_3$ into Dyson's S-matrix formula

  $$S^c = 1 + \int_{-\infty}^{\infty} dt[(V_1 + R_2 + W_3) + (V_1 + R_2 + W_3)\,(V_1 + R_2 + W_3) + \cdots]$$

  - all renorm terms cancel out
  - physical terms appear, which agree with experiment very well.

Multiplication table of operators in QED

	P	V	R
P	P	$P+V$	P
V	$P+V$	$P+V+R$	V
R	P	V	R

P = "physical"
V = "virtual"
R = "renorm"

- Physical terms describing real scattering events appear from the product of virtual terms in the S-matrix: $VV = P + V + R$
- Products of virtual terms result also in renorm terms: $VV = P + V + R$. Their removal from the S-matrix was the main purpose of renormalization. So, the necessity of renormalization is explained by the presence of virtual terms in interaction.

Stability conditions

- *Vacuum stability condition:*
  - **Vacuum state (no particles) has no time dependence:**

$$e^{-iHt/\hbar}\,|vac\rangle = |vac\rangle$$

  - $|vac\rangle$ is an eigenstate of the total Hamiltonian with eigenvalue zero:

$$H|vac\rangle = 0$$

- *Particle stability condition*
  - **States of single stable particles (electrons, protons, photons) remain one-particle states at all times**
  - time evolution keeps vector $a^\dagger|vac\rangle$ within one-particle sector:

$$e^{-iHt/\hbar}a^\dagger|vac\rangle \sim a^\dagger|vac\rangle$$

  - one-particle sector remains invariant with respect to the total Hamiltonian

$$Ha^\dagger|vac\rangle \sim a^\dagger|vac\rangle$$

Stability conditions

- Non-interacting Hamiltonian satisfies both stability conditions

$$H_0 \,|vac\rangle \sim (a^\dagger a + c^\dagger c + \cdots)|vac\rangle = 0$$

$$H_0 a^\dagger |vac\rangle \sim (a^\dagger a + c^\dagger c + \cdots) a^\dagger |vac\rangle = a^\dagger a a^\dagger |vac\rangle \sim a^\dagger \,|vac\rangle$$

- Vacuum stability condition is violated in QED

$$V^c |vac\rangle \sim (a^\dagger b^\dagger c^\dagger + \cdots)|vac\rangle \neq 0$$

- Particle stability condition is violated in QED

$$V^c a^\dagger |vac\rangle \sim (a^\dagger c^\dagger a + \cdots) a^\dagger |vac\rangle \nsim a^\dagger |vac\rangle$$

- Renormalized QED breaks stability conditions due to the presence of virtual interactions.
- Virtual interactions are inevitable in **all** quantum field theories:

$$V \sim (\alpha + \alpha^\dagger)(\alpha + \alpha^\dagger)(\alpha + \alpha^\dagger)(\alpha + \alpha^\dagger) = \alpha^\dagger \alpha^\dagger \alpha^\dagger \alpha^\dagger + \alpha^\dagger \alpha^\dagger \alpha^\dagger \alpha + \cdots$$

Stability conditions

- Physical interactions satisfy both stability conditions

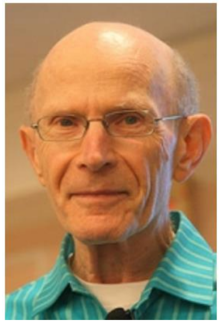
$$P|vac\rangle \sim (a^\dagger a^\dagger a a + a^\dagger c^\dagger a c + \cdots)|vac\rangle = 0$$

$$P a^\dagger|vac\rangle \sim (a^\dagger a^\dagger a a + a^\dagger c^\dagger a c + \cdots) a^\dagger|vac\rangle = 0$$

- This means that physical interaction turns on only when there are two or more particles. This is true in the rest of physics, but not in QFT!
- Stability conditions can be satisfied only in a theory with physical interaction operators.

Greenberg-Schweber dressed particle theory

Oscar Greenberg



Silvan Schweber

Idea: reformulate theory in such a way that its interaction contains only physical terms (no virtual or renorm terms)

Benefits:

- Interactions will correspond to physically realizable processes.
- Vacuum and particle stability conditions will be satisfied.
- Renorm terms will not appear in the S-matrix, because $P \times P = P$. So, mass renormalization will not be needed (hopefully, charge renormalization can be avoided as well).

	P	V	R
P	P	$P+V$	P
V	$P+V$	$P+V+R$	V
R	P	V	R

Greenberg-Schweber dressed particle theory

- What right do we have to change the interaction (Hamiltonian) of QED?
  - This interaction has divergent counterterms, so it is clearly unphysical.
- S-matrix of renormalized QED is extremely accurate. Can we keep this accuracy if the Hamiltonian is transformed?
  - Yes! The S-matrix will remain unperturbed if the new (dressed) Hamiltonian H^d is obtained as a unitary transform of the renormalized QED Hamiltonian:

    $$H^d = e^{i\Xi} H^c e^{-i\Xi}$$

    where Ξ is an Hermitian operator satisfying few simple conditions.
  - Then the new dressed theory will enjoy exactly the same agreement with experiment as the old renormalized QED.

Perturbative calculation of the physical dressed interaction V_d

- Apply unitary dressing transformation to the renormalized Hamiltonian H^c

$$H^d \equiv H_0 + V^d = e^{i\Xi} H^c e^{-i\Xi} \quad (2.16)$$

$$\equiv e^{i\Xi}(H_0 + V^c)e^{-i\Xi}$$

$$= (H_0 + V^c) + i[\Xi, (H_0 + V^c)] - \frac{1}{2!}[\Xi, [\Xi, (H_0 + V^c)]] + \cdots, \quad (2.17)$$

- Represent dressing generator Ξ as a perturbation series

$$\Xi = \Xi_1 + \Xi_2 + \cdots. \quad (2.18)$$

- Collect terms of equal perturbation orders and demand "physical" character of left hand sides

$$V_1^d = V_1 + i[\Xi_1, H_0]$$

$$V_2^d = R_2 + i[\Xi_2, H_0] + i[\Xi_1, V_1] - \frac{1}{2!}[\Xi_1, [\Xi_1, H_0]]$$

$$V_3^d = W_3 + i[\Xi_3, H_0] + i[\Xi_2, V_1] + i[\Xi_1, R_2] - \frac{1}{2!}[\Xi_2, [\Xi_1, H_0]] - \frac{1}{2!}[\Xi_1, [\Xi_2, H_0]] - \frac{1}{2!}[\Xi_1, [\Xi_1, V_1]] - \frac{1}{3!}[\Xi_1, [\Xi_1, [\Xi_1, H_0]]]$$

...

Perturbative calculation of the physical dressed interaction V_d

- **First perturbation order:**

$$V_1^d = V_1 + i[\boldsymbol{\Xi}_1, H_0]$$

- V_1 is purely virtual, so Ξ_1 is calculated from the condition that the right hand side is zero. Then $\boldsymbol{V_1^d = 0}$
- **Second perturbation order:**

$$V_2^d = R_2 + i[\boldsymbol{\Xi}_2, H_0] + i[\Xi_1, V_1] - \frac{1}{2!}[\Xi_1, [\Xi_1, H_0]]$$

- Substitute Ξ_1 into the right hand side and choose Ξ_2 to cancel all virtual terms on the right hand side. Then all renorm terms on the right hand side cancel out automatically(!). $\boldsymbol{V_2^d}$ **is purely physical**, as desired.
- **Higher perturbation orders:**
- The same calculation steps are repeated, so that finally

$$V^d = V_2^d + V_3^d + V_4^d + \cdots$$

where each term is physical. No renorm or virtual contributions in V^d.

Relativistic invariance of dressed theory

- To ensure relativistic invariance of the dressed particle QED, the generator of transformation should be invariant with respect to translations and rotations

$$[\Xi, \boldsymbol{P}_0] = 0$$
$$[\Xi, \boldsymbol{J}_0] = 0$$

- Then, by applying the same transformation to the ten Poincaré generators of renormalized QED

$$\boldsymbol{P}_0 = e^{i\Xi}\boldsymbol{P}_0 e^{-i\Xi}$$
$$\boldsymbol{J}_0 = e^{i\Xi}\boldsymbol{J}_0 e^{-i\Xi}$$
$$\boldsymbol{K}^d = e^{i\Xi}\boldsymbol{K}^c e^{-i\Xi}$$
$$H^d = e^{i\Xi}H^c e^{-i\Xi}$$

- and taking into account that unitary transformations preserve commutators, we conclude that $(\boldsymbol{P}_0, \boldsymbol{J}_0, \boldsymbol{K}^d, H^d)$ is a valid representation of the Poincaré Lie algebra.

All terms in the dressed Hamiltonian have clear physical meanings

Some components of the complete dressed Hamiltonian $H^d = H_0 + V_2^d + V_3^d + V_4^d + \ldots$.

Operator	Example	Physical meaning, manifestations in Nature
H_0	$a^\dagger a, c^\dagger c$	relativistic kinetic energy of particles
V_2^d	$a^\dagger b^\dagger ab$	Coulomb potential, magnetic potential
	$a^\dagger c^\dagger ac$	Compton-type interactions
	$c^\dagger c^\dagger ab$	electron-positron annihilation
V_3^d	$a^\dagger b^\dagger c^\dagger ab$	photon emission
	$a^\dagger a^\dagger aac$	photon absorption
V_4^d	$a^\dagger b^\dagger ab$	anomalous magnetic moment, Lamb shift

Example: **electron-proton** interaction in the 2nd perturbation order V_2^d

- Dressed interaction potential in the low-velocity approximation coincides with the Darwin-Breit potential developed independently.

Coulomb — contact — Darwin (magnetic) — spin-orbit — spin-spin

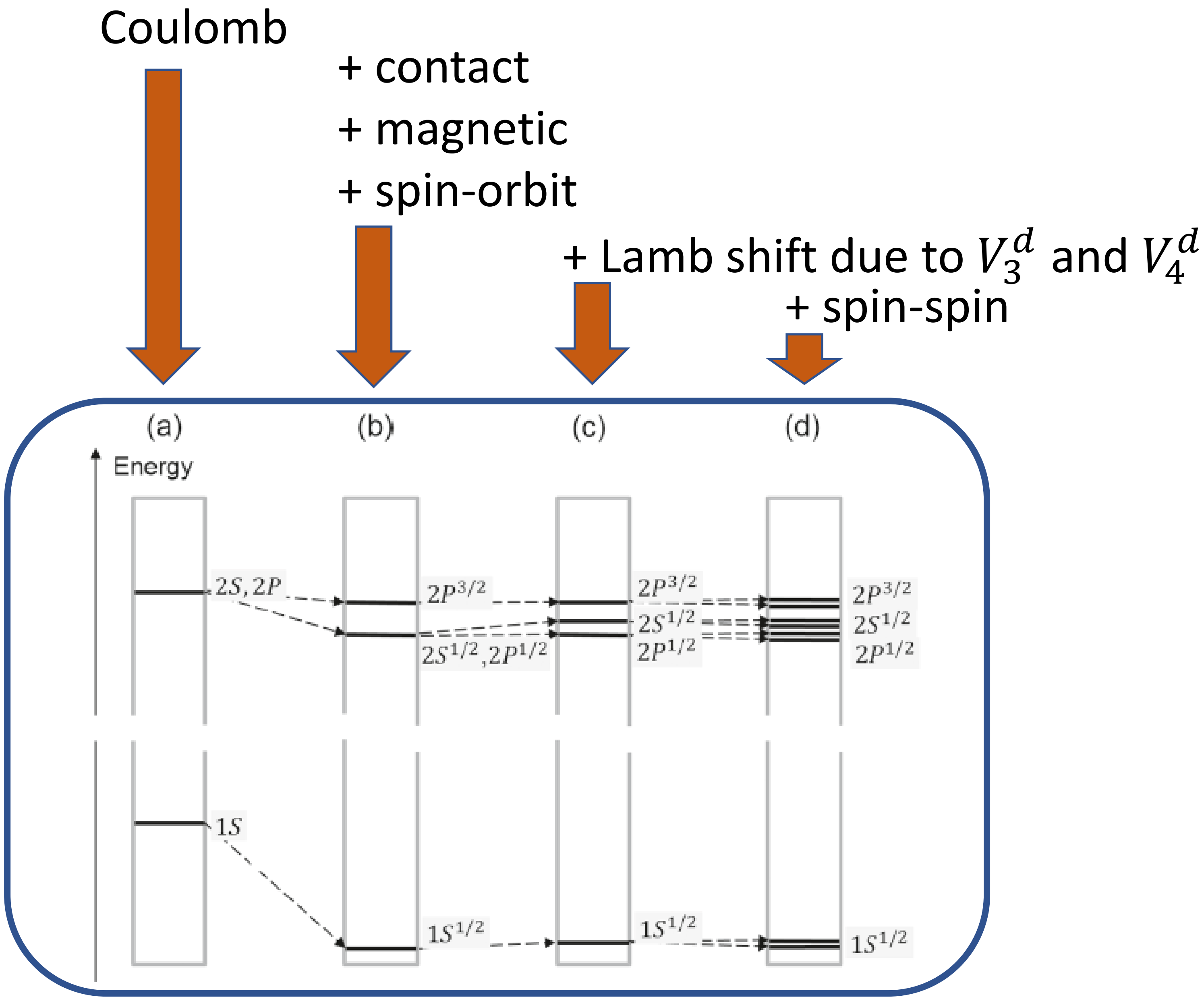
$$\hat{V}_2^d[d^\dagger a^\dagger da] = -\frac{e^2}{4\pi r} + \frac{e^2\hbar^2}{8c^2}\left(\frac{1}{m_p^2}+\frac{1}{m_e^2}\right)\delta(\mathbf{r})$$
$$+\frac{e^2}{8\pi m_p m_e c^2 r}\left[\hat{\mathbf{p}}\cdot\hat{\mathbf{q}} + \frac{(\mathbf{r}\cdot\hat{\mathbf{p}})(\mathbf{r}\cdot\hat{\mathbf{q}})}{r^2}\right] - \frac{e^2[\mathbf{r}\times\hat{\mathbf{p}}]\cdot\mathbf{s}_{pr}}{8\pi m_p^2 c^2 r^3}$$
$$+\frac{e^2[\mathbf{r}\times\hat{\mathbf{q}}]\cdot\mathbf{s}_{el}}{8\pi m_e^2 c^2 r^3} + \frac{e^2[\mathbf{r}\times\hat{\mathbf{q}}]\cdot\mathbf{s}_{pr}}{4\pi m_p m_e c^2 r^3} - \frac{e^2[\mathbf{r}\times\hat{\mathbf{p}}]\cdot\mathbf{s}_{el}}{4\pi m_p m_e c^2 r^3}$$
$$+\frac{e^2}{m_p m_e c^2}\left(-\frac{\mathbf{s}_{pr}\cdot\mathbf{s}_{el}}{4\pi r^3} + 3\frac{(\mathbf{s}_{pr}\cdot\mathbf{r})(\mathbf{s}_{el}\cdot\mathbf{r})}{4\pi r^5} + \frac{2}{3}(\mathbf{s}_{pr}\cdot\mathbf{s}_{el})\delta(\mathbf{r})\right). \quad (3.3)$$

Sir
Charles Galton Darwin
KBE MC FRS

Gregory Breit
Григорий Альфредович Брейт-Шнайдер

- The Darwin-Breit Hamiltonian reproduces all non-radiative electromagnetic phenomena on par with Maxwell's electrodynamics.

Low-energy states of the hydrogen atom (electron + proton) in dressed QED



Dressed particle QED vs. renormalized QED

- In the dressed particle theory, primary ingredients are particles, which are not affected by self-interaction. Their masses and charges are exactly those measured in experiments.
- The dressed QED has a finite Hamiltonian
$$H^d = H_0 + V_2^d + V_3^d + V_4^d + \cdots$$
- This Hamiltonian satisfies Wigner-Dirac requirements of relativistic invariance.
- Presently, it is known at low orders (≤ 4). Higher order terms could be calculated as well.
- The particles interact with each other by action-at-a-distance potentials that depend on instantaneous positions and velocities.

Dressed particle QED vs. renormalized QED

- Higher-order interactions describe also particle emission and absorption.
- S-matrix calculated with H^d is **exactly** the same as S-matrix of the renormalized QED. All experimental predictions of the latter are reproduced **exactly**.
- While renormalized QED is limited to S-matrix calculations, dressed particle theory can go beyond that: it can calculate *wave functions* of bound states and *time evolution* of state vectors and observables.
- We can project (approximately) H^d on a few-particle sector of the Fock space, diagonalize this projection and obtain a realistic energy spectrum as well as wave functions of bound states. Such calculations are impossible with the Hamiltonian H^c of the renormalized QED.
- In the next lecture we will discuss possible experimental tests of the dressed QED and its relationships with Einstein's special relativity.

Thank you!