Renormalizability without infinities

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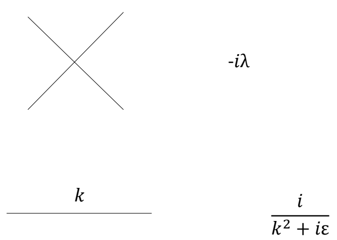
1 What we'll calculate

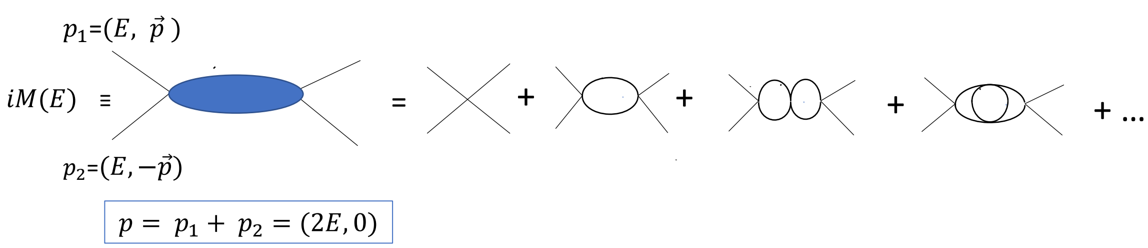
We'll follow Schwartz section 15.4. Take a scalar field theory for simplicity.

$$V_I = \frac{\lambda}{4!}\phi^4 \tag{1}$$

Feynman rules:



Probability amplitude of scattering (collision) of 2 particles:



The theory has one parameter to be set by experiment, λ. Then compute $M(E)$.

2 How to calculate

2.1 Two key assumptions

- QFT calculations are insensitive to physics at very small distances (large momenta)
- Therefore we can pick a cutoff (small distance or large momentum), calculate $M(E)$ with that cutoff, and then show that $M(E)$ is cutoff-insensitive for sufficiently small distances or large momenta.

2.2 Steps

- Each vertex has a factor of λ.
- Each loop involves an integral $\int d^4k = \int |k|^3 d|k| \int d\Omega$
- Don't include large momenta in the integral. i.e., $\int_0^\Lambda |k|^3 d|k| \int d\Omega$. Λ is called the 'cutoff'.
- **Very important**. Set the value of λ by comparing the theoretical result (which may depend on the cutoff) to some measurement.
- Compute $M(E)$ to some order in perturbation theory.

2.3 Digression on general perturbation theory

Suppose we have a perturbation expansion

$$f(\lambda) = a_1\lambda + a_2\lambda^2 + ... \tag{2}$$

and then suppose we introduce $\lambda^{'}$ so that

$$\lambda^{'} = \lambda + b\lambda^2. \tag{3}$$

Assuming λ is small, we can invert this equation to look like

$$\lambda = \lambda^{'} - b\lambda^{'2} + ... \tag{4}$$

Then we can write f in terms of $\lambda^{'}$ as

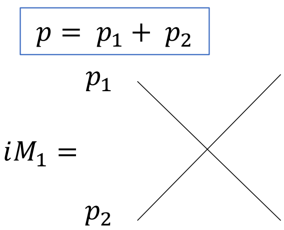
$$\begin{aligned} f(\lambda) &= a_1\lambda + a_2\lambda^2 + ... \\ &= a_1\left(\lambda^{'} - b\lambda^{'2} + ...\right) + a_2\left(\lambda^{'} - b\lambda^{'2} + ...\right)^2 + ... \\ &= a_1\lambda^{'} + (a_2 - a_1 b)\,\lambda^{'2} + ... \end{aligned} \tag{5}$$

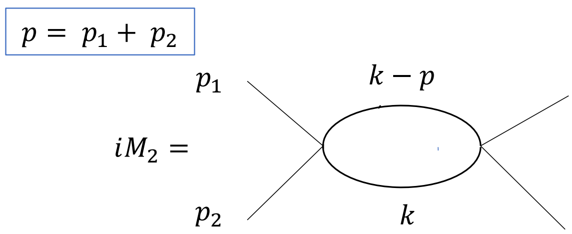
We see that the expansion coefficient for λ^2 is different from the expansion coefficient for $\lambda^{'2}$. **BOTH EXPANSIONS OF f ARE LEGITIMATE PERTURBATION EXPANSIONS BUT THEIR COEFFICIENTS DEPEND ON THE DEFINITION OF λ.**

We are going to exploit this simple fact.

3 Compute

Compute to second order in λ.[1]





Apply the Feynman rules $iM(E) = iM_1 + iM_2 + ...$

$$iM_1 = -i\lambda \tag{6}$$

[1] There is actually another bubble diagram in second order but we will ignore this for now.

$$iM_2 = \frac{(-i\lambda)^2}{2}\int_0^\Lambda \frac{|k|^3}{(2\pi)^4} d|k| \int \frac{i}{k^2}\frac{i}{(p-k)^2} d\Omega \tag{7}$$

where Ω represents the angular coordinates in 4-space[2]..

Next, decide on a measurement to determine the value of λ.

- Pick an energy E_R and measure the 2-particle scattering amplitude when each particle, in the center-of-mass, has energy E_R. i.e., **Measure $\mathbf{M(E_R)}$**.
- Compute $M(E_R)$ as a function of λ.

  $$iM(E_R) = -i\lambda + \frac{(-i\lambda)^2}{2}\int_0^\Lambda \frac{|k|^3}{(2\pi)^4} d|k| \int \frac{i}{k^2}\frac{i}{(p_R-k)^2} d\Omega + ... \tag{8}$$

  where $p_R = (2E_R, \vec{0})$
- Now for notational ease, **define** $\lambda_R = -M(E_R)$, so $-i\lambda_R = -i\lambda + \frac{(-i\lambda)^2}{2}\int_0^\Lambda ... + ...$
- We see that $\lambda_R = \lambda + \alpha\lambda^2 + ...$ where

  $$\alpha = \frac{-i}{2}\int_0^\Lambda \frac{|k|^3}{(2\pi)^4} d|k| \int \frac{i}{k^2}\frac{i}{(p_R-k)^2} d\Omega \tag{9}$$

We're now ready to make a prediction of $M(E)$ for arbitrary energy E in terms of λ. First, remember our digression on perturbation theory. Since λ_R is a power series in λ just as in eq. (3), then $M(E)$ can be expressed as a series in λ_R following the same logic we used for eq. (5). The result is

$$-M(E) = \lambda_R + (\xi(E) - \alpha)\lambda_R^2 + ... \tag{10}$$

where

$$\xi(E) = \frac{-i}{2}\int_0^\Lambda \frac{|k|^3}{(2\pi^4)} d|k| \int \frac{i}{k^2}\frac{i}{(p-k)^2} d\Omega \tag{11}$$

with $p = (2E, \vec{0})$.

[2] You might notice that I treat the 4-vectors as though they are Euclidean, for example $k^2 = \sum_\mu k_\mu^2$. I'm allowed to do this by analytically continuing the 0 components from k_0 to ik_0 and using the $i\epsilon$ to avoid poles. This trick is known as Wick rotation, but for our purposes the details aren't important.

$\xi(E)$ and α are identical integrals except that one has a denominator with p_R and the other had a denominator with p. Combine the integrals to get

$$-M(E) = \lambda_R + \left[\frac{i}{2}\int_0^{\Lambda} \frac{|k|^3}{(2\pi)^4|k|^2}\int\left(\frac{1}{(p-k)^2} - \frac{1}{(p_R-k)^2}\right)d\Omega\right]\lambda_R^2 + ... \tag{12}$$

It turns out that the integral is finite even if $\Lambda \to \infty$! So everything is insensitive to the cutoff.

To see this, write

$$\frac{1}{(p-k)^2} - \frac{1}{(p_R-k)^2} = \frac{p^2 + p_R^2 - (p_R + p)\cdot k}{(p-k)^2(p_R-k)^2} \tag{13}$$

For very large values of $|k|$, the RHS is proportional to $\frac{1}{|k|^3}$. The complete integrand is then proportional, at large $|k|$, to $\frac{1}{|k|^2}$ and therefore is convergent as $\Lambda \to \infty$.

$$-M(E) = \lambda_R + \left(\chi(E) + \mathcal{O}(\frac{1}{\Lambda})\right)\lambda_R^2 + ... \tag{14}$$

where $\chi(E)$ is independent of Λ.

It is worth repeating that we do **NOT** take Λ to ∞. Rather, Λ characterizes the scale of new physics. Even though the equation above has a term $\mathcal{O}(\frac{1}{\Lambda})$, this is a bit misleading because we need a dimensionless quantity and therefore must multiply $\frac{1}{\Lambda}$ by some coefficient with dimensions of energy. But the only candidate is E, so really we should write $\mathcal{O}(\frac{E}{\Lambda})$. As long as we are only interested in physics at energies $E << \Lambda$, that term will be negligibly small.

4 What happened to infinities and counterterms?

The infinities went away because we used a cutoff. That's nothing new. What's 'new' is the viewpoint. Our viewpoint is that physics should be insensitive to what goes on at small distances and that we can test this premise by showing that our computations of physical quantities are insensitive to high-energy terms.

What may seem surprising, is that our program was successful!

In the ϕ^4 example, the procedure used is a bit unwieldy, especially when generalized to other theories and higher orders. For example, we had to combine integrands as in eq. (13). That integral is messy, even though it converges.

Instead, it's often more convenient to do the integrals BEFORE we combine them, so for example we would perform the integral

$$\int_0^\Lambda \frac{|k|^3}{(2\pi)^4} d|k| \int \frac{i}{k^2} \frac{i}{(p-k)^2} d\Omega. \tag{15}$$

This integral is proportional to $\log(\Lambda)$ which blows up as $\Lambda \to \infty$. So that's the famous 'infinity' that has come up before. And how do we cancel that infinity? By subtracting the integral evaluated at p_R, namely

$$\int_0^\Lambda \frac{|k|^3}{(2\pi)^4} d|k| \int \frac{i}{k^2} \frac{i}{(p_R-k)^2} d\Omega. \tag{16}$$

That's the counterterm and it also blows up. When the calculations are done this way, integrals have to be regulated, and counterterms have to be added. The technique works because the 'real' perturbation expansion in terms of renormalized parameters (e.g. λ_R) doesn't have divergent integrals.

Is this kind of thing some brand new QFT-specific kind of mathematics? Not really. We can summarize these manipulations by saying that we are looking at quantities like $\lim_{\Lambda\to\infty}(f_1(\Lambda) - f_2(\Lambda))$. This is well-defined and, in fact, is the only legitimate way to treat integrals whose upper bound is ∞. If, on the other hand, we tried to rewrite the above as $\lim_{\Lambda\to\infty} f_1(\Lambda) - \lim_{\Lambda\to\infty} f_2(\Lambda)$, we'd potentially end up with $\infty - \infty$, which is nonsense.

It turns out we've encountered precisely this kind of thing in Newtonian mechanics, where we look at quantities like $\lim_{\epsilon-\to 0} \frac{f(x+\epsilon)-f(x)}{\epsilon}$ which is conveniently known as $f'(x)$. If we took the limits before taking the ratio, so $\frac{\lim_{\epsilon-\to 0}(f(x+\epsilon)-f(x))}{\lim_{\epsilon-\to 0}\epsilon}$, we'd end up with $\frac{0}{0}$ which is also nonsense. We can make this look even more like the terms we get in QFT, by speaking of $\lim_{\frac{1}{\epsilon}\to\infty}$.

There's nothing a priori obvious that calculus should be useful for dealing with laws of classical physics. There's an implicit assumption that the behavior of systems at VERY small distances, really doesn't affect the physics of ordinary distances. If we wanted to approximate space by a grid with very small spacing, we'd get results that are extremely close to correct. Indeed, that's why quantum mechanics wasn't discovered until the 20th century.