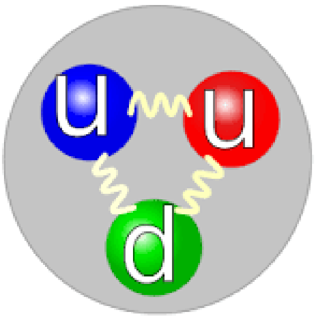
Thomson Chapter 10 SU(3)-color: Lagrangian and Review

Bill Celmaster

January 15, 2023



1 Outline

- Start with something new, the quark-gluon interaction Lagrangian (aka SU(3)-color aka QCD).
  - Feynman rules for SU(3)-color
  - Some interesting Feynman diagrams
  - Some comments
- Review the motivation for the QCD Lagrangian
  - History of quark flavors and group theoretic scattering relationships

  - Local (gauge) symmetry in QED
  - Generalization of gauge symmetry to SU(3)

- Advanced topics
  - Regulators revisited: renormalization and ghosts
  - Asymptotic freedom: The 1970's revolution – the interaction term is small for high energy interactions. QCD validation begins.
  - Confinement and lattice validation
  - Experiments

2 QCD

2.1 The Lagrangian

(See Thomson 10.1.1 and 10.2 for some overlap with what follows.)

Here is the Lagrangian for the interaction of one flavor of quark with gluons. **The Lagrangian also includes terms of the interaction of one scalar with gluons but we will not be concerned with this.** For multiple flavors of quarks and scalars, there is an identical (other than for quark mass) Lagrangian to be added for each flavor.

The first step in performing perturbative calculations in a non-Abelian gauge theory is to work out the Feynman rules. The SU(N)-invariant Lagrangian for a set of N fermions and N scalars interacting with non-Abelian gauge fields is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 + (\partial_\mu \bar{c}^a)(\delta^{ac}\partial_\mu + gf^{abc}A^b_\mu)c^c \\ &+ \bar{\psi}_i(\delta_{ij}i\partial\!\!\!/ + g A\!\!\!/^a T^a_{ij} - m\delta_{ij})\psi_j \\ &+ \left[(\delta_{ki}\partial_\mu - igA^a_\mu T^a_{ki})\phi_i\right]^\star \left[(\delta_{kj}\partial_\mu - igA^a_\mu T^a_{kj})\phi_j\right] - M^2\phi_i^\star\phi_i, \end{aligned} \tag{26.1}$$

where c^a and $\bar{c}^a$ are the Faddeev–Popov ghosts and anti-ghosts respectively and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu. \tag{26.2}$$

We have included scalars in this Lagrangian for generality, even though we have observed no scalar states in nature that are colored (charged under QCD). Many theories, such as supersymmetric QCD, do have colored scalars. The Higgs doublet in the Standard Model is an example of a scalar field charged under the weak gauge group SU(2).

The kinetic terms from the QCD Lagrangian are

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 + \bar{\psi}_i(i\partial\!\!\!/ - m)\psi_i - \phi_i^\star(\Box + M^2)\phi_i - \bar{c}^a\Box c^a. \tag{26.3}$$

- The index 'a' ranges from 1 to 8 and represents coefficients of the 8-dimensional *adjoint* representation of SU(3).
- Greek (vector-)indices range from 0 to 3 (time and space)
- 'A' are vector-boson fields known as **gluon** fields, analogous to the EM vector field known as the photon field.
- 'c' and '$\bar{c}$' are ghost and anti-ghost fields (to be introduced in the future), which are gluon artifacts (they don't correspond to particles).
- 'ξ is an arbitrary constant related to the ghosts. Physical measurements are independent of ξ.
- 'f^{abc}' are color structure constants (see below)

- 'ψ_i' are the quark fields. Each is a 4-spinor whose spinor indices aren't shown. The indices 'i' and 'j' range from 1 to 3 (blue, red and green) and represent coefficients of the 3-dimensional *fundamental* representation of SU(3).
- Slash notation is defined for objects X_μ with vector-indices as $\not{X} = X_\mu \gamma^\mu$ where the γ^μ are 4x4 Dirac matrices acting on spinor objects.
- g is the QCD coupling constant
- m is the quark mass.
- T^a_{ij} are the components of a set of 8 (indexed by 'a') 3x3 color matrices (see below)
- We will ignore the last line having to do with scalar fields.

The 8 matrices $\mathbf{T}^a = \mathbf{T}_a = \frac{1}{2}\boldsymbol{\lambda}_i$ are the generators of SU(3). These form the basis of the SU(3) Lie Algebra.

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

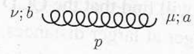
The generators satisfy the algebraic relation which specifies the SU(3) algebra

$$[\mathbf{T}_a, \mathbf{T}_b] = \mathbf{T}_a \mathbf{T}_b - \mathbf{T}_b \mathbf{T}_a = i f^{abc} \mathbf{T}_c. \qquad (1)$$

The f^{abc} are known as the structure constants of the algebra.

2.2 Feynman Rules

Since the kinetic term for the gauge bosons is just the sum over $N^2 - 1$ free gauge boso[ns,] the propagator for each should be just the same as the propagator for a photon. Since [we] chose the basis of group generators to be orthogonal there is no kinetic mixing betwe[en] gluons. So the propagator is



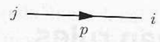
$$= i\frac{-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}}{p^2 + i\varepsilon}\delta^{ab}. \quad (26.4)$$

The gluon propagator is the photon propagator with an extra δ^{ab} factor. When gluo[ns] appear as intermediate states, one must sum over all possible gluons, which gives a su[m] over a and b.

The ghost propagator is

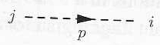
$$= \frac{i\delta^{ab}}{p^2 + i\varepsilon}. \quad (26.5)$$

The propagators for colored fermions,



$$= \frac{i\delta^{ij}}{\not{p} - m + i\varepsilon}, \quad (26.6)$$

and for colored scalars,



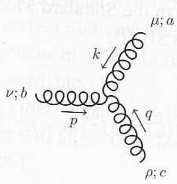
$$= \frac{i\delta^{ij}}{p^2 - M^2 + i\varepsilon}, \quad (26.7)$$

are the same as in QED but with δ_{ij} factors, where i, j refer to fundamental color indices. These δ^{ab} and δ^{ij} factors just say that the color that comes in is the same as the color that comes out – color is conserved. As with the gluon, we must sum over colors when these propagators appear as intermediate states.

The interactions are

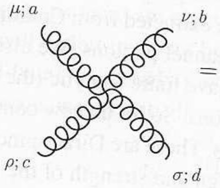
$$\begin{aligned}\mathcal{L}_{\text{int}} = &-gf^{abc}(\partial_\mu A_\nu^a)A_\mu^b A_\nu^c - \frac{1}{4}g^2\left(f^{eab}A_\mu^a A_\nu^b\right)\left(f^{ecd}A_\mu^c A_\nu^d\right) + gf^{abc}(\partial_\mu \bar{c}^a)A_\mu^b c^c \\ &+ gA_\mu^a \bar{\psi}_i \gamma^\mu T_{ij}^a \psi_j + igA_\mu^a T_{ij}^a(\phi_i^\star \partial_\mu \phi_j - \phi_j \partial_\mu \phi_i^\star) + g^2 \phi_i^\star A_\mu^a T_{ik}^a T_{kj}^b A_\mu^b \phi_j,\end{aligned} \quad (26.8)$$

where we have used that $(T_{ij}^a)^\star = T_{ji}^a$ for SU(N). For the triple-gluon vertex, the derivative can act on any of the gluons, giving the Feynman rule



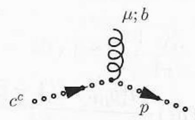
$$= gf^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right]. \quad (26.9)$$

Note that we take all the momentum incoming, so $p+k+q=0$. This is different from the convention we used for QED, where all momenta were going to the right. The four-gluon vertex gives



$$= -ig^2 \times \big[f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \big]. \tag{26.10}$$

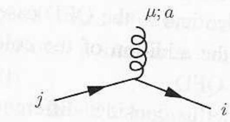
The ghost vertex Feynman rule is



$$= -g f^{abc} p^\mu. \tag{26.11}$$

Note that there is only one contraction (since ghosts and anti-ghosts are different), in contrast to the scalar QED vertex.

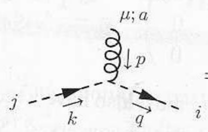
There is one vertex for interaction with a fermion, which gives



$$= ig\gamma^\mu T^a_{ij}. \tag{26.12}$$

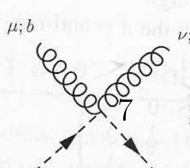
As in QED, the orientation of the vertex in a Feynman diagram does not matter. The vertex gets a factor of $ig\gamma^\mu T^a_{ij}$, with i the color of the quark with the arrow pointing away from the vertex and j the other color.

Finally, there are two vertices for the scalar, just as in scalar QED. These are



$$= ig(k^\mu + q^\mu)\, T^a_{ij} \tag{26.13}$$

and



$$= ig^2 T^a_{ik} T^b_{kj} g^{\mu\nu}. \tag{26.14}$$

There are a few additional rules.

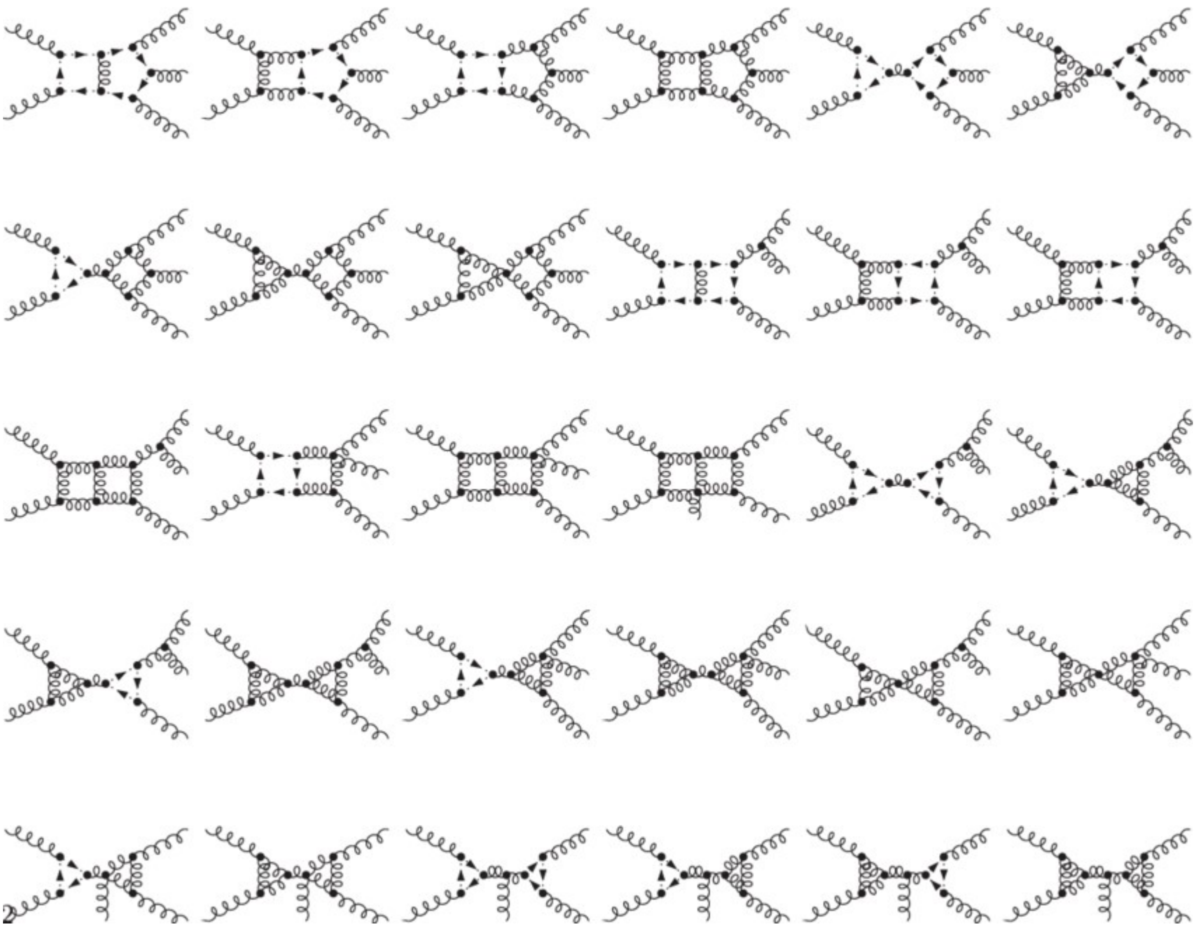
- Propagator rules apply to lines joining two vertices
- 4-momenta are conserved at each vertex.
- Each loop has one undetermined momentum k which, together with the momenta of other lines arriving at the vertex, determine the momentum of each line in the loop. You must then integrate over k. The integrand $f(k,...)$ is obtained from the above Feynman rules

$$\int d^4k \frac{f(k,...)}{(2\pi)^4}. \tag{2}$$

- Don't forget that the integral must be regulated (e.g. 'cut off') and that terms must be combined as we discussed in the section on renormalization (alternatively but equivalently, counterterms can be added to the Feynman rules) and then the regulator can be 'removed' (e.g. cut-off goes to infinity).
- Lines free at one end represent ingoing or outgoing particles and should be multiplied by the polarization vectors for those particles.

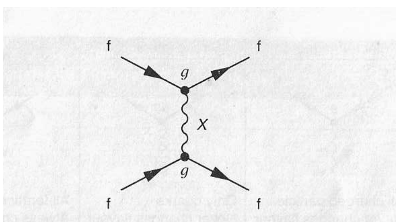
2.3 Some interesting Feynman diagrams

Here is a partial set of Feynman diagrams for two incoming and three outgoing gluons, just to give you an idea.



2.4 Some comments

- The lowest order (g^2) diagrams are *tree* diagrams that look like these.



The Feynman rules for this diagram is similar to the Feynman rules for photon-exchange by electrons (same propagator). **But these lead to a Coulomb force** which dies off at large distances. So we'd expect we could break apart neutrons etc. into quarks, just like we can break off the electrons in atoms. That doesn't happen. Why not?

- The perturbation series can be useful **only** only if higher-order terms are smaller. Prior to the early 1970's, it was accepted wisdom that the coupling constant g is large. That was inferred from interactions amongst nucleons, known as *strong interactions*. As a result, **the tree diagrams do NOT dominate the series, and we don't know what the force is between quarks.**
- When QCD was first hypothesized, physicists gave arm-waving arguments that the inter-quark potential was linear, and therefore quarks were *confined*. Attempts were made to put these arguments on firmer footing based on new kinds of strong-coupling approximation methods – which eventually decades later, led to some successes of lattice QCD.
- What is remarkable is that in the late 60's and early 70's, QCD was simply a huge leap of faith based on some beautiful mathematical ideas with no useful mathematical methods for comparing the theory to experiment.

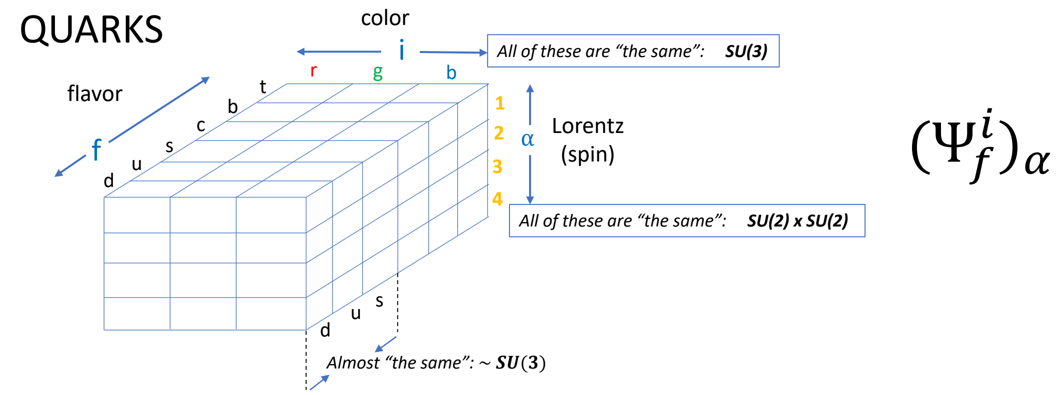
3 Review: Motivating the QCD Lagrangian

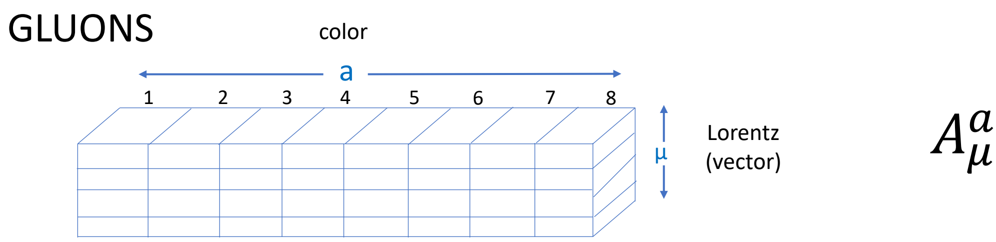
3.1 Symmetries

- E.g. rotational symmetry. "The laws of nature are invariant under rotations". i.e. Equations of motion are invariant under rotations.
  - Suppose e.o.m. looks like
$$F(x, y) = 0 \tag{3}$$
  - Then let (x', y') be a rotation of (x, y).
$$\begin{aligned} x' &= x\cos(\theta) + y\sin(\theta) \\ y' &= -x\sin(\theta) + y\cos(\theta) \end{aligned} \tag{4}$$
  - If we know that the e.o.m. is invariant under rotations, then we can rewrite it as
$$\tilde{F}(x^2 + y^2) = 0. \tag{5}$$

– That's a great simplification.

- Takeaway: Symmetries simplify the equations of motion and generally give relationships between experiments.
- Rotational symmetry is one of the Poincare symmetries.
- There are other symmetries. The earliest symmetry studied was *charge symmetry*. For now, we'll study flavor and color symmetry, both examples of *internal symmetries*.





- Field theory considerations – an example:
  – Suppose a set of fields X^m transform under group transformations as

$$g : X^m \to \sum_n \alpha^{mn}(g) X^n \tag{6}$$

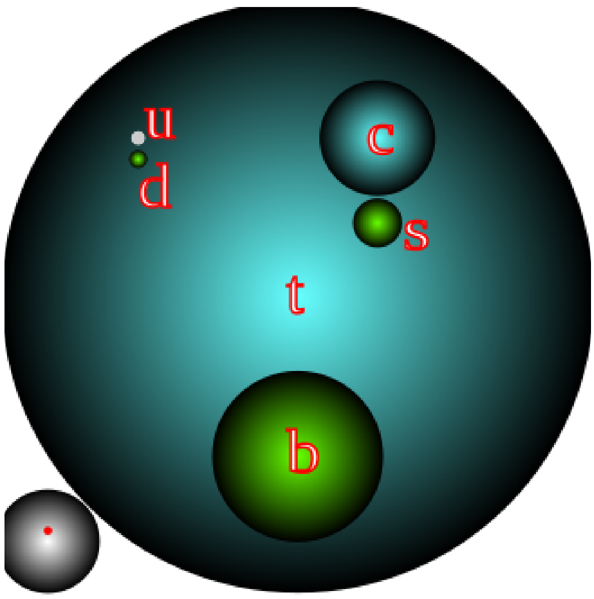
  – Suppose there is a Lagrangian term cubic in X. We want to find coefficients c_{mnp} so that $c_{mnp}X^m X^n X^p$ is invariant under group transformations. Namely

$$g : c_{mnp} X^m X^n X^p \to c_{mnp} \left(\alpha^{mm'}(g) X^{m'}\right) \left(\alpha^{nn'}(g) X^{n'}\right) \left(\alpha^{pp'}(g) X^{p'}\right)$$
$$= c_{mnp} X^m X^n X^p \qquad (7)$$

– We say that $c_{mnp} X^m X^n X^p$ is a scalar. The values of c_{mnp} can be found in tables of *Clebsch-Gordan coefficients*.

3.2 Flavor

The type of quark (there are 6 types) is known as FLAVOR. Although the interactions are symmetric, the quark masses are different and thus break the symmetry.



3.2.1 Fun and games with flavor symmetry

- The up and down quarks have almost the same mass, so there is almost a symmetry in transforming up into down. So, for example, the proton is *uud* and the neutron is *ddu* so by transforming ups into downs, the physics of neutrons and protons is (so far as strong interactions go) similar.

- Most ‘common’ particles are made of up and down quarks, so physicists learned to make predictions based on so-called isospin symmetry, where ups are transformed to downs. That symmetry is mathematically SU(2).

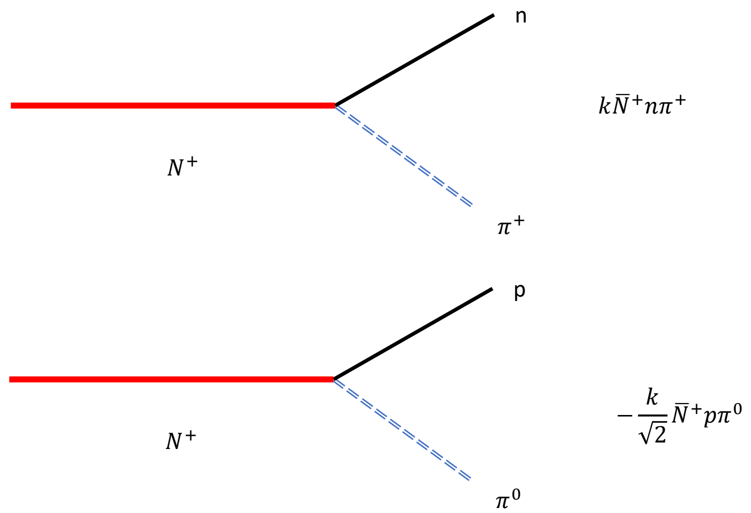
- **One important set of predictions has to do with “clustering” of particle masses. This is analogous to the clustering of spectral lines of the hydrogen atom (aka eigenvalues of the Hamiltonian) originating from spherical symmetry.**

- Another set of predictons has to do with scattering ratios. Consider the heavy baryons known as Roper resonances N^0 and N^+. The two Roper resonances form an isospin doublet (the ‘fundamental’ representation of SU(2)) just like the neutron-proton system. These interact with pions, that form an isospin triplet (the ’adjoint’ representation of SU(2)).

- If we want to write an isospin-invariant (i.e. “scalar”) effective Lagrangian, limited to 3-field interactions that can represent the interactions between Roper-resonances, ordinary nucleons, and pions, the unique SU(2)-invariant interaction is

$$\mathcal{L}_I = k\left[\bar{N}^+ n\pi^+ + \bar{N}^- p\pi^- - \frac{1}{\sqrt{2}}\left(\bar{N}^+ p\pi^0 + \bar{N}^0 n\pi^0\right)\right] + \text{herm. conj.}. \tag{8}$$

- We can then use Feynman diagrams to compute decay rates. For example, here are the Feynman diagrams for positive Roper decays into a nucleon and pion. The Feynman decay diagrams look for example, like this:



These diagrams are similar to one another but what is important is the vertex coefficient. For the first decay, the coefficient is k and for the second decay, the coefficient is $-k/\sqrt{2}$.

Therefore the ratio of decay amplitudes is $-\sqrt{2}$ and the ratio of probabilities is 2.

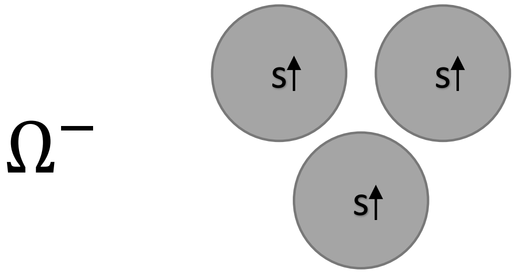
We therefore predict that the rate of decay of $N^+ \to n + \pi^+$ is 2 times the rate of decay of $N^+ \to p + \pi^0$.

- These symmetry considerations can be extended to include the strange quark, even though it has a significantly higher mass than the up and down quarks. The symmetry in that case is SU(3) and is known as SU(3)-flavor because it acts on the flavors. Reasonable predictions, **including mass-clustering patterns** can be made using SU(3) representation theory.

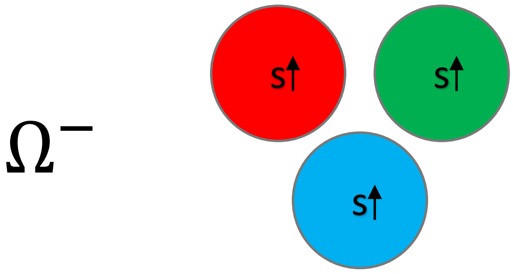
- For heavier quarks, their masses are so much different than the up, down and strange quark masses, that symmetry predictions (based on SU(4) or SU(5) etc.) are more or less useless.

3.3 Color

3.3.1 Why color?



- The Pauli exclusion principle prohibits this, since it isn't possible to have two identical fermions (much less 3 particles) in the same state.
- Resolved if we hypothesize that each of the quarks is actually in a different "state" whose distinct property we call "color".

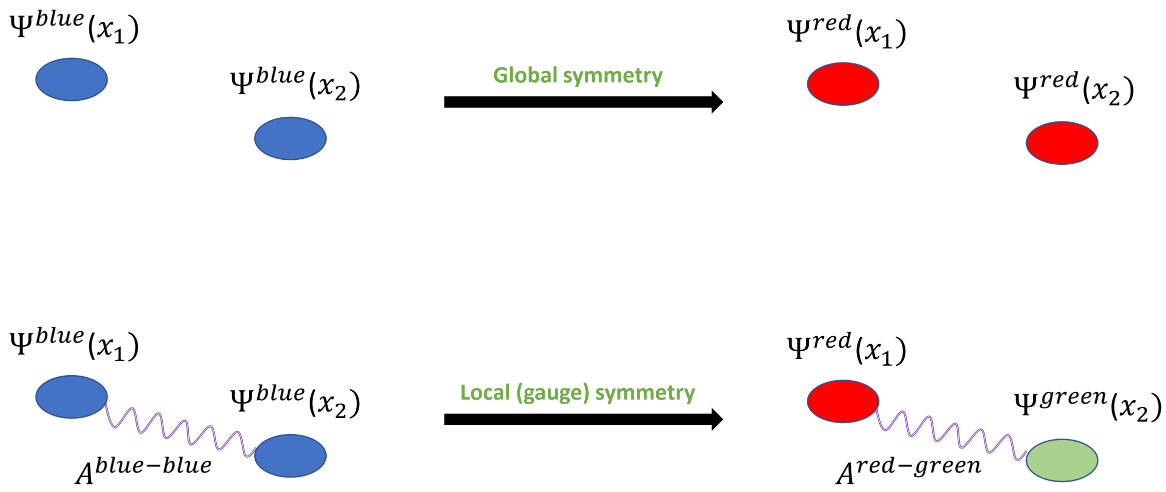


- Unlike flavor symmetry, color symmetry is **exact**.

3.4 From global to local symmetries

Flavor symmetry was a kind of digression because it is **only an approximate symmetry**. Previously, physicists examined Poincare (Lorentz) symmetry and charge symmetry – both examples of **exact symmetries**. Weyl observed that both symmetries could be promoted from "global" to "local", by means of a "connecting" field which effectively communicates the symmetry from one point of spacetime to another.

It was hypothesized that color symmetry was the same way.



3.5 How this works for a scalar theory with charge symmetry

Consider a simple complex scalar field theory whose Lagrangian consists only of kinetic and mass terms.

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\phi^*(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi^*(x)\phi(x). \tag{9}$$

This theory is invariant under the charge-symmetry transformation

$$\phi(x) \to e^{i\alpha}\phi(x). \tag{10}$$

To see this, note that $\phi^*(x) \to e^{-i\alpha}\phi^*(x)$ so that

$$\begin{aligned}\frac{1}{2}\partial_\mu\phi^*(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi^*(x)\phi(x) &\to \frac{1}{2}\partial_\mu\phi^*(x)e^{-i\alpha}e^{i\alpha}\partial^\mu\phi(x) - \frac{1}{2}m^2e^{-i\alpha}e^{i\alpha}\phi^*(x)\phi(x)\\ &= \frac{1}{2}\partial_\mu\phi^*(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi^*(x)\phi(x).\end{aligned} \tag{11}$$

This symmetry is true for any choice of α.

Charge symmetry is a **global symmetry**.

IMPORTANT: To prove the invariance, we relied on the fact that

$$e^{-i\alpha}\partial_\mu\left(e^{i\alpha}\phi(x)\right) = e^{-i\alpha}e^{i\alpha}\partial_\mu\phi(x). \tag{12}$$

This is true only because $e^{i\alpha}$ is a constant.

Now imagine that we want to extend charge symmetry to be a **local** symmetry so that a different α could be chosen for each point in spacetime. That is,

$$\phi(x) \to e^{iq\chi(x)}\phi(x) \tag{13}$$

This will be tricky because the kinetic term has a derivative, which 'connects' fields at 'adjacent' points, and spoils the symmetry. **It will turn out that we need to introduce a new field that compensates for the derivative.**

So... let's call the field A_μ. (For a moment, pretend we never heard of electromagnetism.) Using suggestive notation, we'll define

$$D_\mu\phi(x) = (\partial_\mu + iqA_\mu)\phi(x). \tag{14}$$

This resembles the definition, in geometry, of a covariant derivative and indeed we will call it a covariant derivative. Consider the following transformation rules for both the ϕ and A_μ fields:

$$\begin{aligned}\phi(x) &\to e^{iq\chi(x)}\phi(x)\\ A_\mu(x) &\to A_\mu(x) - \partial_\mu\chi(x).\end{aligned} \tag{15}$$

By applying these local transformation rules we can show that

$$D_\mu\phi^*(x)D^\mu\phi(x) \to D_\mu\phi^*(x)D^\mu\phi(x). \tag{16}$$

We now have a new Lagrangian which is locally charge invariant.

$$\mathcal{L}'(x) = \frac{1}{2} D_\mu \phi^*(x) D^\mu \phi(x) - \frac{1}{2} m^2 \phi^*(x) \phi(x). \tag{17}$$

But we're not done. Since we introduced a new field, A_μ, we have to add a kinetic term for A_μ – a quadratic term involving one or two derivatives. It turns out that the appropriate locally-invariant term is just $F_{\mu\nu} F^{\mu\nu}$, the product of electromagnetic field strengths.

Presto ... out of whole cloth we've created **scalar QED** with a Lagrangian uniquely determined by the requirement of local charge symmetry!

3.6 Local color symmetry

The quark color transformations are a generalization of charge transformations. Instead of multiplication by a phase factor, we now multiply by a complex matrix.

Consider the blue, green and red up-quarks. Think of these designators as vector components. Then

$$\begin{pmatrix} b' \\ g' \\ r' \end{pmatrix} = U \begin{pmatrix} b \\ g \\ r \end{pmatrix} \tag{18}$$

where U is any 3x3 unitary matrix of determinant 1 and can be expressed as

$$U = e^{i\alpha \cdot \hat{\mathbf{T}}}, \tag{19}$$

where the 8 matrices $\hat{\mathbf{T}}_i = \frac{1}{2} \boldsymbol{\lambda}_i$ are the generators of SU(3). These form the basis of the SU(3) Lie Algebra.

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Now, we'll change notation slightly. Instead of writing

$$\begin{pmatrix} b \\ g \\ r \end{pmatrix}, \tag{20}$$

we'll write ψ where we define ψ to be the 3-component object

$$\psi = \begin{pmatrix} \psi_b \\ \psi_g \\ \psi_r \end{pmatrix} \tag{21}$$

Then the SU(3)-color transformations act on ψ.

The Lagrangian for this theory will need to be invariant under these SU(3) transformations – that is, the individual terms are 'scalars' with respect to SU(3) transformations.

So far, the SU(3) transformations are independent of space and time. They are global. We make the symmetry local following the same approach used for local scalar charge symmetry.

- A new field, A_a^μ is introduced. This is called the gluon field.
- It not only has a Lorentz index μ but also a color index a which ranges from 1 to 8.
- Then we modify the quark kinetic term by changing the derivative to a covariant derivative involving the gluon field.

- Furthermore, we add a gluon kinetic term.
- All terms are constrained so that the Lagrangian is invariant under local color symmetries.

The final result was the Lagrangian we studied last time.

The first step in performing perturbative calculations in a non-Abelian gauge theory is to work out the Feynman rules. The $\mathrm{SU}(N)$-invariant Lagrangian for a set of N fermions and N scalars interacting with non-Abelian gauge fields is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 - \frac{1}{2\xi}\left(\partial_\mu A^a_\mu\right)^2 + \left(\partial_\mu \bar{c}^a\right)\left(\delta^{ac}\partial_\mu + g f^{abc} A^b_\mu\right)c^c \\ &+ \bar{\psi}_i\left(\delta_{ij} i\not{\partial} + g \not{A}^a T^a_{ij} - m\delta_{ij}\right)\psi_j \\ &+ \left[\left(\delta_{ki}\partial_\mu - igA^a_\mu T^a_{ki}\right)\phi_i\right]^\star \left[\left(\delta_{kj}\partial_\mu - igA^a_\mu T^a_{kj}\right)\phi_j\right] - M^2\phi_i^\star\phi_i, \qquad (26.1)\end{aligned}$$

where c^a and $\bar{c}^a$ are the Faddeev–Popov ghosts and anti-ghosts respectively and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \qquad (26.2)$$

We have included scalars in this Lagrangian for generality, even though we have observed no scalar states in nature that are colored (charged under QCD). Many theories, such as supersymmetric QCD, do have colored scalars. The Higgs doublet in the Standard Model is an example of a scalar field charged under the weak gauge group $\mathrm{SU}(2)$.

The kinetic terms from the QCD Lagrangian are

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 - \frac{1}{2\xi}\left(\partial_\mu A^a_\mu\right)^2 + \bar{\psi}_i\left(i\not{\partial} - m\right)\psi_i - \phi_i^\star\left(\Box + M^2\right)\phi_i - \bar{c}^a\Box c^a. \qquad (26.3)$$

The above Lagrangian actually also includes tri-colored scalars (no Lorentz index) in case we wanted to add scalar particles into our theory.

The Lagrangian also includes yet other fields known as ghosts. These are color artifacts and will be explained next time.

4 Advanced Topics

4.1 Regulators revisited

4.1.1 Philosophical review

Like all of physics, the mathematics involves limits.

- In classical physics, we typically refer to 'a limit as $\epsilon \to 0$ of some ratio'.
  - That's calculus. Call this ϵ-regularization
- In QFT, refer to a cutoff limit, 'a limit as $\Lambda \to \infty$ of some integral'
  - Call this Λ-regularization
- Mathematical rigor will demand that the limits be carefully defined and that theorems be carefully proven.
  - For QFT, the mathematical rigor is still incomplete.
  - In QFT, other kinds of regularizations are also used, and proven to be equivalent order by order in various kinds of expansions.
- Regularization works because of an implicit law of physics: the laws of macro-scale nature are insensitive to the details of micro-scale nature.

4.1.2 More on Λ-regularization

- Suppose A and B are constants. Then clearly

$$\frac{\int_0^2 A dx}{\int_0^2 B dx} = \frac{2A}{2B} = \frac{A}{B} \tag{22}$$

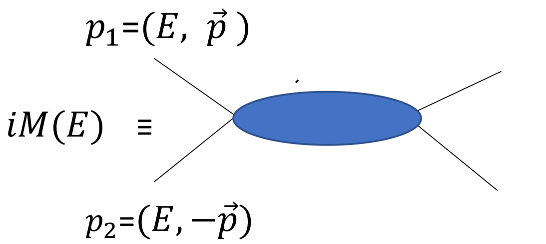
- What if the upper limit of each integral is ∞? We introduce the regulator Λ.
- Then we have

$$\lim_{\Lambda\to\infty} \frac{\int_0^\Lambda A dx}{\int_0^\Lambda B dx} = lim_{\Lambda\to\infty} \frac{A\Lambda}{B\Lambda} = \frac{A}{B}. \tag{23}$$

- This technique works only if the limits are taken after the ratios.

4.1.3 Renormalization (review)

Consider a generic QCD scattering amplitude for 2 incoming and 2 outgoing particles (For simplicity, suppress spin-dependence). This depends on the unknown parameter g.



$$p = p_1 + p_2 = (2E, 0)$$

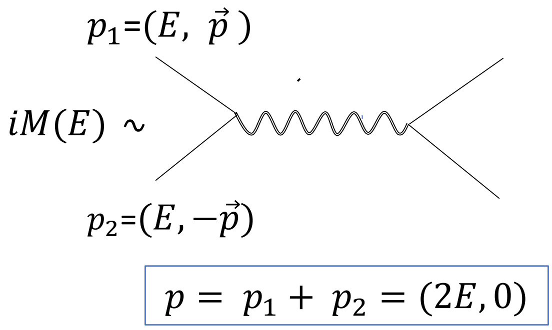
- In what follows, pick Λ and at the end $\Lambda \to \infty$.
- Assume that QCD can predict $R(E, E_R) \equiv \frac{M(E)}{M(E_R)}$
- The prediction of $M(E)$ depends on g. Similarly for $M(E_R)$. So we write $M(E, g)$ and $M(E_R, g)$ so $R(E, E_R) \equiv \frac{M(E,g)}{M(E_R,g)}$
- (Renormalization) Assume we measured $M(E_R, g)$ so now we know its value. $M_{ref} = M(E_R, g)$.
  - Invert: $g = M^{-1}(E_R, M_{ref})$
  - Then
$$R(E, E_R) = \frac{M(E, M^{-1}(E_R, M_{ref}))}{M_{ref}} \tag{24}$$
- The only unmeasured quantity on the RHS is the numerator, and it doesn't involve g.
- We can derive an order by order expansion in M_{ref} or some related parameter. **If that parameter is small we can hope the series is a good approximation.**
- All intermediate steps involve the regulator (e.g. Λ-cutoff) which we can remove at the end.

- **BOOKKEEPING**: The above procedure is arithmetically challenging. A good trick is to introduce intermediate quantities called counterterms. Formally they are additional terms in the Lagrangian leading to new Feynman diagrams and Feynman rules. **THIS IS JUST A TRICK**.

4.1.4 Choosing the coupling constant – asymptotic freedom

See Thomson 10.5 for a version of this material, with many details filled in, and with a comparison to QED – which is **not** asymptotically free.

- Return to the diagram that defines $M(E)$. Suppose that the lowest-order Feynman diagram for QCD scattering of quarks looks like



  If we apply the Feynman rules for this diagram, we get something like

$$iM(E) = -ig^2 \frac{1}{p^2} + \mathcal{O}(g^4) = -ig^2 \frac{1}{4E^2} + \mathcal{O}(g^4) \qquad (25)$$

  where g is the coupling constant that appears in the QCD Lagrangian.

- Here's the trick: Define $g(E_R)$ via

$$iM(E_R) = -i\,(g(E_R))^2\,\frac{1}{p^2} = -i\,(g(E_R))^2\,\frac{1}{4E^2} \qquad (26)$$

  This looks almost identical to eq.(25) EXCEPT that it's missing the order-g^4 terms. **We've defined $g(E_R)$ to be EXACTLY the 'coupling' measured at the reference energy E_R.**

- In the last section, we saw that we can compute $M(E)$ directly from $M(E_R)$ without using g. Now we see, with our definition of $g(E_R)$ in terms of $M(E_R)$ that **we can compute $M(E)$ in terms of $g(E_R)$**.

- As it turns out, this leads to a perturbation series very similar to our original one. $g(E)$ **is known as 'the renormalized QCD coupling constant defined at energy E'.**

- Now, what happens when we compute $M(E)$ through $\mathcal{O}(g^4)$? Here's the answer:

$$iM(E) = -i\frac{1}{4E^2}g^2(E_R)\left(1 + \frac{\beta}{4\pi}g^2(E_R)\log(\frac{E^2}{E_R^2})\right) \tag{27}$$

  where

$$\beta = \frac{2}{3}n_f - 11 \tag{28}$$

  and n_f are the number of flavors.

- β is known as the beta-function. With 6 flavors, **the beta-function is negative!!!!!** THIS WAS THE STUPENDOUS DISCOVERY KNOWN AS 'ASYMPTOTIC FREEDOM'. Why is it so important? Because if $E > E_R$, then $M(E) < M(E_R)$ and that means $g(E) < g(E_R)$. That means perturbation theory is more reliable when we pick a renormalized coupling constant defined at a higher energy.

- But wait! Are we getting something for nothing? Why can't we then do low-energy calculations but using $g(E_R)$ for some large value of E_R? The answer is subtle but amounts to the following: when a process has a typical scattering energy of E, the perturbation calculations typically have terms that look like powers of $\log(E/E_R)$ where E_R is the reference energy used for defining $g(E_R)$. If E is much different than E_R, then the logs cause terms to be large even though $g(E_R)$ might be small. To control these logs, we need to pick a reference energy close to the energy of interest. If the reference energy is small, then $g(E_R)$ is large, so we're out of luck. But if the reference energy is large, then $g(E_R)$ is small, so as long as we're computing process with energies close to E_R, the calculations are reliable.

- The discovery that $\beta < 0$ – and its significance – were first noted independently by Politzer at Harvard, and Gross and Wilczek at Princeton in 1973. This revolutionized physics because it meant perturbative QCD could be used to predict processes at high energies. These predictions have proven to be extremely accurate leading to excellent tests of QCD.

4.1.5 Color confinement

This topic is covered quite well in Thomson 10.4.

Why has no-one ever seen a quark or gluon? If we imitate QED, we'd expect to learn that gluons cause a calculable force between quarks, and that we can then solve the Dirac equation to find

- The bound states of quarks
- Ionization energies of those bound states – i.e., energies at which we can break apart the bound states into their constituent quarks

In the last section we discovered why we can't apply the Dirac equation to a QED-like force. Namely, the QED-force is a manifestation of the lowest-order perturbation theory result that photons cause electron to scatter with an amplitude proportional to $1/p^2$. The typical binding energies are very small. But for QCD, small binding energies correspond to the region where either $g(E)$ is large or, if we chose a larger reference energy E_R our perturbative terms would involve factors of $\log(E/E_R)$ which are large. So perturbation theory isn't applicable and we can't use it to compute the binding force.

Thomson describes a kind of intuitive explanation that we might expect a linear potential rather than a Coulomb potential. The explanation very arm-waving but tends to be regarded as a 'decent' guess. Various approximation techniques were developed in the early days for putting these intuitions onto a firmer basis and ultimately, lattice methods confirmed the intuitions.

The interaction force, according to this hand-waving argument grows as the quarks are pulled apart, so they can't ionize. At best, if you apply enough energy, you'll create severa bound-states where you only had one to begin with (remember, in QFT you can end up with more fundamental particles than you started with).

The rule ends up being that the only observable particles are those with no net color – i.e., 'color singlets'. The group-theoretic way to say this is "N quarks can be observed in a bound state only if their tensor product is in the scalar (1-D) representation of SU(3)". Here are some examples of tensor-product decompositions. Only those which include a 1-D representation are candidates for observable bound states. Recall that quarks are in the 3 representation and antiquarks are in the $\overline{3}$ representation. Both are 3-dimensional but different.

$$\begin{aligned} 3 \otimes \bar{3} &= 1 \oplus 8 \\ 3 \otimes 3 &= 6 \oplus \bar{3} \\ 3 \otimes 3 \otimes 3 &= 8 \oplus 1 \\ 3 \otimes 3 \otimes \bar{3} \otimes \bar{3} &= 1 \oplus 1 \oplus 8 \oplus 8 \oplus 8 \oplus 8 \oplus 10 \oplus 10 \oplus 27 \end{aligned} \tag{29}$$

Bound observable particles can be made only from quark/antiquark combinations (on the LHS of the equations) for which a 1-dimensional (irreducible) representation appears on the RHS. So, for example in the first line, a linear combination can be found of quark-antiquark colors, so that the combination is invariant under color transformations (i.e., it's a scalar). Such a combination is known as a meson. In the second line, we see that there is no invariant color-combination of a quark-quark pair. So no bound states consist of two quarks. On the other hand, three quarks can combine into a color-invariant state which is known as a baryon. The final example here is a so-called 'tetraquark'. Several of these have been discovered in the past decade. As can be seen here, there are two distinct invariant color combinations of 2 quarks plus 2 antiquarks.

4.1.6 Ghosts

(Fadeev and Popov – 1967) This topic is new but also involves a kind of regularization. The issue has to do with the enormous path-redundancy of gauge theories.

- Simple introduction.
  - Formally write $R(f,g)$ as

$$R(f,g) = \frac{\int_0^\infty f(x)dx}{\int_0^\infty g(x)dx} \tag{30}$$

We could define this as a Λ-limit. Recall that if $f(x) = A$ and $g(x) = B$, where A and B are constant, then

$$\lim_{\Lambda\to\infty} \frac{\int_0^\Lambda Adx}{\int_0^\Lambda Bdx} = lim_{\Lambda\to\infty} \frac{A\Lambda}{B\Lambda} = \frac{A}{B}. \tag{31}$$

- Suppose, instead of constant functions, f and g are periodic: $f(x) = f(x + 2n\pi)$. Then f can be expressed as $f(x) = \tilde{f}(\cos(x))$. Similarly with g. The Λ-limit becomes something like

$$\lim_{\Lambda\to\infty} \frac{\int_0^\Lambda f dx}{\int_0^\Lambda g dx} \approx lim_{\Lambda\to\infty} \frac{\Lambda/(2\pi)\int_0^{2\pi} f(x)dx}{\Lambda/(2\pi)\int_0^{2\pi} g(x)dx} = \frac{\int_0^{2\pi} f(x)dx}{\int_0^{2\pi} g(x)dx}. \quad (32)$$

- Rather than defining $R(f, g)$ as a Λ limit, we could simply do the integrals from $0 \to 2\pi$.
- In our example, the problem has *redundancy*. It's 'natural', for periodic functions, to take ratios only by integrating over the period of the function. **In this case, we are integrating over path space and not over** x.

- Integration over a period:
  - For the example of periodic functions, we regulate the theory by limiting the integral to the period. Formally, we can write regulated integrals for periodic functions as

$$I_r(f) = \int_0^\infty dx f(x)\delta_{[\frac{x}{2\pi}]0} \quad (33)$$

  where the floor function $[\frac{x}{2\pi}]$ is the greatest integer less than $\frac{x}{2\pi}$.
  - Notice the trick of integrating over all values but using a delta function to restrict the integral to a single period.

- Analogy between gauge (local) symmetry and periodicity in path integration:
  - Note that the analysis of ghosts is much simpler using path integrals than using canonical quantization methods.
  - The path integral over vector (gluon) fields is written as

$$Z = \int [\mathcal{D}A_a^\mu] e^{i\int d^4x L(A_a^\mu)} \quad (34)$$

  This is an integral over paths.
  - Gauge invariance can be expressed as

$$\int d^4x L\left(\mathcal{G}(A_a^\mu)\right) = \int d^4x L(A_a^\mu) \quad (35)$$

  where $\mathcal{G}(A_a^\mu)$ is a gauge transformation of A_a^μ. That's analogous to a translation by $2n\pi$.

- Path space can be formally expressed as a set-product of a set ('slice') of gauge-inequivalent paths, with the full set of gauge transformations, $\mathcal{I} \times \mathcal{G}$. Here $\mathcal{G}$ is analogous to the n that appears in the periodicity $2n\pi$ and $\mathcal{I}$ is analogous to the interval $[0, 2\pi)$.
- So, in imitation of eq. (33) for defining a regulated integral I_r of periodic functions, we define

$$Z_r = \int [\mathcal{D}A_a^\mu] e^{i \int d^4x (A_a^\mu)} \delta(\mathcal{G}^{fix}(A_a^\mu)) \tag{36}$$

where $\mathcal{G}^{fix}(A_a^\mu) = 0$ is the gauge-fixing equation analogous to $[\frac{x}{2\pi}] = 0$ in eq. (33), which selects a slice of path space where all paths are gauge-inequivalent.

- From delta function to ghosts:

We see that our path integrals have been regulated by inserting a delta-function term. Can we rewrite this to put things in the original form? Namely,

$$Z_r =? \int [\mathcal{D}A_a^\mu][\mathcal{D}B] e^{i \int d^4x (A_a^\mu) + \int d^4x L_g(A_a^\mu, B)} \tag{37}$$

where B are some new fields that are introduced and L_g is an extra term in the Lagrangian.

The answer is 'yes'. Here's a hint. A delta-function can be converted to an exponential integral.

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk. \tag{38}$$

We've converted the delta function into an integral over an exponential function.

Things are a bit hairier when the delta function appears in the form $\delta(h(x))$. Then you need to change variables to $y = h(x)$ and you end up with an exponential integral times an inverse Jacobian determinant (for the change of variables). This too, can be converted into an exponential form but that's more subtle.

The end result is that we have a path integral with some new fields and a modified Lagrangian. The new fields are known as **ghost fields**.

Just like counterterms, ghost fields are used as a trick. Fadeev and Popov invented this technique in order to simplify calculations. The gauge theory is Lorentz invariant, so at all sorts of steps in the calculations, expressions can be simplified using Lorentz invariance. However, in order to properly regulate the calculations, the gauge must be fixed. For example, in QED we sometimes pick a gauge where $A^0 = 0$. In this gauge the Lorentz invariance is no longer apparent (we say the theory is *not manifestly Lorentz covariant*) and calculations can't be simplified using the methods of Lorentz invariance. Fadeev and Popov were able to use their method to choose a Lorentz-invariant gauge-fixing expression, but at the expense of introducing ghost fields (which aren't needed in QED).