Thomson Chapter 17 – The Higgs Boson, a first look at Weak Interactions

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The usual treatment of weak interactions, is to start with historical experiments attributed to new physics – neither electromagnetic or strong interactions.

I want to do this differently, building on the theoretical foundations developed for QCD. Many ingredients are the same. But a new ingredient is **the Higgs boson**.

For now, skip Thomson chapters 11 - 16 which cover the experimental implications of the (electro-) weak interaction theory, as well as the basic modern theory MINUS Higgs.

You can read, as review, Thomson sections 17.1 - 17.4. We will eventually pick this up towards the end of section 17.4. However, for now, I want to jump right into the Higgs mechanism.

1 The role of mass terms in perturbation theory

- Remember that QFT can be expressed using path integrals of the form

$$Z(J) = \int [\mathcal{D}\phi] e^{i \int d^4x \mathcal{L}(x) + J(x)\phi(x)} \tag{1}$$

where

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^*)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^* \phi + \text{interaction terms} \tag{2}$$

- The interaction terms are the ones that aren't quadratic in ϕ. The quadratic terms involve derivatives as well as the mass terms. We can solve the theory when the Lagrangian is quadratic (terms with and without derivatives) **For the rest of this discussion, I'll lump the two types of terms together and call them both 'mass terms'**.

- The path integral is essentially a multidimensional version of

$$\mathcal{I}(a) = \int_{-\infty}^{\infty} dx e^{-ax^2} = \frac{\sqrt{\frac{\pi}{a}}}{2} \tag{3}$$

.

- Furthermore, we can analytically do integrals of the form

$$\mathcal{I}(a, x^n) = \int_{-\infty}^{\infty} dx x^n e^{-ax^2} \tag{4}$$

- That means we can do the integral

$$\mathcal{I}(a, \mathcal{F}) = \int_{-\infty}^{\infty} dx \mathcal{F}(x) e^{-ax^2} \tag{5}$$

  for any function $\mathcal{F}$ that can be expanded in a Taylor series – provided that the resulting series converges.

- For example, if $\mathcal{F}(x) = b_0 + b_1 x + b_2 x^2 + ...$ then

$$\mathcal{I}(a, \mathcal{F}) = b_0 \int_{-\infty}^{\infty} dx e^{-ax^2} + b_1 \int_{-\infty}^{\infty} dx x e^{-ax^2} + b_2 \int_{-\infty}^{\infty} dx x^2 e^{-ax^2} + ... \tag{6}$$

  Each integral is a function of a.

- Generalize to the path integral: the quadratic term includes derivatives as well as the mass term and the role of $\frac{1}{\sqrt{a}}$ becomes the Feynman propagator dependent on mass.

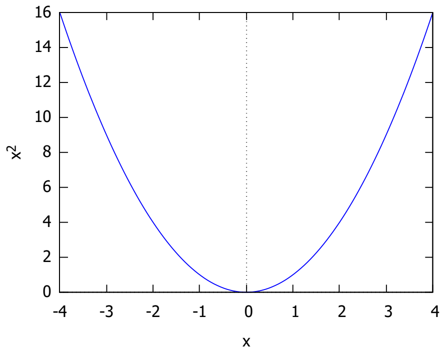
- **Furthermore generalize $\mathcal{F}$ to $exp(\int d^4x \mathcal{L}_I(\phi))$ which is then expanded as a power series in ϕ. $\mathcal{L}_I$ is the non-quadratic part of the Lagrangian. The results are, order-by-order, described by Feynman diagrams.**

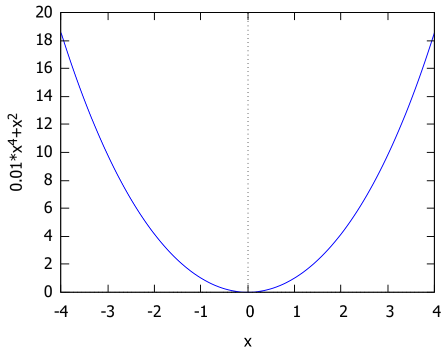
2 When do we expect perturbation theory to work?

- Start with a one-dimensional example.

$$\mathcal{J}(a,\lambda) = \int_{-\infty}^{\infty} dx e^{-\tilde{V}(x)} = \int_{-\infty}^{\infty} dx e^{-ax^2-\lambda x^4} \tag{7}$$

As an example, take $a = 1$ and plot $\tilde{V}(x)$ for $\lambda = 0$ (on the left) and $\lambda = 0.01$.





We see that the two plots look similar, so the quartic term should be a small perturbation.

- Following the approach outlined above, rewrite

$$\int_{-\infty}^{\infty} dx e^{-\tilde{V}(x)} = \int_{-\infty}^{\infty} dx e^{-ax^2} e^{-\lambda x^4} = \mathcal{I}(a, e^{-\lambda x^4}) \tag{8}$$

and then expand $e^{-\lambda x^4}$ as a power series in x.

- $e^{-\lambda x^4} = 1 - \lambda x^4 + \lambda^2 \frac{x^8}{2} - \lambda^3 \frac{x^{12}}{6} + ...$ Note that this is also an expansion in powers of λ.

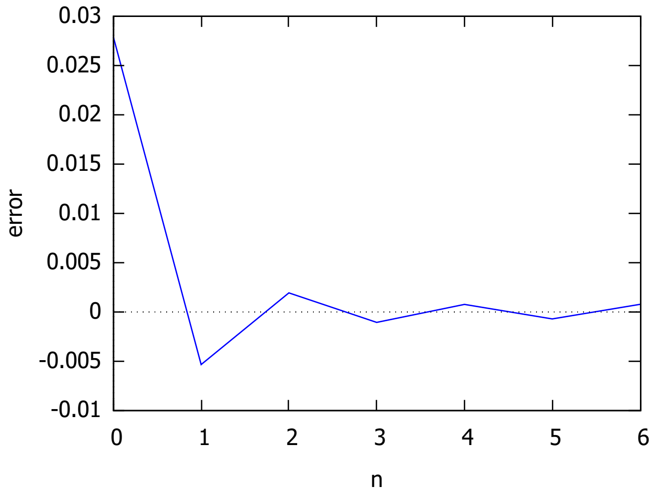
- Now let's integrate:

$$\int_{-\infty}^{\infty} dx e^{-ax^2} \left(1 - \lambda x^4 + \lambda^2 \frac{x^8}{2} - \lambda^3 \frac{x^{12}}{6} + ...\right) = \frac{\sqrt{\pi}}{256 a^{13/2}} \left(128a^6 - 96a^4\lambda + 420a^2\lambda^2 - 3465\lambda^3 + ...\right) \tag{9}$$

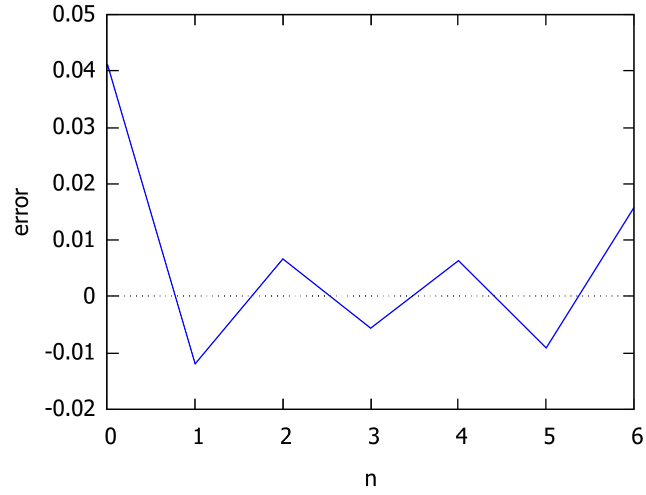
IMPORTANT OBSERVATION: Terms oscillate in sign.

- Next, let's evaluate the quality of the approximation as we expand order by order in n. In the three examples below, set $a = 1$.

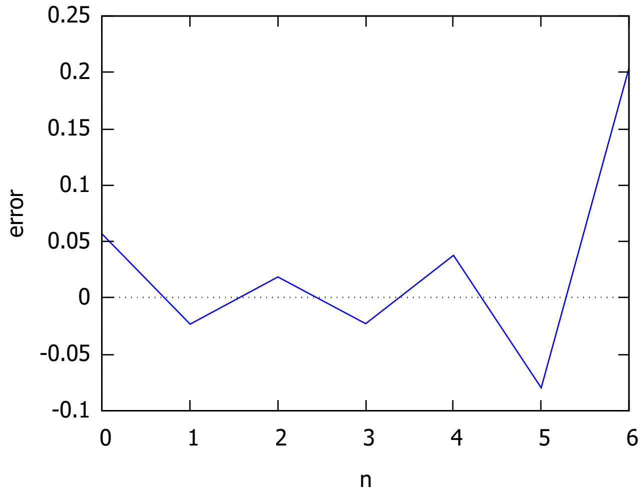
– Set $\lambda = 0.05$.



– Set $\lambda = 0.08$.

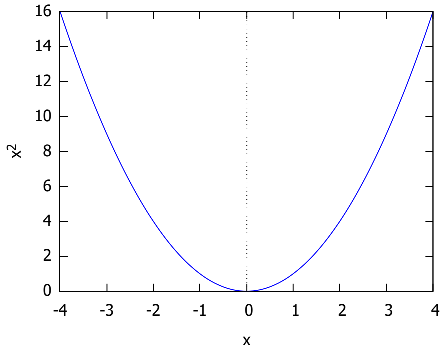


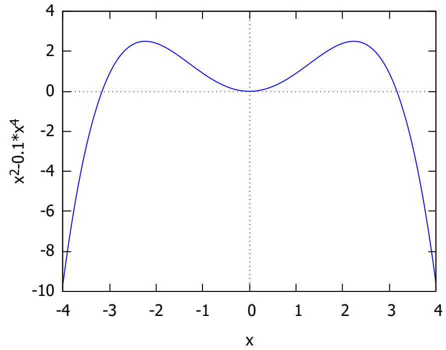
– Set $\lambda = 0.12$.



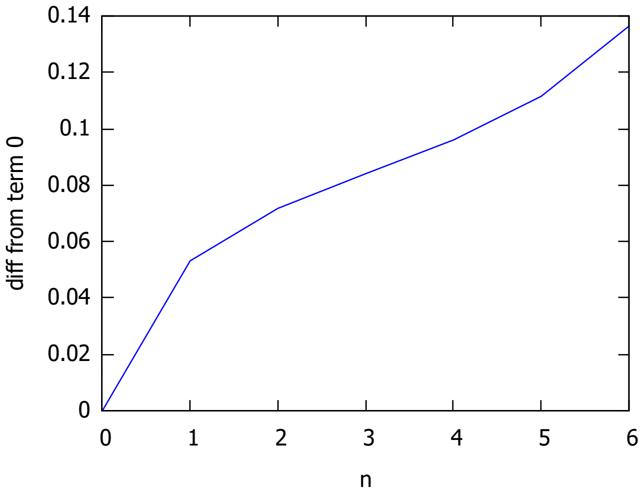
– The behavior is characteristic of an asymptotic series rather than a convergent perturbation series. In each case, the error shown is the difference between the exact result (computed numerically) and the perturbative result through order λ^n. In each case, the first term or two in the expansion gets closer to the right answer, but for smaller values of λ, the series begins to diverge at higher order of λ^n. This tells us that for very small values of λ, we can compute effectively by going out to many orders in the series.

- Now what happens if λ is negative? Recall that we were looking at $\int_{-\infty}^{\infty} dx e^{-\tilde{V}(x)} = \int_{-\infty}^{\infty} dx e^{-ax^2-\lambda x^4}$. As before, take $a = 1$ but this time plot $\tilde{V}(x)$ for $\lambda = 0$ (on the left) and $\lambda = -0.01$.





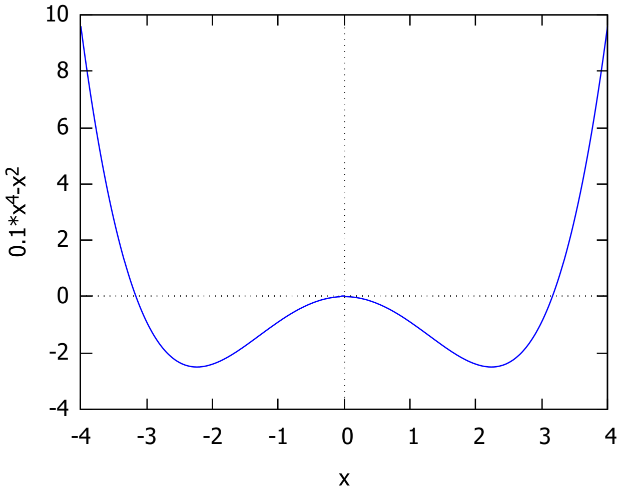
- The two graphs are qualitatively different. **In fact, if λ is negative, the exponential becomes exponentially big for large x and the integral diverges!** So, for sure, the perturbation series can't converge.



This clearly isn't asymptotic since no finite set of terms is close to ∞.

3 The curious case of the double well

- So far, we've let $a = 1$. More generally, we've let a be positive. But on general grounds, nothing prevents us from considering a general potential $\tilde{V}(x) = -ax^2 + \lambda x^4$ where $a > 0$, $\lambda > 0$. The graph is:



- **This is known as a double-well potential.** For this case, the exponential is very small at large x so the integral $\int_{-\infty}^{\infty} dx e^{-\tilde{V}(x)} =$

$\int_{-\infty}^{\infty} dx e^{-\left(-ax^2+\lambda x^4\right)}$ converges. But it appears to have a negative squared-mass term. So what does this mean?

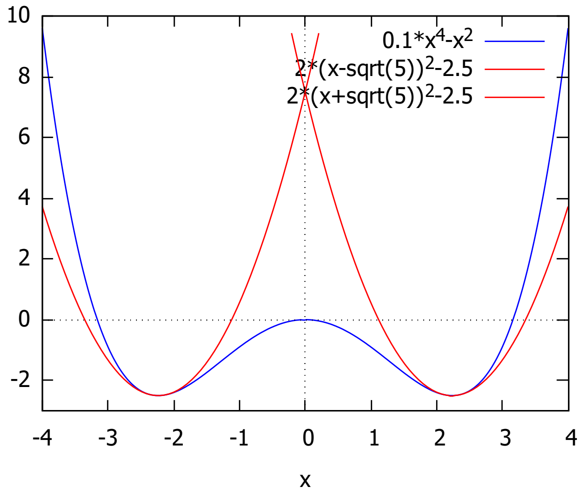
- The trick is to realize that the integral gets most of its contribution from the minimum values of $\tilde{V}$.
  - For $a = 1$ and $\lambda = .1$, we get the numerical value $\int_{-\infty}^{\infty} dx e^{-\tilde{V}(x)} =$ **33.92**.
  - For $a = 1$ and $\lambda = .1$, the minimum values are at $x = -\sqrt{5}$ and $x = \sqrt{5}$.
  - Now expand $\tilde{V}$ around the minimum value $x_m = \sqrt{5}$. We get

$$\tilde{V}(x) = -2.5 + 2(x - \sqrt{5})^2 + ... \tag{10}$$

  - We can shift variables and evaluate the integral of just the quadratic part of $\tilde{V}(x)$ expanded around the minimum. Note that naively we have to multiply by 2 because there is an equal contribution from each of the 2 minima (however, see comment below).

$$\int_{-\infty}^{\infty} dx' e^{-\tilde{V}(x')} \approx 2 \int_{-\infty}^{\infty} dx e^{-\left(-2.5+2x'^2\right)} = \mathbf{30.54}. \tag{11}$$

- This is a pretty good approximation considering it only expands out to quadratic order.
- But here is the magic: **We've end up evaluating a new potential with a positive quadratic term of** $2x^2$**!**
- You might object that we had to multiply by 2. But that turns out to be *sort of* OK because we always take ratios of path integrals, and the multiplier will cancel.
- It's not completely OK because, as you might notice, all integrals go from $-\infty$ to $+\infty$, so there is overlap between the quadratic expansion around the first minimum and the quadratic expansion around the second minimum.



The red curves, when they deviate from the blue curve, are exponentially damped, so in the mismatch region, they are damped.

- **In practice, we don't attempt to double the contributions since we don't have a usable method for computing the suppressed terms. Instead, we assume that it suffices to expand around either minimum, and then hope that perturbation theory will restore the contribution of the other minimum. Actually, we don't really hope that, since we know (or assume) the series is asymptotic and not convergent. However, an asymptotic series, for appropriate expansion parameters, can be quite accurate through several orders of expansion and thus can be expected to account somewhat for the existence of the other minimum.**

- Notice that whereas the original theory is symmetric under $x \to -x$, the modified theory after shifting variables around a minimum, now favors that minimum. **In other words, the symmetry appears broken**. We'll see this in detail later.

4 The Mexican Hat Potential

See Thomson Chapter 17.5.2.

So far we've considered a single scalar field. Let's generalize to two real scalar fields ϕ_1 and ϕ_2, organized as a single complex field $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. Consider a Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - \hat{V}(\phi^*, \phi) \tag{12}$$

where

$$\hat{V} = \frac{1}{2}\mu^2(\phi^*\phi) + \lambda(\phi^*\phi)^2. \tag{13}$$

- We assume $\lambda > 0$ so that the Hamiltonian is bounded from below. Just as for the double-well potential, the shape of $\hat{V}$ depends on whether or not $\mu^2 > 0$. The two cases are illustrated in Fig. (17.7) of Thomson.

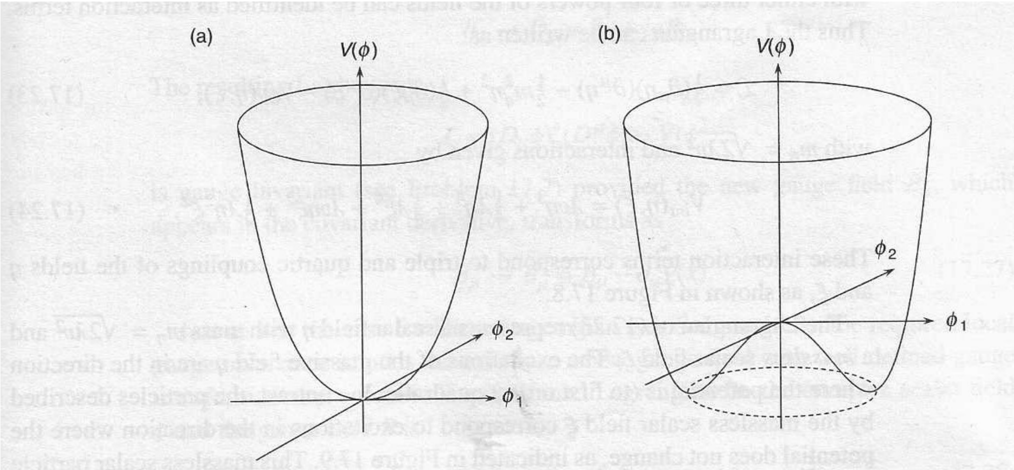


Fig. 17.7 The $V(\phi) = \mu^2(\phi^*\phi) + \lambda(\phi^*\phi)^2$ potential for a complex scalar field for (a) $\mu^2 > 0$ and (b) $\mu^2 < 0$.

- The picture on the right is known as *the Mexican Hat potential*, based on the way the lower part of the diagram resembles a sombrero.
- Both potentials are invariant under the global U(1) transformation

$$\phi \to e^{i\alpha}\phi \tag{14}$$

which resembles the global charge symmetry.

- In the diagram, this symmetry transformation represents a rotation by α around the z-axis.
- In particular, the minimum of the Mexican hat potential, is a circle. i.e., there are a continuous infinity of minima.

- Just as for the double-well potential, we pick one minimum and expand the potential around that minimum. See Thomson for arithmetic details.
  - Let $v = \sqrt{-\mu^2}/\lambda$ (remember that $\mu^2 < 0$). Then one of the minimal points occurs for $(\phi_1, \phi_2) = (v, 0)$. That point is on the positive real axis. We'll expand around that point.
  - By convention, we define new fields $\eta(x) = \phi_1(x) - v$ and $\xi(x) = \phi_2(x)$. Then
  $$\phi = \frac{1}{\sqrt{2}}(\eta + v + i\xi). \tag{15}$$
  - Then rewrite Eq. (12) in terms of these new fields as
  $$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \frac{1}{2}m_\eta^2 \eta^2 + \frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi) - V_{int}(\eta, \xi) \tag{16}$$
  where $m_\eta = \sqrt{2\lambda v^2}$ and
  $$V_{int}(\eta, \xi) = \lambda v \eta^3 + \frac{1}{4}\lambda \eta^4 + \frac{1}{4}\lambda \xi^4 + \lambda v \eta \xi^2 + \frac{1}{2}\lambda \eta^2 \xi^2. \tag{17}$$
  - We see that all the terms of V_{int} are either cubic or quartic in the fields, so this is a traditional interaction potential (without quadratic terms).
  - **IMPORTANT:** We also see that if we chose the minimum at $(\phi_1, \phi_2) = (-v, 0)$, that the form of V_{int} would change. The two terms proportional to v would be sign-reversed. So for example, $\lambda v \eta^3 \to -\lambda v \eta^3$. **It appears that the U(1)-symmetry has been broken.** Since we started with a symmetric theory, we know that the physics ultimately doesn't care about the sign of the terms proportional to v, but in this formulation, that isn't obvious.
  - We also see that the remaining terms of the Lagrangian are quadratic. The η field has a bona fide mass term but the ξ field does not.
  - In summary, by picking a specific minimum around which we expand the Lagrangian, we end up with one massive scalar field (η) and one massless scalar field (ξ). Of course, by picking a minimum we appear to have broken the symmetry.
  - The massless scalar is known as the **Goldstone boson**.

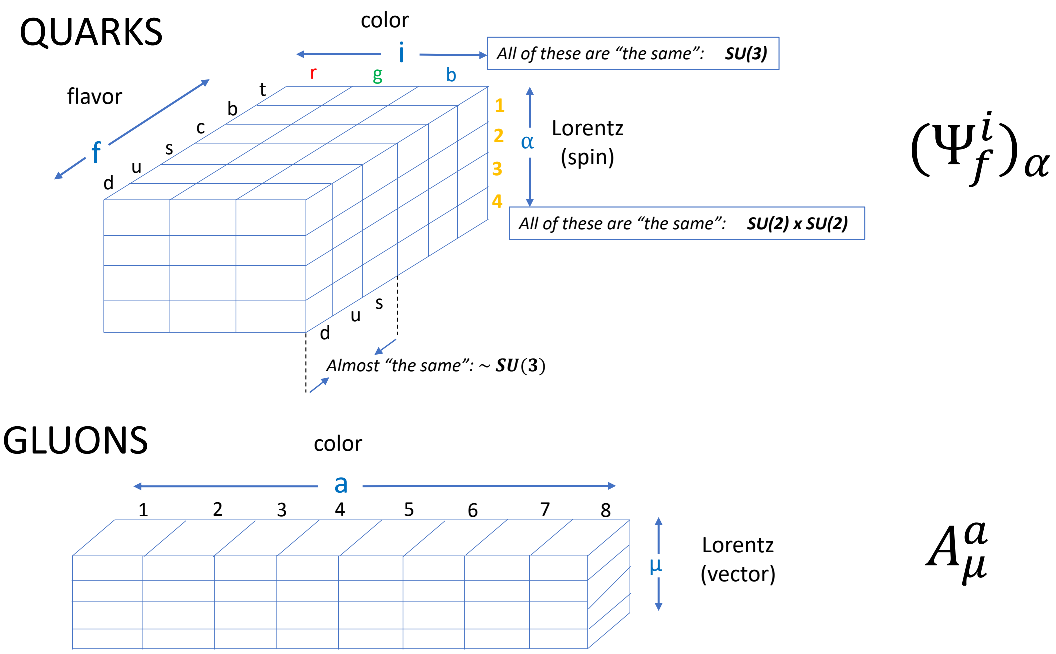
- Shortly we will discover that the Goldstone boson can be eliminated by using a gauge transformation. That leaves a single massive scalar boson known as the **Higgs boson**. First we return to the topic of gauge symmetry.

5 Building a Lagrangian to satisfy local (gauge) SU(2) x U(1) invariance

The essential difference – other than Higgs – between strong and (electro-) weak interactions, is that the symmetry group for strong interactions is SU(3) and the symmetry group for electro-weak interactions is SU(2) x U(1).

NOTE: the full theory of weak interactions turns out to be entwined with the electromagnetic interactions.

Later on, when we return to earlier Thomson chapters, we'll see how the experimental weak and electromagnetic interactions are symmetric under the group SU(2) x U(1).



For now, think of SU(2) x U(1) to simply be a different group than SU(3) and whose differences can be encapsulated in the group generators (there are 4 generators instead of 8). We want to construct a Lagrangian that is locally invariant under SU(2) x U(1). The form of the Lagrangian is identical to the QCD Lagrangian below except that there are different values of the structure constants f_{abc}, the generators T^a_{ki} and the bounds on the various group indices.

The first step in performing perturbative calculations in a non-Abelian gauge theory is to work out the Feynman rules. The SU(N)-invariant Lagrangian for a set of N fermions and N scalars interacting with non-Abelian gauge fields is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 + (\partial_\mu \bar{c}^a)(\delta^{ac}\partial_\mu + g f^{abc} A^b_\mu) c^c \\ &+ \bar{\psi}_i(\delta_{ij} i\partial\!\!\!/ + g A\!\!\!/^a T^a_{ij} - m\delta_{ij})\psi_j \\ &+ \left[(\delta_{ki}\partial_\mu - igA^a_\mu T^a_{ki})\phi_i\right]^\star \left[(\delta_{kj}\partial_\mu - igA^a_\mu T^a_{kj})\phi_j\right] - M^2 \phi_i^\star \phi_i, \end{aligned} \tag{26.1}$$

where c^a and $\bar{c}^a$ are the Faddeev–Popov ghosts and anti-ghosts respectively and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{26.2}$$

We have included scalars in this Lagrangian for generality, even though we have observed no scalar states in nature that are colored (charged under QCD). Many theories, such as supersymmetric QCD, do have colored scalars. The Higgs doublet in the Standard Model is an example of a scalar field charged under the weak gauge group SU(2).

The kinetic terms from the QCD Lagrangian are

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 + \bar{\psi}_i(i\partial\!\!\!/ - m)\psi_i - \phi_i^\star(\Box + M^2)\phi_i - \bar{c}^a \Box c^a. \tag{26.3}$$

This time, we are going to care – A LOT – about the scalar field ϕ. We are also going to tweak the scalar term in the Lagrangian by changing the sign of M^2 and by adding a quartic term. A few other things:

- The scalar field will be called THE HIGGS FIELD
- The fermions will include quarks and also will include leptons such as neutrinos, electrons, muons, etc.
- There are 4 vector fields (a ranges from 1 to 4) and they end up being linear combinations of the fields known as Z_μ, W_μ^+, W_μ^- and A_μ. YES, that last one is the EM field!

6 What is mass?

- In classical mechanics, the kinetic energy is $\frac{mv^2}{2}$ and the potential energy is denoted V.
- In relativistic mechanics, the kinetic energy is $\sqrt{m^2+v^2}$ where we set $c=1$.
- In the Lagrangian formulation of a scalar field theory, we have

  $$L = \int d^4x \mathcal{L}(x) \tag{18}$$

  where

  $$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^*)(\partial^\mu \phi) - \frac{1}{2}m^2 \phi^* \phi \tag{19}$$

- In the Feynman diagrams corresponding to such a Lagrangian, connected lines with momentum k correspond to a term called the Feynman propagator,

  $$\frac{i}{k^2 - m^2 + i\epsilon} \tag{20}$$

In all the examples above, m is known as the mass. In the various limits of experimental interest, m has the property that we call 'mass'. Furthermore in relativistic physics, mass characterizes irreducible representations of the Poincare group.

7 Why photons and other gauge bosons should be massless

This follows Thomson 17.4.

- Recall that the Lagrangian must be invariant under the symmetry transformation (gauge transformation)

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi. \tag{21}$$

- The quadratic derivative term $F_{\mu\nu}F^{\mu\nu}$ can be shown to have this invariance.

- However, suppose you had a mass term. It would have to look like $\mu A_\nu A^\nu$ since this is the only quadratic term without derivatives and which is Lorentz invariant. But

$$A'_\nu A'^\nu = (A_\nu - \partial_\nu \chi)(A^\nu - \partial^\nu \chi) \neq A_\nu A^\nu \tag{22}$$

  which is not invariant.

- A similar argument appears to exclude masses for any bosons that appear as a result of imposing a local symmetry (e.g. gluons).

- **There is a loophole – the Higgs mechanism.**

8 The Higgs mechanism

See Thomson chapter 17.5.3 for details.

- Thomson begins the analysis by extending the scalar global U(1) symmetry to a local U(1) symmetry. This leads to a gauge Lagrangian similar to what we showed before, but considerably simpler. I'll skip most of the details although they are very similar to what we did for QED.

- We generalize the scalar-field global transformation to

$$\phi(x) \to e^{ig\alpha(x)}\phi(x), \tag{23}$$

  which is a local transformation.

- We introduce a new vector field B_μ which serves as the gauge field and transforms as

$$B_\mu \to B_\mu - \partial_\mu \alpha(x). \tag{24}$$

- Then we can show that the following Lagrangian is invariant under the above local transformations.

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (\partial_\mu\phi)^*(\partial^\mu\phi) + \mu^2\phi^*\phi - \lambda(\phi * \phi)^2 \\ &- igB_\mu\phi^*(\partial^\mu\phi) + igB_\mu\phi(\partial^\mu\phi^*) + g^2B_\mu B^\mu\phi^*\phi. \end{aligned} \tag{25}$$

  where $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. **I changed things slightly from Thomson and made μ^2 positive and changed the sign of that term to be negative. I found it too confusing to make μ^2 negative.**

- As anticipated in the previous section, there are no mass terms (quadratic without derivatives) for B_μ. On the other hand, the last term in the Lagrangian is quadratic in B and also quadratic in ϕ. We regard this as a quartic term. But remember that we're going to shift ϕ so that we expand around a minimum. This will result in a term quadratic in B but without the ϕ fields. Hence a mass will emerge for B.

- Carry out the field shift by writing $\phi = \frac{1}{\sqrt{2}}(\eta + v + i\xi)$ as we did for the Mexican hat potential. Substitute in the above expression for the Lagrangian, leading to Thomson eq. (17.31)

$$\mathcal{L} = \underbrace{\tfrac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \lambda v^2\eta^2}_{\text{massive } \eta} + \underbrace{\tfrac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi)}_{\text{massless } \xi} - \underbrace{\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}g^2v^2B_\mu B^\mu}_{\text{massive gauge field}} - V_{int} + gvB_\mu(\partial^\mu\xi),$$

  As before, V_{int} consists of terms that are products of 3 or 4 fields. We have terms that were explained for the Mexican hat potential – namely one massive and one massless scalar. But most notably we now have a vector meson that is massive. This is different than either the gluon or photon, both of which are massless because of the gauge symmetries of the theory. However, by selecting a non-zero minimum around which to expand the potential, we appear to have broken the symmetry and therefore are allowed to have a non-zero mass for the gauge field.

- There is one more trick up our sleeves. Remember that our original Lagrangian was invariant under the local (gauge) transformations eqs. (23) and (24). This was true for any real-valued function $\alpha(x)$. So, let

$$\alpha(x) = -\frac{1}{gv}\xi(x). \tag{26}$$

In other words, the local gauge phase α is set to be proportional to the field ξ. When expanded around the minimum $(\phi_1, \phi_2) = (v, 0)$, the form of the Lagrangian appears to change under the gauge transformation, because we've broken the symmetry. In particular, the transformed Lagrangian no longer involves the field ξ.

- We say that the Goldstone boson has been "eaten" by the gauge field. Also, by convention, we rename the field η to be h, and call that the Higgs field. See eq. (17.34) in Thomson for details.
- The net result of all this, is that we end up with one real scalar field h with a mass $m_H = \sqrt{2\lambda}v$ and one vector field B_μ with a mass $m_B = gv$. All of this is known as the **Higgs mechanism**.
- We've now set up the formalism, but we're not done. The group of interest to us in studying weak and electromagnetic interactions isn't U(1). Rather, it is SU(2) x U(1). That's more complicated but we can generalize the Higgs mechanism to that group.