Energy momentum tensor

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Reference: Sean Carroll Spacetime and Geometry 1st edition pages 164,165 (in section 4.3)

Einstein's equation refers to the 'energy momentum tensor' $T_{\mu\nu}$. Usually, we start by giving information about that tensor, and then deriving the metric tensor – which gives us the geometry.

So what is that tensor and how do we know or measure it?

1 Historical comments

1.1 Similarity of Newton's laws and Einstein's equation

Newton's 2nd law:

$$m\vec{a} = \vec{F}. \tag{1}$$

Einstein's equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2}$$

Newton told us "if you know the force on an object, then you can calculate its acceleration". Einstein tells us "if you know the energy-momentum tensor somewhere, then you can calculate how spacetime is shaped (and therefore motions of objects in spacetime)".

The equations are analogous but there are some conceptual differences that needn't concern us now.

In Newton's day we could have spoken about 'designer forces'. For example, suppose I want my wagon to experience a particular motion (speed, acceleration, distance etc. along a path). By plugging into the LHS, I can figure out the force. Then, with vast experience of how forces come into being, I could engineer an engine that caused the desired motion.

The same sort of thing could be done with Einstein's equation. I look for some desirable properties of spacetime and then that gives me the energy-momentum tensor. The question is whether we can engineer such a tensor?

1.2 Example: perfect fluid

In special relativity, momentum and energy together form a 4-vector. Historically, the EM tensor was defined as the 4-matrix which gives the flow of energy and momentum in 4 perpendicular directions. That kind of thing shows up a lot in fluid dynamics. The very simplest fluid is "dust". By this we mean "a bunch of non-interacting particles whose mass density is ρ. For example, if there are N particles in a volume V, and each particle has mass m, then $\rho = mN/V$. Here is the rest-frame EM tensor for "dust"

$$T_{\mu\nu}^{\text{dust}} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{3}$$

In other frames of reference, the tensor is transformed by Lorentz transformatons.

When the particles interact, the system dynamics can be computed "on average" if we characterize the system by both density and pressure. Here is the rest-frame EM tensor for a "perfect fluid".

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} . \tag{4}$$

ρ is the rest-frame energy density and p is the rest-frame pressure in each of the three directions. In a moving reference frame, all of these quantities are transformed by appropriate Lorentz transformations and some of the 0 components become non-zero.

A mathematically compelling way of deriving that tensor, was from the Lagrangian formulation of mechanics. Noether showed that this tensor (as well as energy and momentum separately) showed up naturally as a consequence of the spacial and temporal invariance of the equations of motion.

For example, for the perfect fluid, the equations of motion can be used to prove the 4 conservation equations

$$\partial^\mu T_{\mu 0} = 0;\ \partial^\mu T_{\mu 1} = 0;\ \partial^\mu T_{\mu 2} = 0;\ \partial^\mu T_{\mu 3} = 0 \tag{5}$$

In somewhat more familiar form, these can be re-cast as the Euler equation of fluid mechanics:

$$\rho\,[\partial_t \vec{v} + (\vec{v}\cdot\nabla)\vec{v}] = -\nabla p \tag{6}$$

where $\vec{v}$ is the velocity of the system relative to the rest frame.

2 According to Carroll ...

2.1 Pure gravity and matter

Inspired by history, let's look at a Lagrangian formulation of physics including gravitation. In the Lagrangian formulation of gravity, we represent the gravitational physics by a gravitational field $g_{\mu\nu}(t,\vec{x})$. **We can think of this as a field in the same way that the electric field $\vec{E}_\mu(t,\vec{x})$ is a field.** . This field can be identified with metric tensor and we can therefore interpret everything geometrically.

Write the action as

$$S = \frac{1}{16\pi G} S_H + S_M \tag{7}$$

where the first term represents *pure gravity* and the second term represents *matter*. What we mean by *matter* is "all the stuff in the universe which we thought was there except for gravity". For example, electromagnetic fields, quarks, fluids (i.e, aggregation of more basic forms of matter) etc. Then *pure gravity* is all the rest.

Digression on the distinction between gravity and the other stuff

By analogy, think of the electromagnetic field as *pure electromagnetism* and treat it separately from everything else (for example, electrons and other charged matter but **ALSO** uncharged stuff like neutrons).

In the full gravitational action, we regard S_M as 'the source of gravity'. Unlike electromagnetism, where only charged objects are sources of electricity, in a gravitational theory, **all** stuff (except 'pure gravity') is a 'source of gravity'.

One natural question is then "why isn't gravity, like other stuff, a source of pure gravity?" Loosely thinking, you regard *mass* as a source of gravity, and you equate energy to mass, so surely the gravitational energy ought to be a source of gravity. After all, an electromagnetic field is a source of gravity, so why shouldn't the gravitational field be such a source?

It turns out that this question is largely one of semantics. The LHS of Einstein's equation is 'nonlinear' in the gravitational field. In derivations of Einstein's equations, it readily becomes apparent that 'gravity acts on itself'. However, both gravity **and** its source become tangled with one another in such a way that the net result is captured in the LHS of Einstein's equation. So in fact, gravity **is** a source for gravity. But it isn't possible (or at least not obvious how) to isolate the gravitational source so that it can be sensibly included as a term on the RHS of the Einstein equation.

Einstein's great insight, was that the laws of physics should look the same regardless of the coordinate system. **This statement is a sleight of hand!** What it really means is this: find another way of expressing the laws of physics so that instead of using familiar quantities like ∂_μ, you use new quantities like D_μ – the covariant derivative. These new quantities are picked so that they depend on coordinates in such a way that under coordinate transformations, the old equations look like the new ones.

So far, there isn't any content here. What matters, is that these new quantities have to be constrained by geometrical considerations, and that one can always find a set of coordinates where those new quantities look just like the old ones (that's the principle of equivalence) . This is all highly nontrivial and not even well-defined unless you insist on some simplicity (or other) assumptions.

Be that as it may, here's an example: Consider a pre-gravitation physical theory of scalar fields.

$$S_\phi = \int d^4x \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] \tag{8}$$

If you change coordinates, that expression changes form. However, the following expression maintains its form when changing coordinates.

$$S_M = \int \sqrt{-g} d^4x \left[-\frac{1}{2} D_\mu \phi D^\mu \phi - V(\phi) \right] \tag{9}$$

where, by definition, $D_\mu \phi D^\mu \phi \equiv g^{\mu\nu} D_\mu \phi D_\mu \phi$ and g (appearing in the square-root) is defined as the determinant of the metric tensor (the determinant is negative) and D_μ is a covariant derivative defined using the Christoffel (connection) coefficients.

Moreover, when the metric is flat, it turns out that $S_M = S_\phi$.

2.2 Variation of the action

Remember that the equations of motion are obtained by varying the fields or paths in the action and finding the extrema. So, in particular, if we vary the action by changing the metric (aka gravitational field), we'll end up with a set of equations of motion. If we look at the example of S_M, we see that the metric tensor appears in the Lagrangian, so we have to see what happens when we change $g_{\mu\nu}$. Of course, there is also a change in the S_H part of the action, but ignore that for a moment.

We use the notation $\frac{\delta}{\delta g^{\mu\nu}}$ to indicate the variation with respect to the field. This terminology can be made mathematically precise.

Then we **define**

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \tag{10}$$

That's the energy-momentum tensor. When we also vary S_H we end up with Einstein's equation.

2.3 Why is this called the energy-momentum tensor?

I think this is where Carroll shines. He acknowledges that this definition of the EM tensor is non-traditional. He explains how it connects to the traditional definition. First, it has the right dimensions (mass) for a gravity-source. Second, it's symmetric (like traditional definitions). Third, it's a tensor which transforms under Minkowski transformations, just the same way as the traditional EM tensor. Fourth, and probably most important, in Minkowski space (flat space without gravity), it is equal the traditional EM tensor.

What is the traditional EM tensor? It shows up in various equations of motion, but formally it is obtained through a conservation law. Noether showed a general theorem for Lagrangian systems. When the Lagrangian (more generally, the action) is invariant under a symmetry transformation (transformations of fields or paths), then there is an associated quantity known as a 'conserved current', from which one can derive a conservation law. For translations in space and time, the 4 conserved currents are the 4 rows of the EM tensor, and the conserved quantities – energy and momentum – are integrals associated with each row.