Overview of General Relativity

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"But even aside from the potential impact of general relativity on astronomy and other branches of physics, the theory in its own right makes many remarkable statements concerning the structure of space and time and the nature of the gravitational field. After one has learned the theory, one cannot help feeling that one has gained some deep insights into how nature works." - Robert M. Wald

1 The Formalism of General Relativity

The general theory of relativity assumes the following:

- Spacetime is a 4-dimensional continuum (manifold).
- The geometry of spacetime is determined by Einstein's field equation.
- In the absence of (non-gravitational) forces, the trajectories of test particles are the geodesics of the geometry.

Einstein's equation and the geodesic equation are disucssed in Sections 1.1 and 1.2, respectively.

1.1 Einstein's Field Equation

In SI units, Einstein's field equation is written as follows:

$$G_{\mu\nu} = \frac{8\pi \mathsf{G}}{c^4} T_{\mu\nu} \tag{1}$$

where:

- G is Newton's gravitational constant.
- c is the speed of light.
- $G_{\mu\nu}$ is the Einstein tensor, defined by

  $$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{2}$$

  where

- The fields $g_{\mu\nu}$, $R_{\mu\nu}$, and R are elements of Riemannian geometry.
- The field $T_{\mu\nu}$ represents physical quantities, i.e. matter and energy.

- The names associated with these fields are as follows:
  - $g_{\mu\nu}$ is called the *metric tensor* (a.k.a the *fundamental tensor*)
  - $R_{\mu\nu}$ is called the *Ricci tensor* (a.k.a. the *Ricci curvature*)
  - R is called the *Ricci scalar* (a.k.a. the *scalar curvature*)
  - $T_{\mu\nu}$ is called the *energy-momentum tensor* (a.k.a the *stress-energy tensor*)

Formally, the fields $g_{\mu\nu}$, $R_{\mu\nu}$, and $T_{\mu\nu}$ are tensor fields of type (0,2), as indicated in index notation by the presence of the two subscripts. By contrast, an abitrary tensor field of type (2,0) would be written as $A^{\mu\nu}$ and a tensor field of type (1,1) would be written as A^{μ}_{ν}or A_{μ}^{ν}. In this terminology, the scalar field R is a tensor field of type (0,0), i.e. no indices. On a spacetime manifold, it is customary for indices to range over the values 0,1,2,3, where the index 0 is associated with the temporal dimension, and the remaining indices are associated with the spatial dimensions.

Any tensor field with two indices may be represented by a 4×4 matrix, in which each element is a smooth, real-valued function of some general spacetime coordinates $q^{\mu},\ \mu = 0, 1, 2, 3$. However, matrices representing different types of tensor fields *cannot* be summed, despite having the same number of rows and columns. We will denote the matrix representation of any such tensor by placing its symbol in square brackets, e.g. the matrix representation of the metric tensor is denoted by $[g_{\mu\nu}]$.

The scalar field R is a single, smooth, real-valued function of the spacetime coordinates..

1.1.1 The Metric Tensor

For the sake of clarity, the matrix representation of a generic metric tensor is written out explicitly as

$$[g_{\mu\nu}] = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{3}$$

Note that the metric tensor is symmetric, i.e. $g_{\mu\nu} = g_{\nu\mu}$, which is reflected in the fact that $[g_{\mu\nu}]$ is a symmetric matrix. Thus, only $n(n+1)/2$=10 of the $n^2 = 16$ functions are unique. One may regard g_{00} as a purely temporal component, the off-diagonal elements $g_{0i} = g_{i0},\ i = 1, 2, 3$ as spatio-temporal components, and the remaining elements as purely spatial components. (The same break-down applies to the fields $R_{\mu\nu}$ and $T_{\mu\nu}$.)

Metric Signature On a Riemannian manifold, the metric tensor is positive-definite at all points and for all coordinate systems. In other words, the eigen values of the metric tensor are positive everywhere on the manifold. By contrast, on a spacetime manifold the metric tensor is indefinite, with the sign of the temporal eigenvalue being opposite to the sign of the spatial eigenvalues. This is referred to as *Lorentz signature* . There are two conventions in use in the literature, either $(+,-,-,-)$ or $(-,+,+,+)$. The choice is arbitrary, but must be used consistently, once made.

Distance on the Manifold Given two points on the manifold, q^μ and $q^\mu + dq^\mu$, where the $dq^\mu,\ \mu = 0,1,2,3$ are differential displacements, the differential distance ds between the two points is given by a quadratic form called the *metric*, as follows:

$$ds^2 = g_{ab} dq^a dq^b \tag{4}$$

Indices appearing as both subscripts and superscripts, like a and b in the above metric, are called *dummy indices*, implying summation over the values 0,1,2,3 (Einstein summation convention), so the expression 4 implies the sum of $n^2 = 16$ terms, i.e.

$$g_{\mu\nu} dq^\mu dq^\nu \Longrightarrow \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} dq^\mu dq^\nu \tag{5}$$

Length of a Vector Given any 4-vector field v^μ, $\mu = 0,1,2,3$, its squared length at any point in spacetime is given by

$$|v|^2 = g_{ab} v^a v^b \tag{6}$$

where the double summation is implied, as described above. Calculated in this manner, the length is automatically corrected for curvature and/or arbitrary choice of coordinates.

On a manifold with positive-definite metric, $|v|^2$ as given by 6 is such that$|v|^2 \geq 0$, where $|v|^2 = 0$ iff v^μ is the zero vector. On the otherhand, on a spacetime manifold, one can have $|v|^2 > 0$, $|v|^2 = 0$, or $|v|^2 < 0$. If using the convention $(+,-,-,-)$, the three possibilites indicate timelike, light-like (a.k.a null), and space-like vectors, respectively. If using the other convention, the definitions of time-like and space-like are swapped. In either case, $|v|^2 = 0$ does *not* imply that v^μ is the zero vector.

Generalized Inner Product Given two 4-vectors fields v^μ and w^μ, their inner product at any point in spacetime, taking into account curvature and/or the choice of coordinates, is given by

$$(v^\mu, w^\nu) = g_{ab} v^a w^b \tag{7}$$

Converting Tensor Fields to Other Types In the language of tensor fields, a 4-vector field v^μ is a tensor field of type (1,0). The metric tensor can be applied so as to convert it to a tensor of type (0,1), as follows:

$$v_\mu = g_{\mu a} v^a$$

where a is a dummy index, implying the sum of four terms. Similarly, given a tensor field $A^{\mu\nu}$ of type (2,0), the metric tensor can be applied to convert it to a tensor of type (1,1), as follows:

$$A_\mu^{\ \nu} = g_{\mu a} A^{a\nu} \tag{8}$$

A second application of the metric tensor will convert the resultant tensor of type (1,1) to a tensor of type (0,2), i.e.

$$A_{\mu\nu} = g_{a\nu} A_\mu^{\ a} \tag{9}$$

In general, the metric tensor is applied once per index to be changed from a superscript to a subscript. The two conversions above can thus be written in one step, i.e.

$$A_{\mu\nu} = g_{\mu a} g_{b\nu} A^{ab} \tag{10}$$

The reverse process is accomplished using the inverse of the metric tensor. Defined by

$$[g^{\mu\nu}] := [g_{\mu\nu}]^{-1} \tag{11}$$

Then,

$$v^\mu = g^{\mu a} v_a \tag{12}$$

and so on.

1.1.2 The Ricci Tensor and Ricci Scalar

The Ricci tensor is, roughly speaking, a measure of how the geometry in the neighborhood of each point in spacetime differs from flatness. Like the metric tensor, it may be regarded as a 4×4 symmetric matrix of smooth, real-valued functions, given by

$$R_{uv} = \partial_\nu \Gamma^a_{\ \mu a} - \partial_a \Gamma^a_{\ \mu\nu} + \Gamma^a_{\ \mu\nu} \Gamma^b_{\ ab} - \Gamma^a_{\ \mu b} \Gamma^b_{\ \nu a} \tag{13}$$

where

- ∂_ν is a short hand for $\partial / \partial q^\nu$
- The quantities$\Gamma^\xi_{\ \mu v}$ are the *Levi-Civita connection coefficients* (a. k.a. *Christoffel symbols*), given by

$$\mathbf{\Gamma}^\xi_{\ \mu\nu} = \frac{1}{2} g^{\xi a} \left(\partial_a g_{\mu\nu} + \partial_\nu g_{\mu a} - \partial_\mu g_{\nu a} \right) \tag{14}$$

where

- The index a is a dummy index, so each term in parenthases expands to 4 terms.
- The connection coefficients are symmetric in their two subscripts.
- One may regard the entire connection, i.e. the collection of $n^3 = 64$ connection coefficients, as a array of 4, symmetric 4×4 matrices.

The Ricci scalar is obtained from the Ricci tensor, as follows:

$$R = \mathrm{tr}\,\{g^{\mu a} R_{a\nu}\} = \mathrm{tr}\,\{R^{\mu}_{\nu}\} = R^{b}_{b} \tag{15}$$

where the indices a and b are dummies. Changing an upper and lower index to the same letter (as on the extreme RHS) results in an implied summation. This is called a *contraction*.

Alternate Method of Calculating the Ricci Tensor The *Riemann-Christoffel curvature tensor* (a.k.a, *Riemann tensor* or *curvature tensor*) is calculated as follows:

$$R^{\xi}_{\mu\sigma\nu} = \partial_\sigma \Gamma^{\xi}_{\mu\nu} - \partial_\nu \Gamma^{\xi}_{\sigma\mu} + \Gamma^{\xi}_{\sigma a}\Gamma^{a}_{\mu\nu} - \Gamma^{\xi}_{\nu a}\Gamma^{a}_{\sigma\mu} \tag{16}$$

Then, the Ricci tensor is obtained by contracting on the first and third indices of $R^{\mu}_{\ \nu\sigma\xi}$, i.e.

$$R_{uv} = R^{a}_{\ \mu a\nu} \tag{17}$$

If the Riemann tensor vanishes, i.e. $R^{\mu}_{\ \nu\sigma\xi} = 0$, everywhere, then the manifold (whether Lorentzian or Riemannian) is intrinsically flat. This is not necessarily the case for the Ricci tensor.

1.1.3 The Energy-Momentum Tensor

The purely temporal, purely spatial, and spatio-temporal portions of the energy-momentum tensor $T_{\mu\nu}$ have different physical interpretations. More precisely, common physical quantities are more readily assocated with $T^{\mu\nu} = g^{\mu a} g^{b\nu} T_{ab}$. Physical meaning may then be ascribed as follows:

- T^{00} is the mass/energy density..
- T^{0i}, $i = 1, 2, 3$ is the ith component of momentum density.
- T^{ij}, $i, j = 1, 2, 3$ are the components of the ordinary stress tensor.

1.1.4 Implications

Based on the discussions in sections 1.1.1 through 1.1.3, Einstein's equation 1 can be reqarded as a matrix equation, which, in turn, may be regarded as a system of 16 coupled, second-order, non-linear, partial differential equations in the metric tensor functions $g_{\mu\nu}$, where only 10 of the equations are unique. Finding a general solution for this system of equations is an analytically intractable problem. However, exact solutions can be found in special physical situations, wherein spacetime symmetries can be utilized to make the problem tractable. We will return to this topic in Section 2.

1.1.5 Other Forms of the Field Equation

Trace-Reversed Field Equation Substituting 2 into 1, yields the form of Einstein's field equation often seen in the literature, i.e.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi \mathsf{G}}{c^4}T_{\mu\nu}. \tag{18}$$

From this, one can derived what is called the *trace-reversed field equation*:

$$R_{\mu\nu} = \frac{8\pi \mathsf{G}}{c^4}\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) \tag{19}$$

where

$$T = \mathrm{tr}\left\{g^{\mu a}T_{a\nu}\right\}. \tag{20}$$

Vacuum Field Equation It is immediately apparent from 19 that, if $T_{\mu\nu} = 0$, then

$$R_{\mu\nu} = 0 \tag{21}$$

Since $T_{\mu\nu} = 0$ implies a vacuum, 21 is called the *vacuum field equation.* Any point in spacetime where 21 holds is called *Ricci flat.* However, this does not necessarily imply that spacetime is intrinsically flat at that point.

1.2 The Geodesic Equation

1.2.1 Curves on Manifolds

In terms of the general coordinates q^μ, a smooth curve on a (pseudo-)Riemannian manifold of dimension n can be represented parametrically by n functions $q^\mu(\lambda), \mu = 1, 2, ..., n$, where λ is a parameter that is monotonically increasing (in one direction or the other) along the curve. The n functions together give the coordinates at each point along the curve. The tangent vector to the curve at each point along the curve is then given by

$$\dot{q}^\mu := \frac{dq^\mu(\lambda)}{d\lambda} \tag{22}$$

The squared length of the tangent vector is then given by 6, i.e.

$$|\dot{q}|^2 = g_{ab}\dot{q}^a\dot{q}^b = g_{\mu\nu}\frac{dq^a(\lambda)dq^b(\lambda)}{d\lambda^2} = \frac{ds^2(\lambda)}{d\lambda^2} \tag{23}$$

In general, the length varies with λ. However, on Riemannian manifolds, there exists a preferred class of parameters that are linearly related to the distance s along the curve, i.e.

$$\lambda = a\,s + b \Longrightarrow d\lambda = a\,ds \Longrightarrow |\dot{q}| = 1/a \tag{24}$$

where $a \neq 0$ and b are constants. Thus, if restricted as in 24, the length of the tangent vector does *not* vary with λ. Under these conditions, λ is called an *affine parameter*.

On a spacetime manifold, things become a bit more complicated due to the metric having Lorentz signature. Specifically, a curve is said to be time-like if the tangent vector is time-like at every point along the curve. Light-like and space-like curves are defined in the analogous manner. A smooth curve cannot switch at some point from one class to another. Material particles travel on time-like curves, massless particles travel on light-like curves, and there are no known particles that travel on space-like curves.

Time-like curves can be affinely parameterized in the same manner as curves on Riemannian manifolds. Typically, the choice is $a = 1/c$. With this choice, $d\lambda = ds/c := d\tau$, i.e. the parameter is the proper time, and $\dot{q}^\mu = dq^\mu/d\tau$ is the 4-velocity of a test particle moving along the curve. To be clear, in general relativity $\dot{q}^\mu$ has a geometric interpretation as the tangent vector to a time-like curve and a physical interpetation as the 4-velocity of a test particle moving along that curve. In the latter case, the invariant quantity $|\dot{q}|^2 = c^2$ can be interpreted as the particle's rest energy per unit mass.

The situation is quite different for light-like curves. Since $ds = 0$ everywhere along light-like curves, λ *cannot* be affinely parameterized as prescribed by 24, because then $d\lambda = 0$, so 22 and 23 are undefined. In fact, 22 tells us that 4-velocity cannot be defined for massless particles. Instead, for light-like curves, we may take λ to be any monotonically increasing parameter that is *not* related to the distance s along the curve. For any such parameter, 23 tells us that $|\dot{q}| = 0$ everywhere along the curve, and therefore any such parameter satisfies the criterion that the tangent vector be of the same length everywhere along the curve. Clearly, for light-like geodesics, $|\dot{q}|^2 = 0$ and can thus we interpreted as rest energy.

The preceding two paragraphs imply that, for all affinely parameterized curves of physical interest, $|\dot{q}|^2$ is a constant of motion - a result that may be summarized by

$$g_{ab}\dot{q}^a\dot{q}^b = K_\varepsilon \tag{25}$$

where $K_\varepsilon = c^2$ for material particles and $K_\varepsilon = 0$ for massless particles.

1.2.2 Geodesics and the Geodesic Equation

The geometry of a Riemannian manifold determines a class of curves, called *geodesics*, that minimize the distance between points on the manifold. For spacetime manifolds, there are time-like, light-like, and space-like geodesics, which are special cases of the time-like, light-like, and space-like curves, respectively, discussed above. Material particles on which no forces act - the effects of gravity are embodied in the geometry - travel on time-like geodesics. Massless particles travel on light-like geodesics. The time-like geodesics actually maximize the distance between points, and of course the distance along all light-like geodesics is zero. The differences between the description of geodesics, above, are subsumed in a more general definition: *geodesics are paths of stationary action.*

Regardless of whether we have a Riemannian manifold or a spacetime manifold, the geodesics are solutions to the *geodesic equation*:

$$\frac{d^2 q^\mu}{d\lambda^2} + \Gamma^\mu_{ab} \frac{dq^a}{d\lambda} \frac{dq^b}{d\lambda} = 0 \tag{26}$$

provided that λ is an affine parameter. Concerning this equation, the following facts are worth noting:

- The geodesic equation 26 is actually a set of 4 coupled, second-order, ordinary differential equations.
- A solution is a set of four functions $q^\mu(\lambda)$, $\mu = 0, 1, 2, 3$.
- In the second term on the left, a and b are dummy indices, so this term expands to 16 terms (in general).
- In 3-dimensional, Euclidean space with $\lambda = t$, where t is Newton's universal time, the geodesic equation 26 is Newton's first law in arbitrary coordinates.
- In spacetime, the geodesic equation is a generalization of Newton's first law, i.e. the equation of motion in the absence of forces.
- The geodesic equation can be derived from the Euler-Lagrange equation

  $$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{q}^\mu}\right) = \frac{\partial L}{\partial q^\mu} \tag{27}$$

  where the Lagrangian L is kinetic energy per unit mass, given by

  $$L = \frac{1}{2}\left|\dot{q}\right|^2 = \frac{1}{2} g_{ab} \frac{dq^a}{d\lambda}\frac{dq^b}{d\lambda} = \frac{1}{2} g_{ab} \dot{q}^a \dot{q}^b \tag{28}$$

It is often very difficult to solve the geodesic equation 26 directly. However, as shown in the next section, symmetries of the spacetime geometry lead to a metric tensor in which many elements are zero, and the remaining are independent of one or more of the spacetime coordinates. For each such coordinate q^a, we have

$$\frac{\partial L}{\partial q^a} = 0 \tag{29}$$

and substituting this into 27, we have

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = 0 \tag{30}$$

so

$$\frac{\partial L}{\partial \dot{q}^a} = K_a \tag{31}$$

where K_a is a constant of motion. If four constants of motion can be found, then the system of 4 *coupled*, second-order, differential equations 26 can be reduced to a system of 4 *separate*, first-order, differential equations. In other words, the equations can be put into the form

$$\dot{q}^\mu = f^\mu(q^b) \tag{32}$$

where μ runs over the 4 spacetime indices and b runs only over only those indices corresponding to spacetime coordinates on which the metric tensor functions depend. Then, the four functions $q^\mu(\lambda)$ can be found by integration of the right-hand side of 32, if possible, or via quadrature.

2 Solving Einstein's Equation

The content of Einstein's field equation is often summarized as follows: the geometry of spacetime (as embodied in the metric tensor $g_{\mu\nu}$) is related, via the field equation, to the matter/energy distribution in spacetime. The first exact solution, found by Karl Schwarzschild in 1916, illustrates why this can be misleading. This solution describes a spacetime outside a static, spherically-symmetric mass, assuming no other sources (matter, energy, etc) exist. The latter stipulaton - that there is no matter/engery in the region of interest - implies that $T_{\mu\nu} = 0$ in this region, which in turn implies it is the vacuum field equation 21 that was solved, under the prescribed conditions. How then do we account for the presence of the central mass in Einstein's equation? The answer is: via constraints on the form of the metric tensor, consistent with the prescribed conditions. Specifically:

- A spherically-symmetric mass implies that the spacetime is invariant under 3-dimensional rotations about the center of mass, so choose a coordinate system best suited to this situation, i.e. $\{q^0 = t, q^1 = r, q^2 = \theta, q^3 = \phi\}$, where $\{r, \phi, \theta\}$ are spherical polar coordinates, with origin at the center of mass.
- With the above choice of coordinates, rotational invariance implies the metric tensor elements $g_{r\theta}$, $g_{r\phi}$ and $g_{\theta\phi}$ must be zero, and that non-zero elements of $g_{\mu\nu}$ *cannot* depend on the angular coordinates θ and ϕ.
- The static condition implies that the spacetime is invariant under time translation and time reversal, which implies that the metric tensor elements g_{tr}, $g_{t\theta}$ and $g_{t\phi}$ are zero, and that non-zero elements *cannot* depend on the time coordinate t.

In summary, the conditions in the problem statement, together with an appropriate choice of coordinates, results in a metric tensor that is diagonal, where the four non-zero elements g_{tt}, g_{rr}, $g_{\theta\theta}$ and $g_{\phi\phi}$ are functions of r only. This, in turn, constrains the form of the connection coefficients, Ricci tensor and Ricci scalar. In particular, many elements are zero, while the non-zero elements are simplified. Not only does this ensure that the effects of the central mass are incorporated into Einstein's equation, it also ensures that the latter becomes analytically tractable. That is, instead of a system of 10, coupled, non-linear, differential equations in four coordinates, we have a system of 4, coupled, non-linear, differential equations in one coordinate: r. In addition, this situation leads to three additional constants of motion (as described in the preceding section), which together with the constant of motion 25, allow the equations of motion to be put into the form 32.

Other important solutions to Einstein's field equation have been obtained through a similar method of attack. That is, the symmetries of the particular situation result in contraints on the metric - typically many or all of the off-diagonal elements turn out to be zero and the remaining elements are functions of a reduced number of spacetime coordinates. For example:

- In the Kerr solution (1963) for the vacuum region (where $T_{\mu\nu} = 0$) of a static, axially-symmetric spacetime, the corresponding constraints on the metric tensor lead to a system of five differential equations in two coordinates: radial coordinate r and angular coordiate ϕ about the axis of symmetry.
- In the cosmological solution of Robertson and Walker (1930s) - a situation in which $T_{\mu\nu} \neq 0$, the symmetries of large-scale homogeneity and isotropy lead to one differential equation in the one coordinate: the time t.

3 Tests of the General Relativity

Verified predictions:

- Precession of Perihilia
- Deflection of Light (Gravitational Lensing)
- Gravitational Time Dilation
- deSitter Precession - precession, due to a central mass, of a vector carried along with an orbiting body.
- Lense-Thirring Precession - precession, due to rotation of a central mass, of a vector carried along with an orbiting body.
- Black Holes
- Gravitational Radiation (Gravity Waves)