# Designer Spacetimes

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## 1 Traditional Usage of Einstein's Field Equation: Review

Review of traditional method of using Einstein's field equation:

1. Choose the physical situation to which the field equation will be applied.
2. Construct a stress-energy-momentum tensor $T_{\mu\nu}$ consistent with the physical situation.
3. Constrain the form of the metric, based on symmetry considerations of the physical situation.
4. Solve the resultant system of differential equations, to obtain the corresponding, exact solution, i.e. a specific metric tensor $g_{\mu\nu}$.

This is difficult.

## 2 Using Einstein's Equation Backwards

Choose (or guess at) an exact metric tensor $g_{\mu\nu}$ with Lorentz signature, and obtain the corresponding $T_{\mu\nu}$, via straight-forward (but long-winded) calculations, as follows:

$$g_{**}\left\{\begin{array}{c} \stackrel{\text{inv}}{\rightarrow} g^{**} \\ \stackrel{\text{dif}}{\rightarrow} \partial_* g_{**} \end{array}\right\} \stackrel{(14)}{\rightarrow} \Gamma^*{}_{**} \stackrel{\text{dif}}{\rightarrow} \partial_*\Gamma^*{}_{**}$$

$$\left\{\begin{array}{c} \Gamma^*{}_{**} \\ \partial_*\Gamma^*{}_{**} \end{array}\right\} \stackrel{(13)}{\rightarrow} R_{**}$$

$$\left\{\begin{array}{c} g^{**} \\ R_{**} \end{array}\right\} \stackrel{\text{raise}}{\rightarrow} R^*_* \stackrel{\text{tr}}{\rightarrow} R$$

$$\left\{\begin{array}{c} R_{**} \\ R \\ g_{**} \end{array}\right\} \stackrel{(2)}{\rightarrow} G_{**}$$

$$G_{**} \stackrel{\div k}{\rightarrow} T_{**}$$

This is easy (but tedius). (There is a more efficient way to do this, which we will come to in the next talk.)

## 3 Energy Conditions

According to the preceding section, we can choose any metric tensor with Lorentz signature, and compute the corresponding stress-energy tensor. Does this mean that every Lorentzian metric is a valid solution to Einstein's equation? The physics community says "NO": according to the literature, there is a consensus that $T_{\mu\nu}$ must be constrained in such a manner as to be physically realistic. Such constraints are called *energy conditions*. Historically, four such conditions have been proposed (based on what seems reasonable):

1. Weak Energy Condition (WEC) - the energy density, as measured by time-like obvservers, is never negative. This is expressed mathematically by

   $$T_{ab}t^a t^b \geq 0 \tag{1}$$

   where $t^a$ is any time-like vector field.

2. Null Energy Condition (NEC) - the energy density, as experienced by massless particles, is never negative. This is expressed mathematically by

   $$T_{ab}n^a n^b \geq 0 \tag{2}$$

   where $n^a$ is any future-pointing, null (light-like) vector field.

3. Strong Energy Condition (SEC) - energy sources, as seen be time-like observers, are always gravitationally attractive. This is expressed mathematically by

   $$\left(T_{ab} - \frac{1}{2}T g_{ab}\right) t^a t^b \geq 0 \tag{3}$$

   where $t^a$ is any time-like vector field.

   NOTE: the SEC is now known to be ruled out due to the existence of dark energy, which is causing the expansion of the universe to accelerate.

4. Dominant Energy Condition (DEC) - the WEC is satisfied and, in addition, energy can never be observed to be flowing faster than light. The latter is expressed mathematically by requiring that, for any causal vector field $\chi^a$, the momentum $-T^{\mu}_{a}\chi^a$ must be a future-pointing, causal vector.

**Crucial Point** In order for a particular energy condition to be satisfied, the corresponding constraint(s) must be satisfied in every frame of reference.

## 4 The Hawking-Ellis Classification Scheme

So, given a stress-energy tensor, how do we determine the status of the energy conditions for all reference frames? There exists a procedure, which depends on a classification scheme originally presented by Hawking and Ellis (1973). Using this scheme, the stress-energy tensor is determined to be one of four possible types, designated simply as I, II, III or IV. This determination may be different at each point in spacetime, so the following analysis is required at every point in the region of interest:

1. Find the eigenvalues and eigenvectors of the stress-energy tensor.

2. If there are only two real eigenvalues (the other two being a complex-conjugate pair), we have Type IV.

3. If there are four real eigenvalues, distinguishing among Types I, II and III depends on the *eigenvalue multiplicities*.

## 4.1 Eigenvalue Multiplicities

Let $A$ be the matrix representation of a linear operator on a vector space of dimension $n$. The eigenvalues of $A$ are not necessarily unique, nor is the total number of linearly independent eigenvectors necessarily equal to $n$. Such conditions can be quantified as follows:

1. The *algebraic multiplicity* $\mu(\lambda_i)$ of an eigenvalue $\lambda_i$ is the number of times $\lambda_i$ appears as a root of the characteristic polynommial
$$\det(A - \lambda I) = 0. \tag{4}$$
2. The *geometric multiplicity* $\mu_g(\lambda_i)$ of an eigenvalue $\lambda_i$ is the number of linearly independent eigenvectors belonging to $\lambda_i$, or equivalently, the dimension of the eigenspace $S(\lambda_i)$ of $\lambda_i$ .

The two multiplicities may differ, but it is always the case that

$$1 \leq \mu_g(\lambda_i) \leq \mu_a(\lambda_i) \tag{5}$$

for all $i = 1, 2, \cdots, m$, where $m$ is the number of distinct eigenvalues and $1 \leq m \leq n$. From (5), it can be seen that, if $n = 4$, then for any eigenvalue $\lambda$ we have the following explicit possibilities:

- If $\mu_a(\lambda) = 1$, then $\mu_g(\lambda) = 1$.
- If $\mu_a(\lambda) = 2$, then it may be that $\mu_g(\lambda) = 1$ or $\mu_g(\lambda) = 2$.
- If $\mu_a(\lambda) = 3$, then it may be that $\mu_g(\lambda) = 1$, $\mu_g(\lambda) = 2$ or $\mu_g(\lambda) = 3$.
- If $\mu_a(\lambda) = 4$, then it may be that $\mu_g(\lambda) = 1$, $\mu_g(\lambda) = 2$, $\mu_g(\lambda) = 3$ or $\mu_g(\lambda) = 4$.

In general, if $\mu_g(\lambda_i) < \mu_a(\lambda_i)$, then:

- $\lambda_i$ is called *defective*, and
- The total number of linearly independent eigenvectors is reduced by $\mu_a(\lambda_i) - \mu_g(\lambda_i)$.
- If one or more eigenvalues are defective, then the linear operator (or its matrix representation) is also called defective.

**A Notation for Eigenvalue Multiplicities** It is convenient to have a compact notation for eigenvlaue multiplicities. In the literature, the Segre notation is most often used. However, this notation is so compact that the meaning can be obscure, unless one spends some time becoming famaliar with it. Here we will use a somewhat less compact, but perhaps more intuitive notation, as follows: for a given operator (or matrix), the complete set of multiplicities is given by two rows of numbers, enclosed in braces, where the top row contains the algebraic multiplicies of each distinct eigenvalue, in descending order, with the corresponding geometric multiplicities written directly underneath. Using this notation, the following table provides an exhaustive list of multiplicities for $n = 4$:

| Non-Defective | Defective |
|---|---|
| $\begin{Bmatrix}1111\\1111\end{Bmatrix}$ | |
| $\begin{Bmatrix}211\\211\end{Bmatrix}$ | $\begin{Bmatrix}211\\111\end{Bmatrix}$ |
| $\begin{Bmatrix}22\\22\end{Bmatrix}$ | $\begin{Bmatrix}22\\21\end{Bmatrix}, \begin{Bmatrix}22\\11\end{Bmatrix}$ |
| $\begin{Bmatrix}31\\31\end{Bmatrix}$ | $\begin{Bmatrix}31\\21\end{Bmatrix}, \begin{Bmatrix}31\\11\end{Bmatrix}$ |
| $\begin{Bmatrix}4\\4\end{Bmatrix}$ | $\begin{Bmatrix}4\\3\end{Bmatrix}, \begin{Bmatrix}4\\2\end{Bmatrix}, \begin{Bmatrix}4\\1\end{Bmatrix}$ |

## 4.2 Hawking-Ellis Types I, II and III

Using the multiplicities described above, we can now distinguish among the Hawking-Ellis types I, II and III, as follows:

- Type I
  - There are no defective eigenvalues.
  - Any of the configurations in the left-hand column of the table can occur.
  - Configuration $\left\{ {4 \atop 4} \right\}$ occurs only at points where spacetime is instrinsically flat.
  - Four independent eigenvectors: 1 time-like and 3 space-like.
- Type II
  - One defective eigenvalue $\lambda$, with $\mu_a(\lambda) = 2$ and $\mu_g(\lambda) = 1$, and eigenvector $E(\lambda)$ is null (light-like).
  - Configurations $\left\{ {211 \atop 111} \right\}$ and $\left\{ {22 \atop 21} \right\}$can occur, but configuration $\left\{ {22 \atop 11} \right\}$ is theoretically forbidden.
  - Three independent eigenvectors: one is the null eigenvector and the remaining two are space-like.
- Type III
  - One defective eigenvalue $\lambda$, with $\mu_a(\lambda) = 3$ and $\mu_g(\lambda) = 1$, and eigenvector $E(\lambda)$ is null (light-like).
  - Only configuration $\left\{ {31 \atop 11} \right\}$can occur, but configureation $\left\{ {31 \atop 21} \right\}$ is theoretically forbidden.
  - Two independent eigenvectors: one is the null eigenvector and the other is space-like.

NOTE: remaining configurations $\left\{ {4 \atop 3} \right\}, \left\{ {4 \atop 2} \right\}, \left\{ {4 \atop 1} \right\}$ are theoretically forbidden.

## 4.3 Determining the Status of the Energy Conditions

**If the stress-energy tensor is Type I**:

- The principle energy density is $\bar{\rho} = -\lambda_0$, where the eigenvector $E(\lambda_0)$ is time-like.
- The principle pressures are $p_i = \lambda_i,\ i = 1, 2, 3$, where the eigenvectors $E(\lambda_i)$ are space-like.

Then, the energy conditions are satisfied as follows:

- NEC: $\bar{\rho} + p_{\min} \geq 0$
- WEC: NEC is satisifed and $\bar{\rho} \geq 0$
- SEC: NEC is satisifed and $T \geq 0$
- DEC: $\bar{\rho} \geq |p_i|\,,\ i = 1, 2, 3$ and $\bar{\rho} \geq 0$

**If the stress-energy tensor is Type II**:

- The principle energy density is $\bar{\rho} = -\lambda_0$, where the eigenvector $E(\lambda_0)$ is null (light-like).
- The principle pressures are $p_i = \lambda_i,\ i = 1, 2$, where the eigenvectors $E(\lambda_i)$ are space-like.
- The stress energy tensor can be put in the following form

$$\begin{bmatrix} v + \bar{\rho} & v & 0 & 0 \\ v & v - \bar{\rho} & 0 & 0 \\ 0 & 0 & p_1 & 0 \\ 0 & 0 & 0 & p_2 \end{bmatrix} \tag{6}$$

Then, the energy conditions are satisfied as follows:

- NEC: Type I NEC is satisfied and $v \geq 0$.
- WEC: Type I WEC is satisfied and $v \geq 0$.
- SEC: Type I SEC is satisfied and $v \geq 0$.
- DEC: Type I DEC is satisfied and $v \geq 0$.

**If the stress-energy tensor is type III or IV**: all energy conditions are violated.

### 4.3.1 Complications for Type I

Complications arise for Type I, when the configuration is $\left\{ {22 \atop 22} \right\}$. In this case, there are two unique eigenvalues $\lambda_1$ and $\lambda_2$, each belonging to a 2D eigenspace. Eigenvalue decomposition algorithms often return a pair of space-like eigenvectors corresponding to each eigenvalue, i.e.

- $E_1(\lambda_1)$ and $E_2(\lambda_1)$ are space-like
- $E_1(\lambda_2)$ and $E_2(\lambda_2)$ are space-like

So where is the required time-like eigenvector? The answer is that:

- One 2D eigenspace must admit a time-like eigenvector, and
- The other 2D eigenspace must NOT admit a time-like eigenvector

so we must determine which is which in order that we may correctly assign:

- $\bar{\rho} = -\lambda_t$ and $p_1 = \lambda_t$, where $\lambda_t$ is the eigenvalue whose eigenspace admits the time-like eigenvector.
- $p_2 = p_3 = \lambda_s$, where $\lambda_s$ is the eigenvalue whose eigenspace does not admit the time-like eigenvector.

We determine which eigenspace admits a null eigenvector as follows: given two space-like eigenvectors $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ belonging to the same eigenspace, said eigenspace admits a time-like eigenvector vector if

$$(\boldsymbol{E}_1, \boldsymbol{E}_2)^2 > (\boldsymbol{E}_1, \boldsymbol{E}_1)(\boldsymbol{E}_2, \boldsymbol{E}_2) \tag{7}$$

where $(*, *)$ indicates inner product.

## 4.4 Eigenvalues and Eigenvectors of Rank-2 Tensors

As discussed above, we must find the eigenvalues and eigenvectors of the stress-energy tensor. This is the same as finding eigenvalues and eigenvectors of any rank-2 tensor, so we will generalize the discussion to an arbitrary rank-2 tensor $\boldsymbol{A}$.

### 4.4.1 How Rank-2 Tensors Act Upon Vectors and Covectors

As we saw last time, there are four different kinds of rank-2 tensor: type (0,2) with index notation $A_{**}$, type (2,0) with index notation $A^{**}$ and type (1,1) with index notation $A^*_{\ *}$ or $A_*^{\ *}$. We will characterize these by how they act upon vectors $v^b \in \mathbb{T}_P$ and covectors $v_b \in \mathbb{T}^*_P$, where $\mathbb{T}_P$ and $\mathbb{T}^*_P$ are dual vector spaces denoting, respectively, the tangent space and cotangent space at a point $P$ on the spacetime manifold. Letting index notation guide us, we have:

- The purely covariant form of $\boldsymbol{A}$, denoted by $\mathcal{A}_{ab}$, acts only upon tangent vectors, mapping them to cotangent vectors, i.e.
$$\mathcal{A}_{ab} v^b = v_a \in \mathbb{T}^*_P \;\Rightarrow\; \mathcal{A}_{ab} : \mathbb{T}_P \to \mathbb{T}^*_P. \tag{8}$$
- The purely contravariant form of $\boldsymbol{A}$, denoted by $\mathcal{A}^{ab}$, acts only upon cotangent vectors, mapping them to tangent vectors, i.e.
$$\mathcal{A}^{ab} v_b = v^a \in \mathbb{T}_P \;\Rightarrow\; \mathcal{A}^{ab} : \mathbb{T}^*_P \to \mathbb{T}_P. \tag{9}$$

- The mixed form of $\boldsymbol{A}$, denoted by either $\mathcal{A}^a_{\ b}$ or $\mathcal{A}_a^{\ b}$, acts upon either tangent or cotangent vectors, mapping them to cotangent vectors or tangent vectors, respectively, i.e.

$$\mathcal{A}^a_{\ b} v^b = v^a \in \mathbb{T}_P \ \Rightarrow \ \mathcal{A}^a_{\ b} : \mathbb{T}_P \to \mathbb{T}_P, \tag{10}$$

$$\mathcal{A}_a^{\ b} v_b = v_a \in \mathbb{T}^*_P \ \Rightarrow \ \mathcal{A}_b^{\ a} : \mathbb{T}^*_P \to \mathbb{T}^*_P. \tag{11}$$

Looking at (8) through (11), we see that a mixed tensor is a linear operator on a vector space - either $\mathbb{T}_P$ or $\mathbb{T}^*_P$, whereas the pure tensors $\mathcal{A}_{ab}$ and $\mathcal{A}^{ab}$ are mappings from one vector space to another.

### 4.4.2 Choosing the Appropriate Type of Rank-2 Tensor

In the theory of vector spaces, if $A : \mathbb{V} \to \mathbb{V}$ is a linear operator on a vector space $\mathbb{V}$ over a field $F$, then a *non-zero* vector $v \in \mathbb{V}$ is called an eigenvector of $A$, if $Av$ is a scalar multiple of $v$, i.e. if

$$Av = \lambda v. \tag{12}$$

where $\lambda \in F$. Then, $\lambda$ is called the eigenvalue of $A$ belonging to the eigenvector $v$.

As shown in Section 4.4.1, only the mixed forms of rank-2 tensors are linear operators on a vector space. Thus, it is no surprise that, if we attempt to rewrite (12) in index notation using $A := \mathcal{A}_{ab}$, we obtain an invalid result, i.e.

$$\mathcal{A}_{ab} v^a = \lambda v^a \quad (invalid) \tag{13}$$

since we cannot equate the covector on the LHS with the vector on the RHS. An attempt to substitute $A := \mathcal{A}^{ab}$ leads to the analogous inconsistency. On the other hand, if we rewrite (12) in index notation using $A := \mathcal{A}^a_{\ b}$, we obtain a valid result, i.e.

$$\mathcal{A}^a_{\ b} v^b = \lambda v^a. \tag{14}$$

This can be used to obtain a version of the characteristic polynomial in index notation, i.e.

$$\begin{aligned}
\mathcal{A}^a_{\ b} v^b - \lambda v^a &= 0 \\
\left(\mathcal{A}^a_{\ b} - \lambda \delta^a_{\ b}\right) v^b &= 0 \\
\left(\mathcal{A}^a_{\ b} - \lambda \delta^a_{\ b}\right) &= 0 \qquad (15) \\
\det\left(\mathcal{A}^a_{\ b} - \lambda \delta^a_{\ b}\right) &= 0 \qquad (16)
\end{aligned}$$

where $\delta^a_{\ b}$ is the Kronecker delta.

Thus, we must use a mixed form of the tensor when computing eigenvalues and eigenvectors. In retrospect, this had to be the case for the stress-energy tensor, for which the forms $T_{**}$ and $T^{**}$ are symmetric, and thus could not possibly have complex eigenvalues, which would then contradict the fact that Hawking-Ellis Type IV is one of the possibilities.

As shown in Section 4.4.1, there are two versions of the mixed tensor. They have the same eigenvalues, but as evident from (10) and (11), the eigenvectors of the first version are vectors, where as the eigenvectors of the second version are covectors. This becomes important, if it becomes necessary to use (7), because

- For eigenvectors $v^a$ and $w^b$

$$\left(v^a, w^b\right) = g_{ab} v^a w^b \tag{17}$$

- But for eigenvectors $v_a$ and $w_b$

$$\left(v_a, w_b\right) = g^{ab} v_a w_b \tag{18}$$

Thus, computation of inner products in (7) must be consistent with the choice of mixed tensor.