General Relativity: 3+1 Formalism

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In general relativity (GR), the 3+1 formalism may be thought of as restoring the Newtonian concept of a 3-dimensional space evolving in time, but in such a way as to preserve the conceptual changes wrought be GR. (This formalism was first introduced in a 1962 paper by Arnowitz, Deser and Misner, and is therefore also refered to as the ADM formalism. However,

1 Time-Like Foliation of the Spactime Manifold

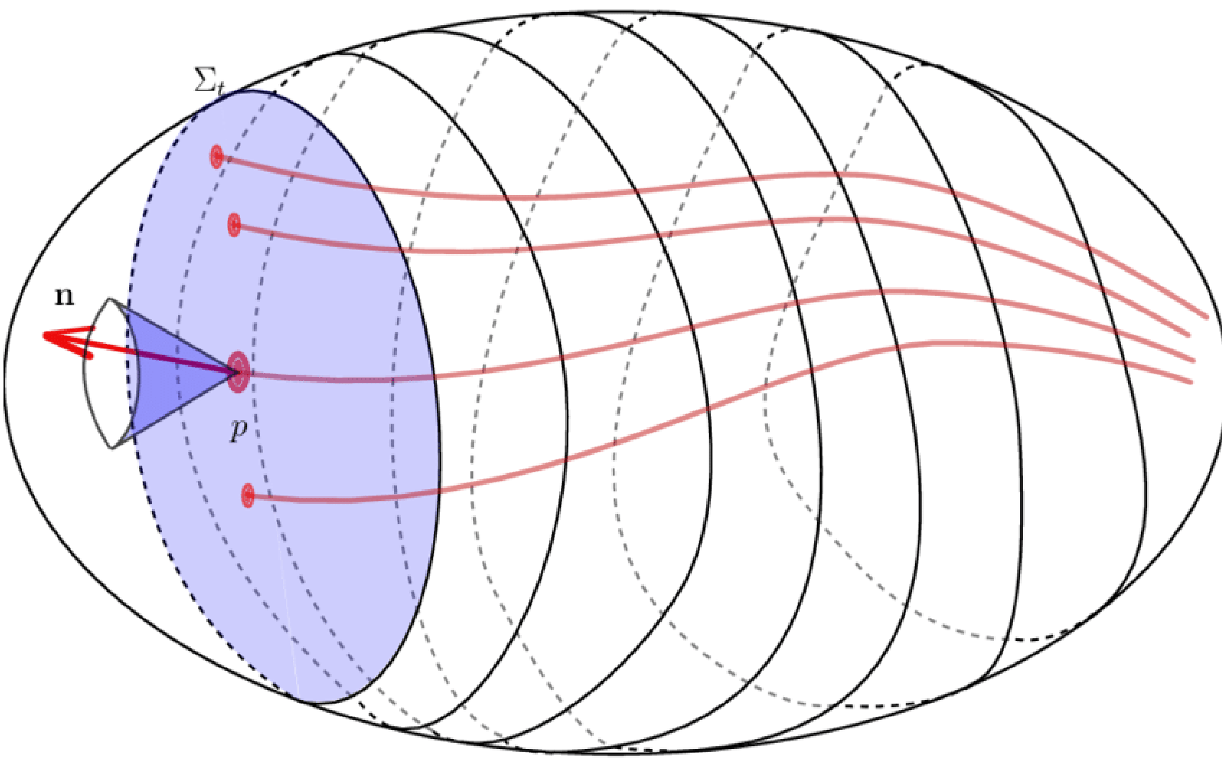

Fig. 1

Description of Fig. 1

- Coordinate time t increases from right to left.
- At each time t, there is a space-like hypersurface Σ_t (disks: 1 spatial dimension is supressed).
- Each space-like hypersurface is a snapshot of all of space at time t.
- There is a family of time-like curves (red), which never intersect.
- Each time-like curve passes through each hypersurface only once.
- The unit normal vector $\boldsymbol{n}$ to any hypersurface at any point p is tangent to time-like curve passing through p, i.e. the time-like curves are everywhere orthogonal to the hypersurfaces.

- Each unit tangent vector (or normal vector) lies within its local light cone;
  otherwise the red curves would not be time-like.
- The unit tangent vector field is also the 4-velocity of a family of observers moving along the time-like curves;
  In the literature these are called "Eulerian" observers.

1.1 Zooming in on a Portion of the Foliated Manifold

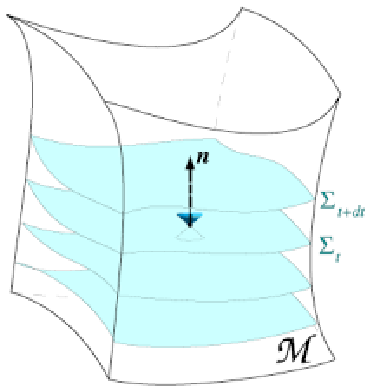

Fig. 2

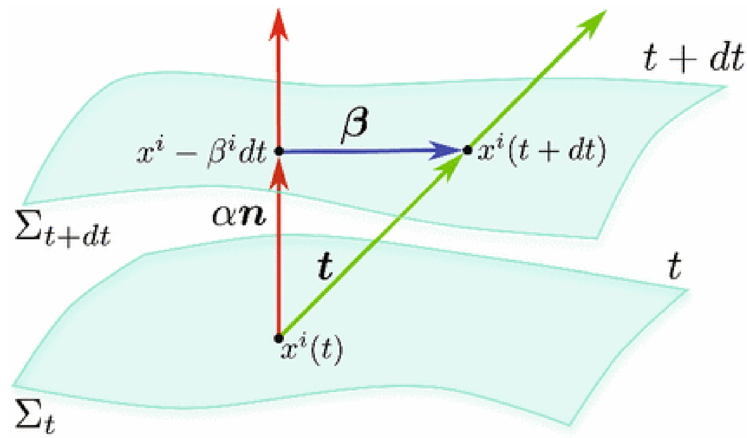

Fig. 3

Description of Fig. 1 and Fig. 2

- Coordinate time is increasing in the upward direction (the conventional way of depicting spacetime).
- The focus is on two neighboring hypersurfaces Σ_t and Σ_{t+dt}, separated by differential time difference dt.
- On each hypersurface, we have a 3-dimensional system of spatial coordinates x^i, $i = 1, 2, 3$,
  which are functions of the time coordinate t.
- In general, the spatial coordinates undergo a shift as $x^i(t) \to x^i\,(t + dt)$, as indicated by the 4-vector $\boldsymbol{t}$.
- The vector $\boldsymbol{t}$ may be decomposed into temporal and spatial parts $\alpha\boldsymbol{n}$ and $\boldsymbol{\beta}$, respectively.
- α is called the *lapse function*: the rate of change of proper time, relative to coordinate time.
- $\boldsymbol{\beta} \coloneqq \beta^i$, $i = 1, 2, 3$ is called the *shift vector*: relates spatial coordinate systems on different hypersurfaces.

Note: there are (unfortunately) two different notations in use for the lapse function and shift vector; many authors use α and $\boldsymbol{\beta}$, as above; but many other authors, e.g. Wald, use the notaton originally introduced by ADM, wherein $\alpha \Longleftrightarrow N$ and $\beta^i \Longleftrightarrow N^i$; henceforth, we use the ADM notation.

1.2 Relation to Conventional GR

The elements of the 3+1 formalism can be related to conventional spacetime by the following considerations:

- The space-like hypersurfaces are submanifolds of the spacetime manifold.
  Thus, they are manifolds in their own right, specifically, 3-dimensional Riemannian manifolds,
  i.e. manifolds with Euclidean signature $(+, +, +)$.
- There is a convenient notation for distinguishing tensors on the full spacetime manifold from those on the submanifolds:
  - On the former, Greek indices run over the values 0, 1, 2, 3.
  - On the latter, Latin indices run over the values 1, 2, 3.
- Examples of the latter are the shift vector N^i and the metric tensor on the submanifolds, which we will denote by h_{ij}.

- Given: a lapse function N, a shift vector N^i, and a spatial metric tensor h_{ij}, the geometry of the spacetime manifold is completely determined.
  That is, the metric tensor of spacetime is given by

$$[g_{\mu\nu}] = \begin{bmatrix} N_i N^i - N^2 & N_1 & N_2 & N_3 \\ N_1 & h_{11} & h_{12} & h_{13} \\ N_2 & h_{21} & h_{22} & h_{23} \\ N_3 & h_{31} & h_{32} & h_{33} \end{bmatrix} \tag{1}$$

  where $N_i = h_{ij} N^j$.

- Construction the metric tensor in this way guarantees that
  - The signature of the metric is Lorentzian.
  - The spactime is *globally hyperbolic*: conditions on any one space-like hypersurface determines conditions on all of them.
    This means that "the entire future and past history of the universe can be predicted (or retrodicted) from conditions at the instant of time represented by [any hypersurface] Σ." - Wald
    The following is also of interest:
    "There are some good reasons for believing that all physically realistic spacetimes must be globally hyperbolic." - Wald

- Alternatively, if a spacetime is globally hyperbolic, then it admits a time-like foliation, as described above.

1.3 Relation to Designer Spacetimes

Once a metric tensor is given by (1), the stress-energy-momentum tensor can be calculated by conventional means, i.e. via the following algorith (discussed last time):

$$g_{**} \left\{ \begin{array}{c} \stackrel{\text{inv}}{\to} g^{**} \\ \stackrel{\text{dif}}{\to} \partial_* g_{**} \end{array} \right\} \stackrel{(14)}{\to} \Gamma^*{}_{**} \stackrel{\text{dif}}{\to} \partial_* \Gamma^*{}_{**}$$

$$\left\{ \begin{array}{c} \Gamma^*{}_{**} \\ \partial_* \Gamma^*{}_{**} \end{array} \right\} \stackrel{(13)}{\to} R_{**}$$

$$\left\{ \begin{array}{c} g^{**} \\ R_{**} \end{array} \right\} \stackrel{\text{raise}}{\to} R^*_* \stackrel{\text{tr}}{\to} R$$

$$\left\{ \begin{array}{c} R_{**} \\ R \\ g_{**} \end{array} \right\} \stackrel{(2)}{\to} G_{**}$$

$$G_{**} \stackrel{\div k}{\to} T_{**}$$

Once we have T_{**}, we can compute the Eulerian energy density, i.e. the energy density as seen by the Eulerian observers via

$$\rho = T_{\mu\nu} n^\mu n^\nu \tag{2}$$

where n^μ are the components of the normal to the hypersurfaces or equivalently the tangent vector to the time-like curves or equivalently the 4-velocity of the Eulerian observes, which is given by

$$n^\mu = \frac{1}{N} \left(1, -N^1, -N^2, -N^3\right). \tag{3}$$

If the Eulerian energy density is negative is some regions, then we already know that the energy conditions are violated, without doing the eigenvalue analysis discussed previously.