The Alcubierre Drive

June 13, 2023

1 Alcubierre's Ansatz

In 1994, Mexican physicist Miguel Alcubierre considered a time-like foliation of spacetime in which:

- The space-like hypersurfaces are i ntrinsically flat.
- The chosen coordinate system on the space-like hypersurfaces is Cartesian, with coordinates (x, y, z), which implies that the spatial metric tensor is given by $h_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker delta, or in matrix notation the 3×3 identity matrix.
- The lapse function is given by $N = 1$, which implies the following equivalent conditions for the family of time-like curves normal to the hypersurfaces:
  - The proper time is the coordinate time;
  - The time-like curves are geodesics;
  - The Eulerian observers experience geodesic motion;
  - The Eulerian observers are in free fall, i.e. they experience no forces.
- The shift vector is given by

$$N_x = -v\frac{\Delta}{C} \tag{1}$$
$$N_y = 0 \tag{2}$$
$$N_z = 0 \tag{3}$$

where

$$C = 2\tanh(R/D) \tag{4}$$
$$\Delta = \tanh(r_+) - \tanh(r_-) \tag{5}$$

where

$$r_\pm = \frac{(r \pm R)}{D} \tag{6}$$

where

$$r = \sqrt{\left(x - x_s(t)\right)^2 + y^2 + z^2} \tag{7}$$

where $x_s(t)$ is an arbitrary function of time and

$$v = \frac{dx_s}{dt}. \tag{8}$$

Concerning the above equations, note the following items:

1. D and R are parameters with dimensions of length, constrained by $D, R > 0$ and $D \lesssim R/4$, but otherwise arbitrary.
2. From item 1, R/D, $r_+$and r_- are dimensionless, as appropriate for arguments to tanh.

3. From (4), we see that, after fixing the parameters D and R, C is a dimensionless constant.

4. From (5), we see that Δ is dimensionless, so Δ/C is dimensionless.

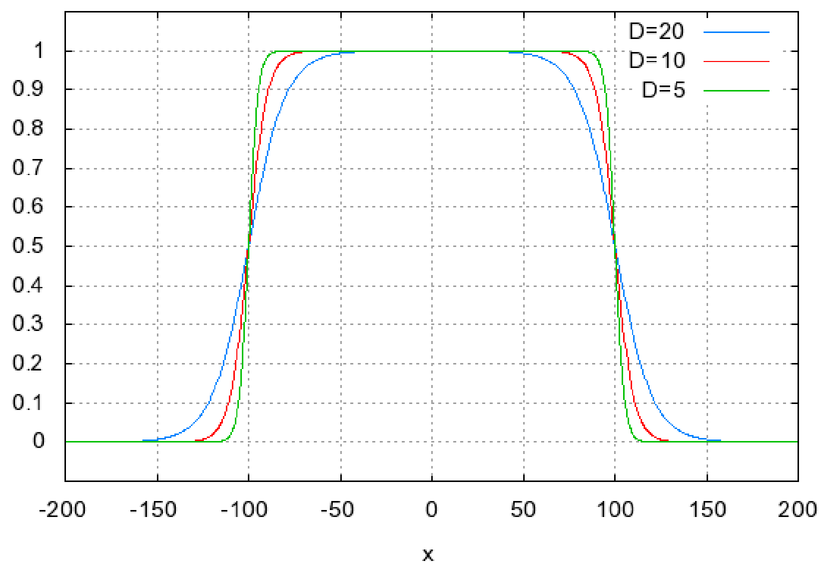
5. From (8), we see that v is a velocity.

Therefore, N_x has dimensions of velocity.

Now, since $x_s(t)$ is arbitrary, let $x_s(t) = act$, where a is an arbitrary positive factor and c is the speed of light. Then, $v = ac$. If we work in natural units, then $v = a$ and $x_s(t) = at$. In either case, we can take $a \geq 0$ as a third parameter.

1.1 Interpretation of the Parameters

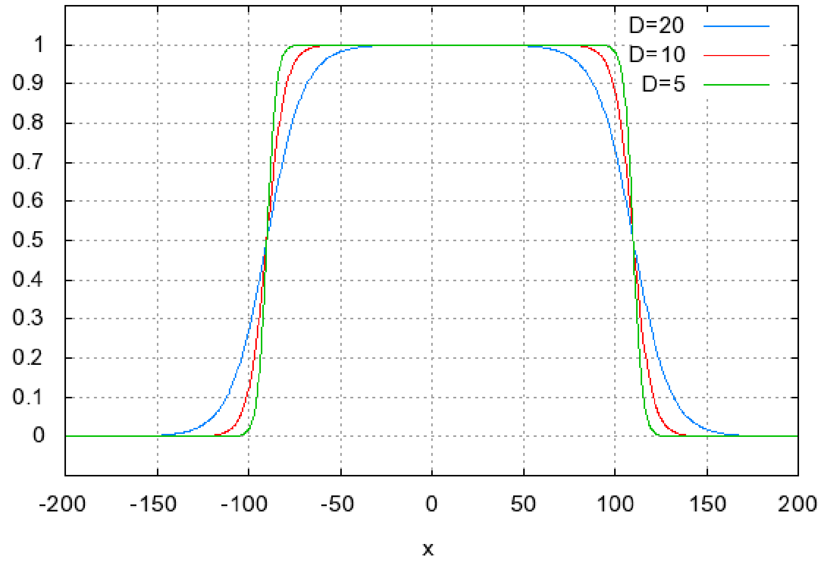
Cross-section of Δ/C at $y = z = t = 0$, for $R = 100$ and 3 values of D:



Note that:

- Δ/C has roughly the shape of a top hat.
- R determines the radius of the hat at the mid-point of its height, i.e. at $\Delta = 0.5$.
- D determines the steepness of the hat's sides; specifically, the sides get closer to vertical as D gets closer to zero.
- The transition from vertical to horizontal is smooth, and is completed at roughly $R \pm 3D$.

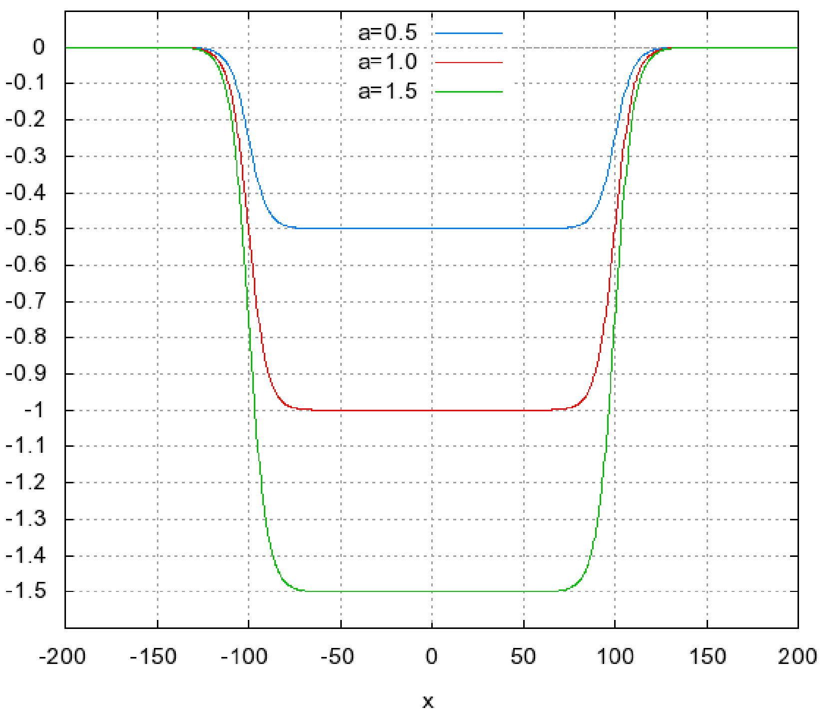
Cross-section of Δ/C at $y = z = 0$ and $t = 10$, for $R = 100$ and 3 values of D:



Note that:

- The hat-like structure moves in the positive x-direction, as time passes.
- The shape of the structure never changes.

Cross-section of $N_x = -a\Delta/C$ at $y = z = t = 0$, for $R = 100$, $D = 10$ and 3 values of a:

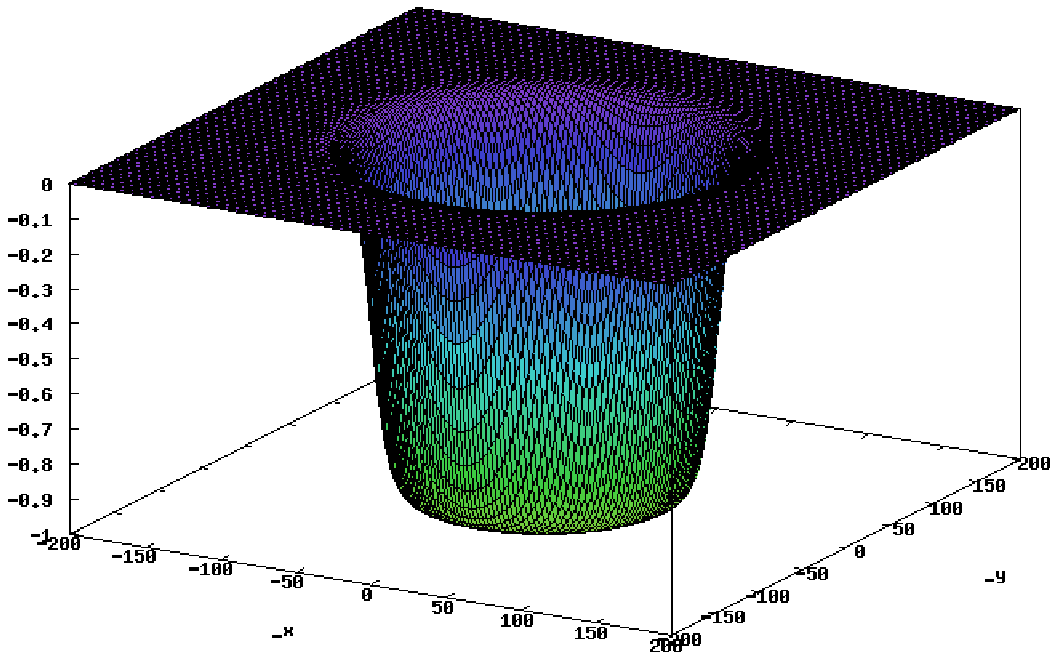


Note that:

- Multiplying Δ/C by $-a$ inverts the hat-like structure and scales it, such that the inverted hat always has depth a.
- The inverted hat also moves in the positive x-direction, as time passes, without changing its shape.

- Since N_x determines the spacetime geometry and does not change its shape with the passage of time, all geometric characteristics can be determined at $t = 0$ without loss of generality.

Embedding diagram of N_x at $z = 0$, for $R = 100$, $D = 10$:



Note that:

- The horizontal plan is the x, y-plane.
- The spatial dimension associated with z is supressed.
- The vertical axis corresponds to values of N_x.

2 Characteristics of Alcubierre Spacetime

We will calculate three characteristics of Alcubierre spacetime:

- The extrinsic curvature of the space-like hypersurfaces
- The expansion factor
- The Eulerian energy density

2.1 Extrinsic Curvature

In all generality, the extrinsic curvature K_{ij} tensor on the space-like hypersurfaces is given (in the 3+1 formalism) by

$$K_{ij} = \frac{1}{2N}\left(\nabla_i^{(3)} N_j + \nabla_j^{(3)} N_i - \frac{\partial h_{ij}}{\partial t}\right),\ i, j = 1, 2, 3 \tag{9}$$

where $\nabla_i^{(3)}$ is the covariant derivative on the 3-dimensional, space-like hypersurfaces.

Since covariant derivatives of tensor fields have not yet been describe (in previous talks), they are introduced here:

The covariant derivative is the ordinary derivative, plus one correction term for each contravariant index and minus one correction term for each covariant index, where the correction terms involve the connection coefficients. In particular, this prescription yields the following equations for contravariant and covariant vectors:

$$\nabla_j \xi^i = \partial_j \xi^i + \mathbf{\Gamma}^i_{\ aj} \xi^a \tag{10}$$

$$\nabla_j \xi_k = \partial_j \xi_k - \mathbf{\Gamma}^a_{\ jk} \xi_a \tag{11}$$

Recall that:

- The connection coefficients are defined in terms of the partial derivatives of the metric tensor along the coordinate directions.
- The 3-dimensional, space-like hypersurfaces are flat and choice of coordinate system is Cartesian, i.e. $h_{ij} = \delta_{ij}$.

Thus, the correction terms are zero, so according to (10) and (11), the covariant derivates are just the ordinary derivatives.

For Alcubierre:

- The covariant derivatives in (9) may be replaced by ordinary diervatives (as discussed in the box).
- In addition, $h_{ij} = \delta_{ij}$ implies $\partial h_{ij}/\partial t = 0$.
- $N = 1$

Thus, (9) simplifies to

$$K_{ij} = \frac{1}{2}\left(\partial_i N_j + \partial_j N_i\right) \tag{12}$$

and recalling that, for Alcubierre, $N_y = 0$ and $N_z = 0$, (12) yields the following components:

$$\begin{aligned}
K_{xx} &= \frac{1}{2}\left(\partial_x N_x + \partial_x N_x\right) = \partial_x N_x \\
K_{xy} &= \frac{1}{2}\left(\partial_x N_y + \partial_y N_x\right) = \frac{1}{2}\partial_y N_x \\
K_{xz} &= \frac{1}{2}\left(\partial_x N_z + \partial_z N_x\right) = \frac{1}{2}\partial_z N_x \\
K_{yx} &= \frac{1}{2}\left(\partial_y N_x + \partial_x N_y\right) = \frac{1}{2}\partial_y N_x \\
K_{yy} &= \frac{1}{2}\left(\partial_y N_y + \partial_y N_y\right) = 0 \\
K_{yz} &= \frac{1}{2}\left(\partial_y N_z + \partial_z N_y\right) = 0 \\
K_{zx} &= \frac{1}{2}\left(\partial_z N_x + \partial_x N_z\right) = \frac{1}{2}\partial_z N_x \\
K_{zy} &= \frac{1}{2}\left(\partial_z N_y + \partial_y N_z\right) = 0 \\
K_{zz} &= \frac{1}{2}\left(\partial_z N_z + \partial_z N_z\right) = 0
\end{aligned}$$

or using matrix representation, the extrinsic curvature is

$$[K_{**}] = \begin{bmatrix} \partial_x N_x & \frac{1}{2}\partial_y N_x & \frac{1}{2}\partial_z N_x \\ \frac{1}{2}\partial_y N_x & 0 & 0 \\ \frac{1}{2}\partial_z N_x & 0 & 0 \end{bmatrix} \tag{13}$$

2.2 Expansion Factor

In all generality, the expansion factor θ is defined by

$$\theta := -NK \tag{14}$$

where K is the trace of the extrinsic curvature tensor. Recall that when taking the eigenvalues of a rank-2 tensor, we must, in general, use the mixed form of the tensor. Since the trace is the sum of the eigenvalues, taking the trace of a rank-2 tensor also requires the mixed form. For the extrinsic curvature, this is given by

$$K^i_j = h^{ia} K_{aj} \tag{15}$$

or in matrix form

$$[K^*_*] = [h^{**}] [K_{**}] \tag{16}$$

and so the trace is given by

$$K = \mathrm{tr}\,[K^*_*] = \sum_{i=1}^{3} K^i_i. \tag{17}$$

However, since (for Alcubierre) $h_{ij} = \delta_{ij}$, we have $h^{ij} = \delta^{ij}$, so using (16) we have

$$[K^*_*] = I_{3\times 3} [K_{**}] = [K_{**}] \tag{18}$$

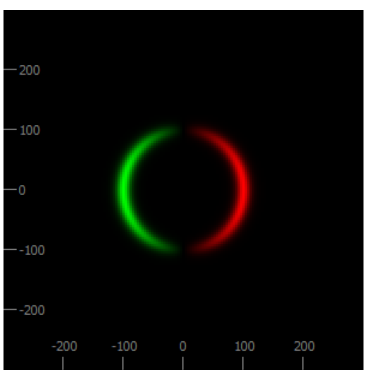
and using (13) and (18), (17) reduces to

$$K = \mathrm{tr}\,[K_{**}] = \partial_x N_x. \tag{19}$$

Finally, substituting (19) into (14) and recalling once again that $N = 1$, we have

$$\theta = -\partial_x N_x. \tag{20}$$

Expansion factor cross-section at $z = 0$, with $D = 10$, $R = 100$ and $a = 1$

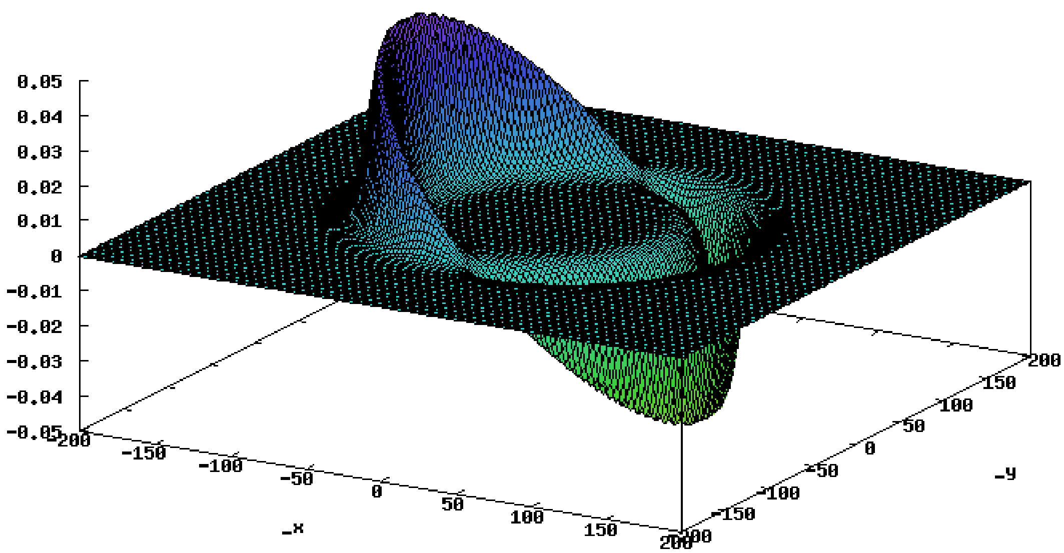


Note that:

- x-axis and y-axis are horizontal and vertical, respectively.
- z-axis is perpendicular to the image.
- The expansion θ is essentially zero in the black region.
- The expansion θ is positive in the green region and negative in the red region.
- The more intense the green the greater the expansion.
- The more intense the red, the greater the contraction.

- $\theta_{\max}$ is at $x = -R$ and $y = 0$; $\theta_{\min}$ is at $x = R$ and $y = 0$;
- The region where $\theta \neq 0$ forms a thin, spherical shell, such that the $\theta > 0$ in the hemisphere where $x < 0$ and $\theta < 0$ in the hemisphere where $x > 0$.
- This spherical shell moves at the speed of light (because $a = 1$) in the positive x-direction.
- A vehicle (with observers) in the interior of the shell is carried along as the shell moves.
- These observers experience no forces and no time dilation.

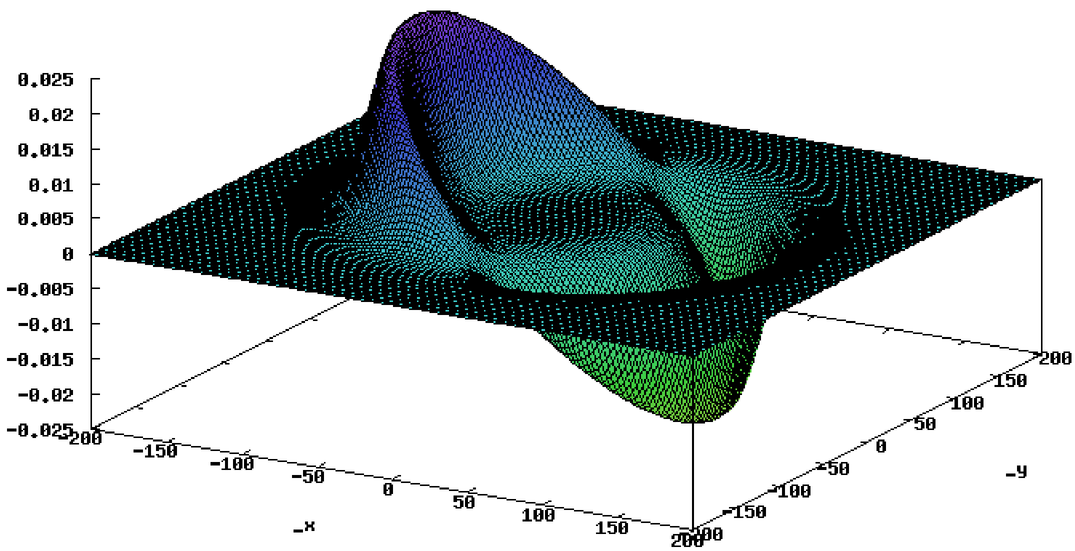
Embedding diagram of expansion factor θ at $z = 0$, for $R = 100$, $D = 10$ and $a = 1$:



Note that:

- The horizontal plane is the xy-plane.
- The spatial dimension corresponding to z is supressed.
- The vertical axis corresponds to θ, so we can see the actual numerical values.

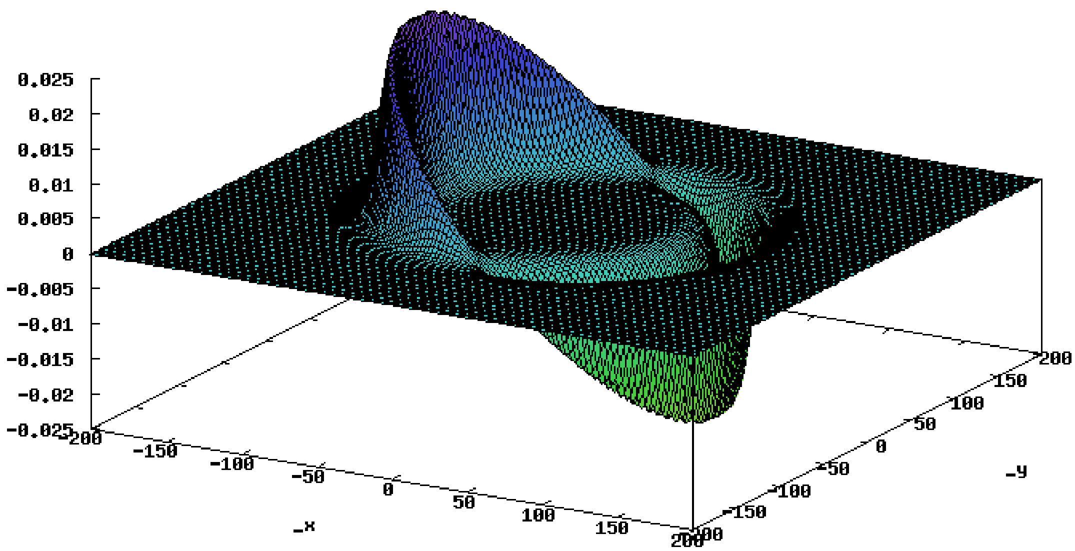
Expansion factor at $z = 0$, with $D = 20$, $R = 100$ and $a = 1$



Note that:

- The shell thickness is proportional to D.
- The maximum and minimum expansion are inversely proportional to D.

Expansion factor at $z = 0$, with $D = 10$, $R = 100$ and $a = 0.5$



Note that, the maximum and minimum expansion are proportional to a.

2.3 Eulerian Energy Density

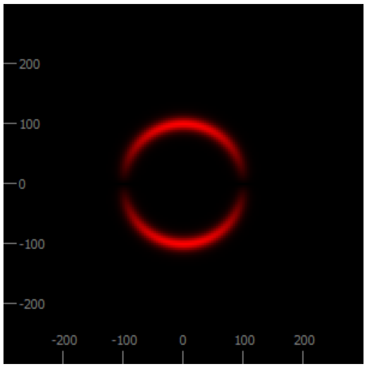
In all generality, the Eulerian energy density ρ is given by

$$\rho = \frac{1}{16\pi}\left(R^{(3)} + K^2 - K_{ij}K^{ij}\right) \tag{21}$$

where $R^{(3)}$ is the Ricci scalar on the space-like hypersurfaces. For Alcubierre, (as we have seen) these are instrinsically flat, which implies that $R^{(3)} = 0$ and $[K^{**}] = [K_{**}]$, so (21) reduces to

$$\rho = \frac{1}{16\pi}\left(K^2 - \sum_{i=1}^{3}\sum_{j=1}^{3} K_{ij}^2\right) \tag{22}$$

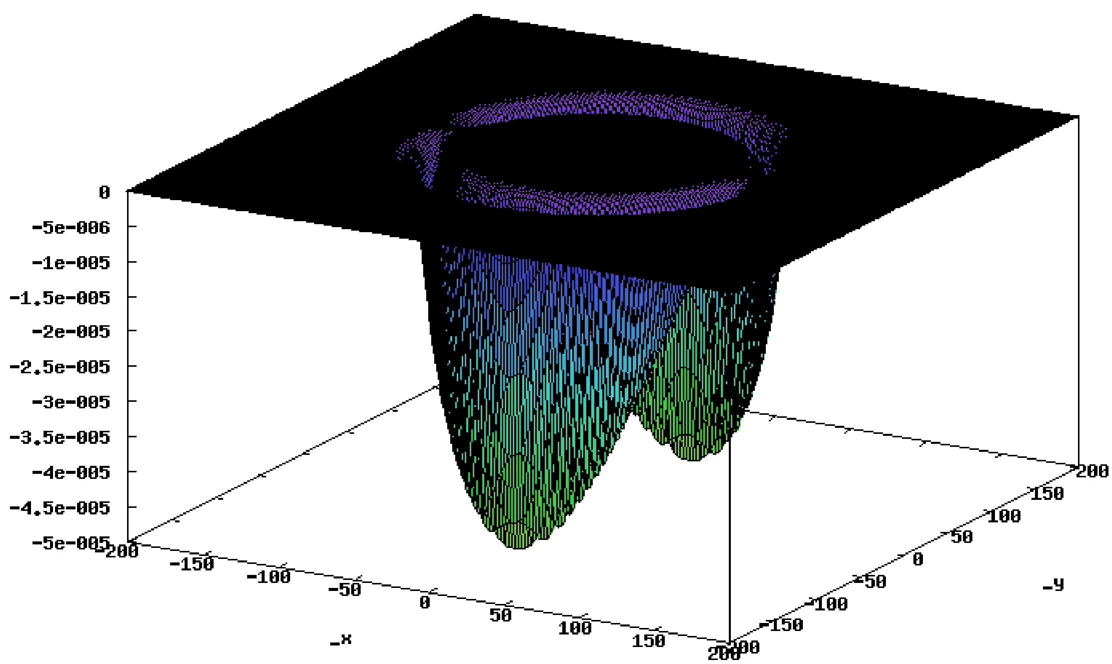
Eulerian energy density cross-section at $z = 0$, with $D = 10$, $R = 100$ and $a = 1$



Note that

- x-, y- and z-axes are defined as expansion factor cross-section.
- Color coding has the same meaning as expansion factor cross-section.
- All energy densities are negative.
- The region where $\rho \neq 0$ is forms a thin, spherical shell, but the regions of greatest intensity (of negative energy density) are perpendicular to the direction of motion, i.e. $|\rho|_{\max}$ is at $x = 0$ and $y = \pm R$.
- This spherical shell is what would have to be engineered in order to make this warp drive a reality.

Embedding diagram of Eulerian energy density ρ at $z = 0$, for $R = 100$, $D = 10$ and $a = 1$:

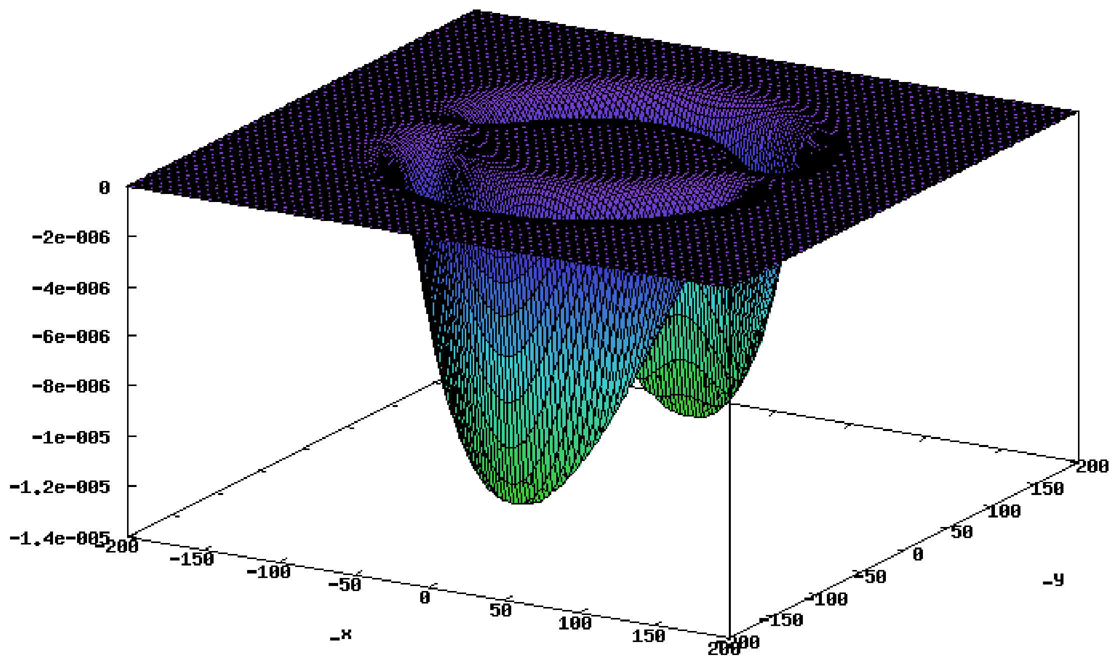


Note that:

- The horizontal plane is the xy-plane.

- The spatial dimension corresponding to z is supressed.
- The vertical axis corresponds to ρ, so we can see the actual numerical values.

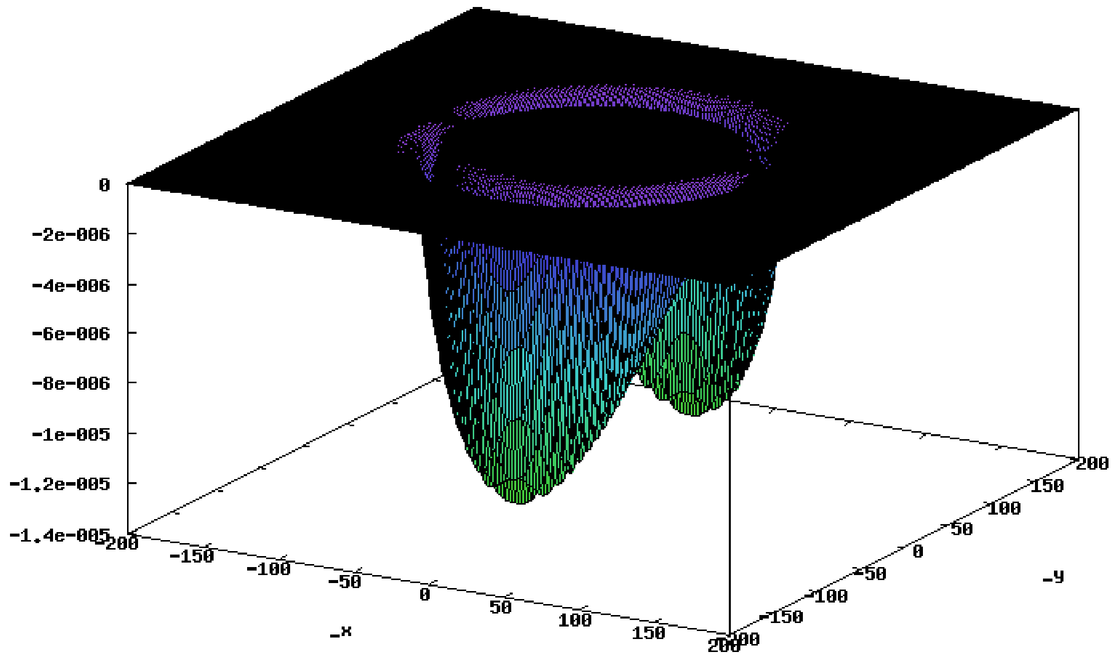
Embedding diagram of Eulerian energy density ρ at $z = 0$, with $D = 20$, $R = 100$ and $a = 1$



Note that:

- The shell thickness is proportional to D.
- The amount of exotic matter required descreases with increasing D.

Embedding diagram of Eulerian energy density ρ at $z = 0$, with $D = 10$, $R = 100$ and $a = 0.5$

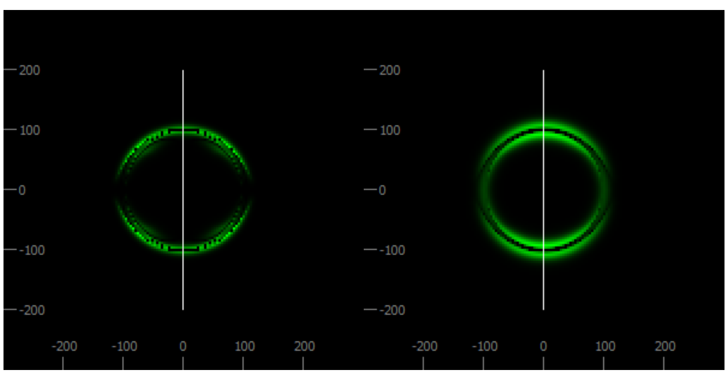


Note that the amoutn of exotic matter required decreases with a.

3 Energy Conditions

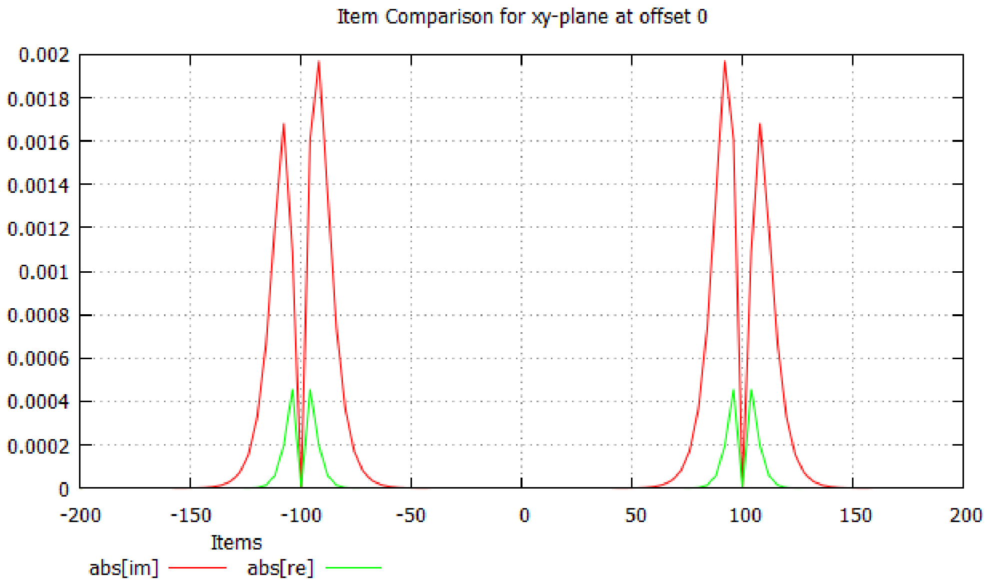
Since the Eulerian energy density is negative within the spherical shell, we already know that the WEC is violated. Suppose, now, that we go ahead and analyze the eigenvalues of the (mixed) energy tensor. The result is that within the spherical shell the there are complex eigenvalues.

Magnitude of real imaginary parts of complex conjugat pair



Images show that non-zero eigenvalues are confined to the spherical shell.

Graph of real and imaginary magnitudes along path shown above (white line)



Where eigenvalues are non-zero, imaginary parts dominate. Thus, there is no doubt that the Hawking-Ellis type is IV, which means all energy conditions are violated.