The Fell-Heisenberg Drive

June 28, 2023

1 Fell-Heisenberg Ansatz

In 2021, physicists Shaun Fell and Lavinia Heisenberg considered a time-like foliation of spacetime which satisifies the Alcubierre conditions

- The space-like hypersurfaces are intrinsically flat.
- The chosen coordinate system on the space-like hypersurfaces is Cartesian, with coordinates (x, y, z)
- The lapse function is given by $N = 1$

but wherein the shift vector is different.

1.1 Fell-Heisenberg Parameters

There is a pair of quantities appearing the F-H paper, which I call $q_+$and q_-, that are defined as follows:

$$q_\pm := \frac{\rho \pm \frac{r^{2P}}{w_\pm(x,y,z)}}{\sqrt{\sigma}} \tag{1}$$

where r is a variable giving radial distance from the origin of the Cartesian coordinate system, i.e.

$$r = \sqrt{x^2 + y^2 + z^2} \tag{2}$$

and

- P is a constant parameter, restricted to the half-open interval $(0, 0.5]$.
- ρ and σ are constant parameters, to be redefined shortly.
- w_+ and w_- are arbitrary, smooth functions of the Cartesian coordinates.

It will be easier to understand (1), if the parameters ρ and σ are recast in terms of the Alcubierre parameters R and D. Recall that, in Alcubierre:

- R the radial distance to the midpoint of the spherical shell, centered at the origin.
- D is rouglhy half the thickness of that shell.

Specifically, define

$$\rho := R^{2P} \tag{3}$$
$$\sigma := D^{4P} \tag{4}$$

Then, (1) can be rewritten as

$$q_{\pm} = \frac{R^{2P} \pm \frac{r^{2P}}{w_{\pm}}}{D^{2P}} \tag{5}$$

Now, if $w_+ = w_- = 1$ and $P = 1/2$, then (5) reduces to

$$q_{\pm} = \frac{R \pm r}{D} \tag{6}$$

which is equivalent to the definitions of r_+ and r_-, respectively, in the Alcubierre ansatz. Thus (5) can be seen as a generalization of (6), with the latter as a limiting case. It will be seen that, despite a significant difference in shift vectors, the F-H ansatz, gives rise to a spherical shell that contains the bulk of the non-zero energy density, spacetime curvature and expansion factor, where the radius and thickness of the shell is controlled via R and D, respectively.

1.2 Fell-Heisenberg Shift Vector Potential

Unlike Alcubierre, the F-H paper does not specify a shift vector directly. Instead, it specifies a scalar field ϕ (over the hypersurfaces), whose gradient is the shift vector. This *shift vector potential* is given (in terms of the recast parameters) by

$$\phi = S \frac{w_+ w_-}{w_+ + w_-} D^{4P} \left\{ \Sigma_1 + \sqrt{\pi} \Sigma_2 \right\} \tag{7}$$

where S is an additional parameter providing an overall scale, and where Σ_1 and Σ_2 are given by

$$\Sigma_1 = \exp\left(-q_-^2\right) + \exp\left(-q_+^2\right) \tag{8}$$

$$\Sigma_2 = q_- \text{erf}\left(q_-\right) + q_+ \text{erf}\left(q_+\right) \tag{9}$$

where exp and erf are the exponential and error functions, respectively, and q_- and q_+ are given by (5).

1.2.1 The Spherically Symmetric Case

It turns out that if $w_+ = w_- = 1$, the shell is spherically symmetric. Also, since S is arbitrary, we can conveniently absorb the constants in (7) into S and set it to 1. That is, we may write

$$\phi = \Sigma_1 + \sqrt{\pi} \Sigma_2. \tag{10}$$

Then, when the shift vector is obtained by taking the gradient of (10), i.e.

$$N_i = \frac{\partial \phi}{\partial x^i} = \frac{\partial \Sigma_1}{\partial x_i} + \sqrt{\pi} \frac{\partial \Sigma_2}{\partial x_i}, \tag{11}$$

we obtain (after much algebra) the form

$$N_i = x_i r^{2P-2} \Delta \tag{12}$$

where

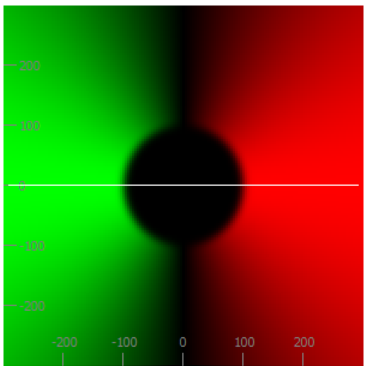
$$\Delta = \text{erf}\left(q_+\right) - \text{erf}\left(q_-\right). \tag{13}$$

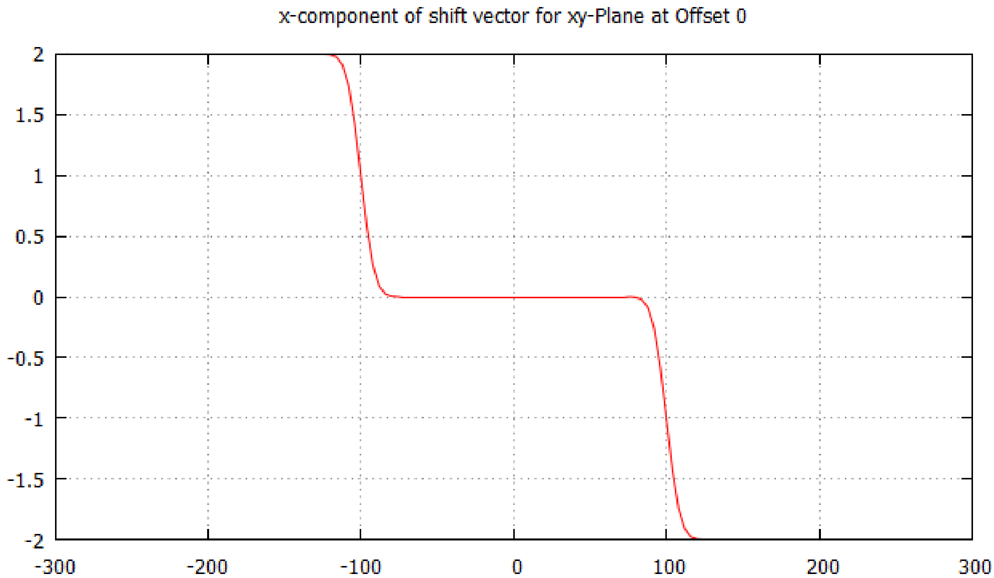
Note that, given N_i, the extrinsic curvature tensor K_{**} and therefore the expansion factor and the Eulerian energy density can be computed as shown in the preceding talk.

1.3 Results for the Spherically Symmetric Case

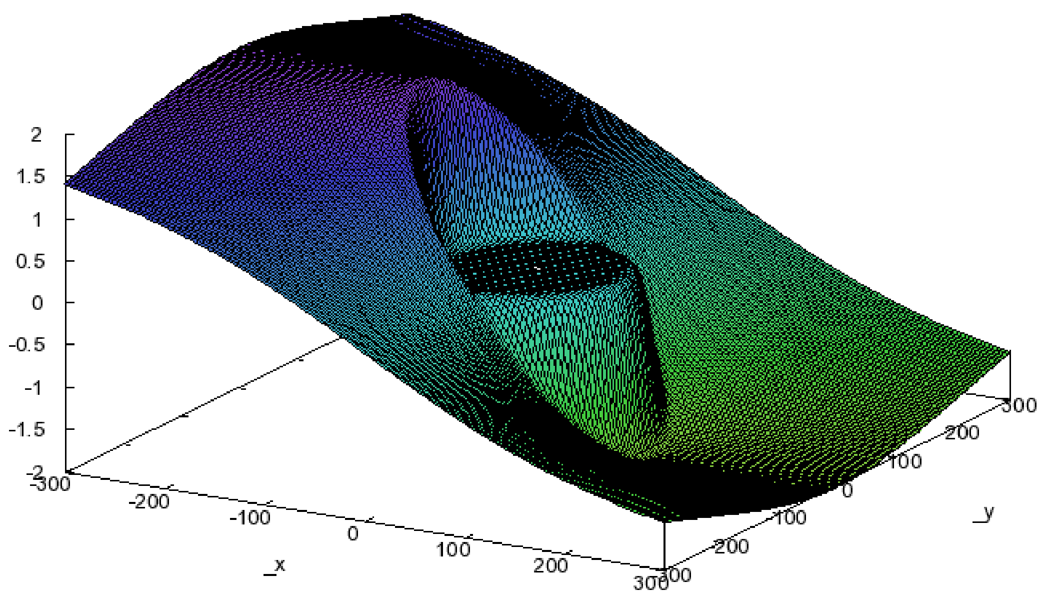
1.3.1 Shift Vector

N_x **cross-section at** $z = 0$**, with** $P = 0.5$**,** $D = 10$**,** $R = 100$



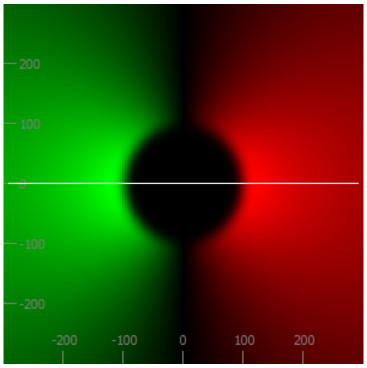


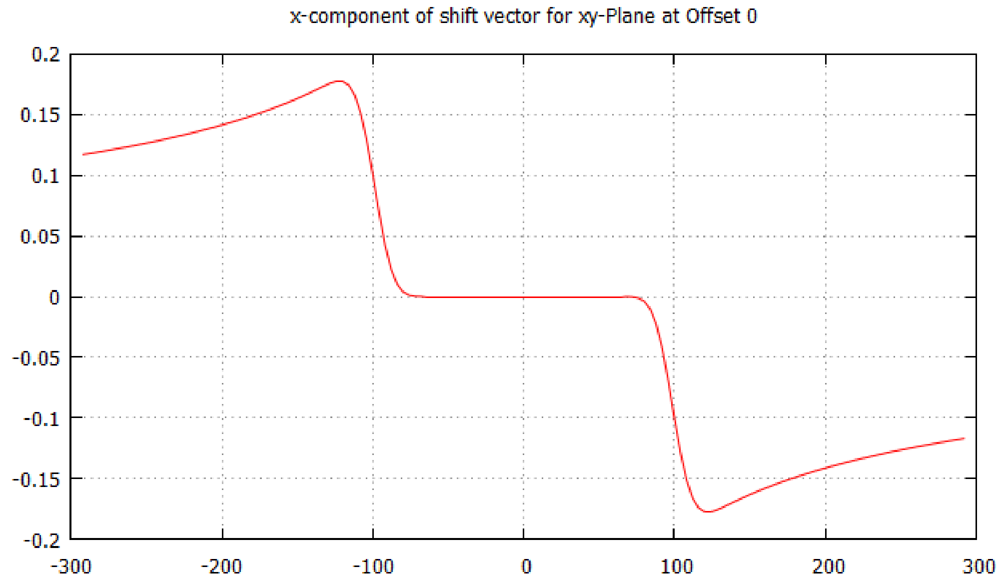
Corresponding embedding diagram



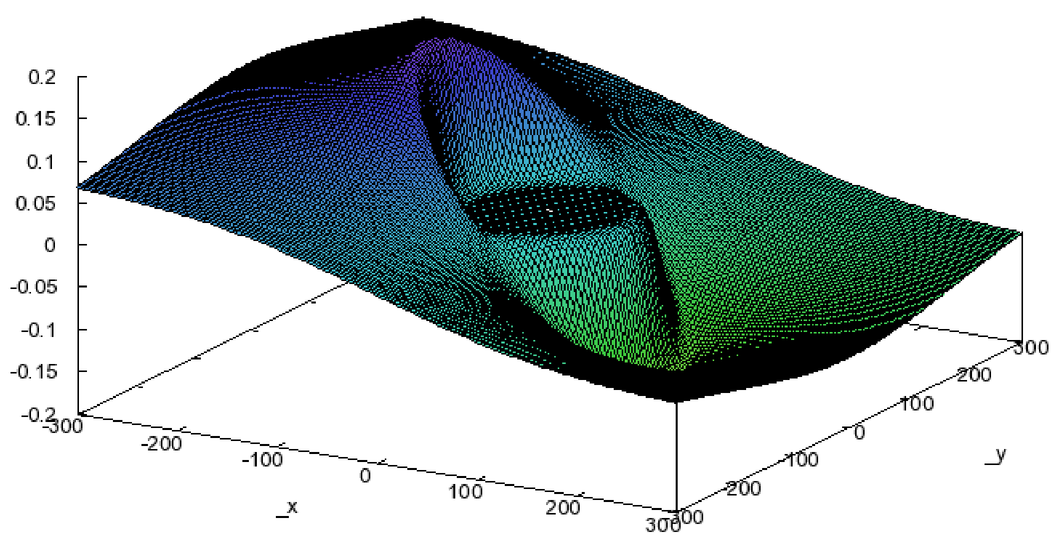
Note that: as $x \to \pm\infty, N_x \to \pm 2$.

N_x cross-section at $z = 0$, with $P = 0.25,\ D = 0.5,\ R = 100$





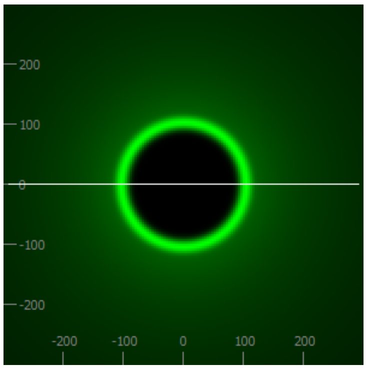
Corresponding embedding diagram

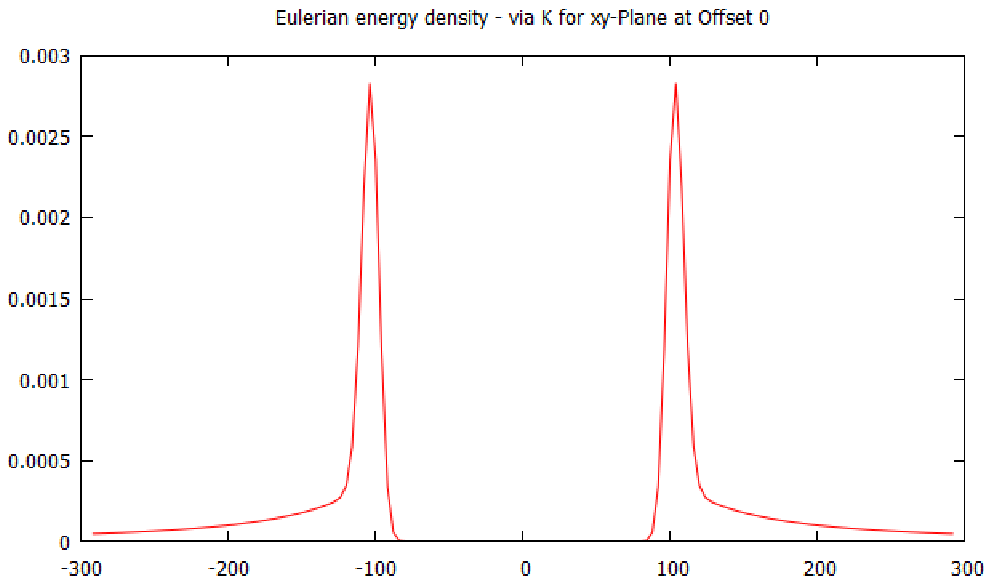


Note that: as $x \to \pm\infty, N_x \to 0$

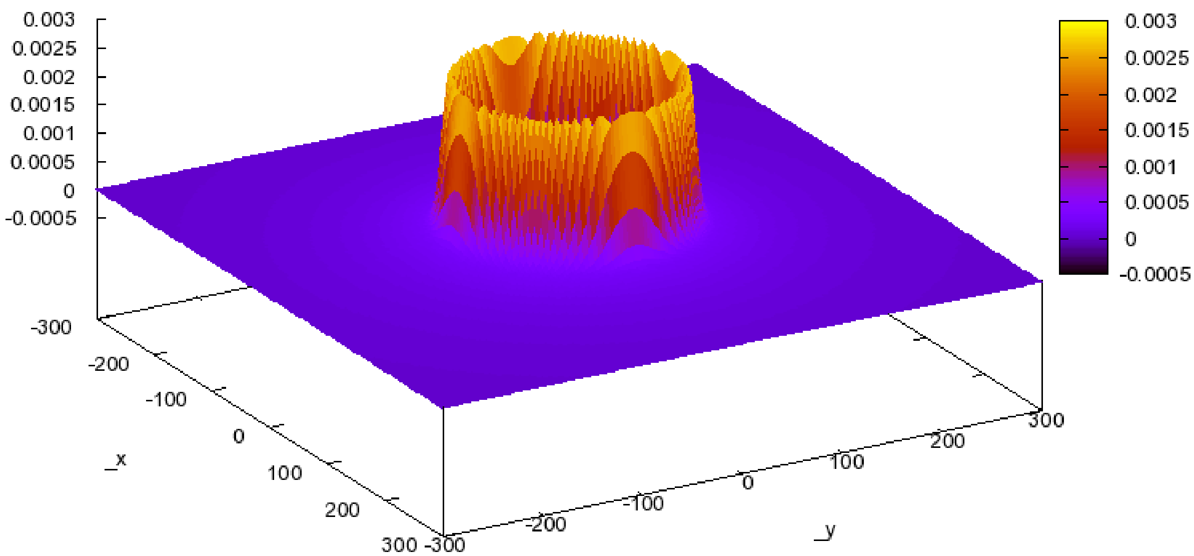
1.3.2 Energy Density

Eulerian energy density cross-section at $z = 0$, with $P = 0.5$, $D = 10$, $R = 100$





Corresponding embedding diagram

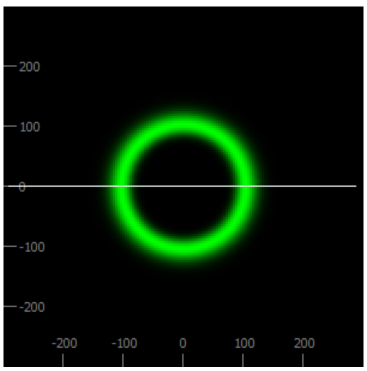


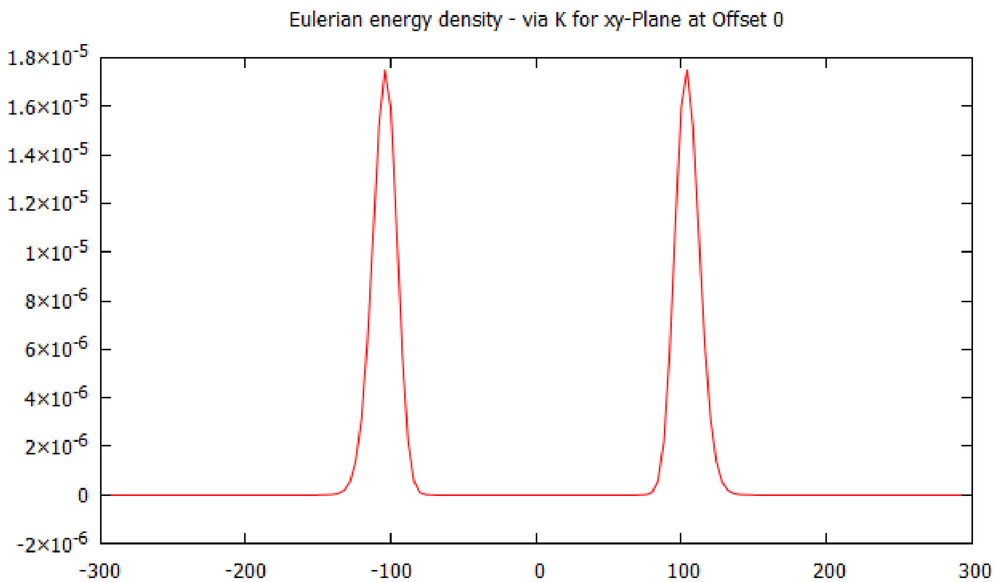
Note that:

- Distribution of energy density is spherically symmetric.

- The bulk of the energy is within the spherical shell at $r = 100$.
- There is, however, a non-zero energy density outside the shell, that tapers off as $r \to \infty$.
- The region enclosed by the shell has essentially zero energy density.

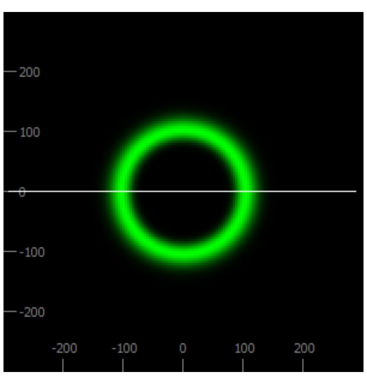
Eulerian energy density cross-section at $z = 0$, with $P = 0.25$, $D = 0.5$, $R = 100$

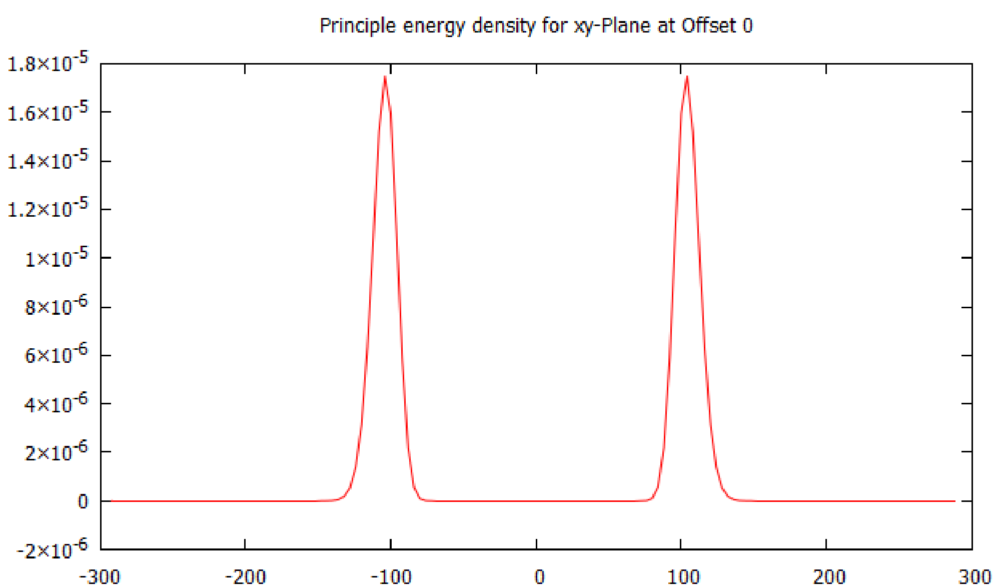




Note that: the energy density is confined within the shell.

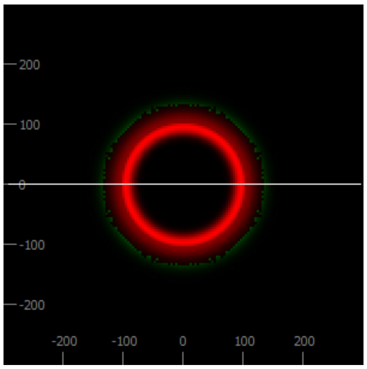
Principle energy density cross-section at $z = 0$, with $P = 0.25$, $D = 0.5$, $R = 100$

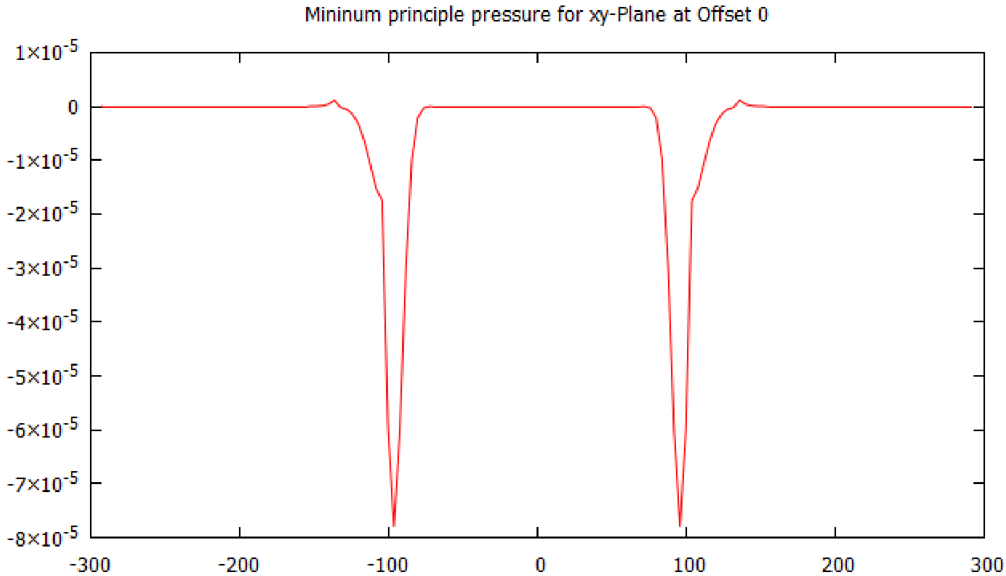




Note that: the principal energy density is identical to the Eulerian energy density.

Minimum principal pressure at $z = 0$, with $P = 0.25,\ D = 0.5,\ R = 100$

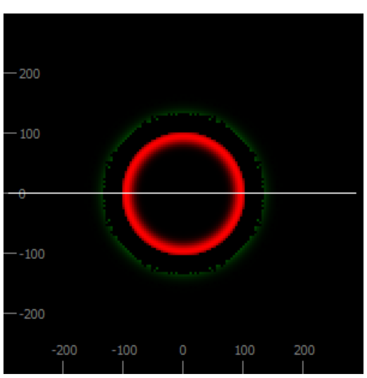


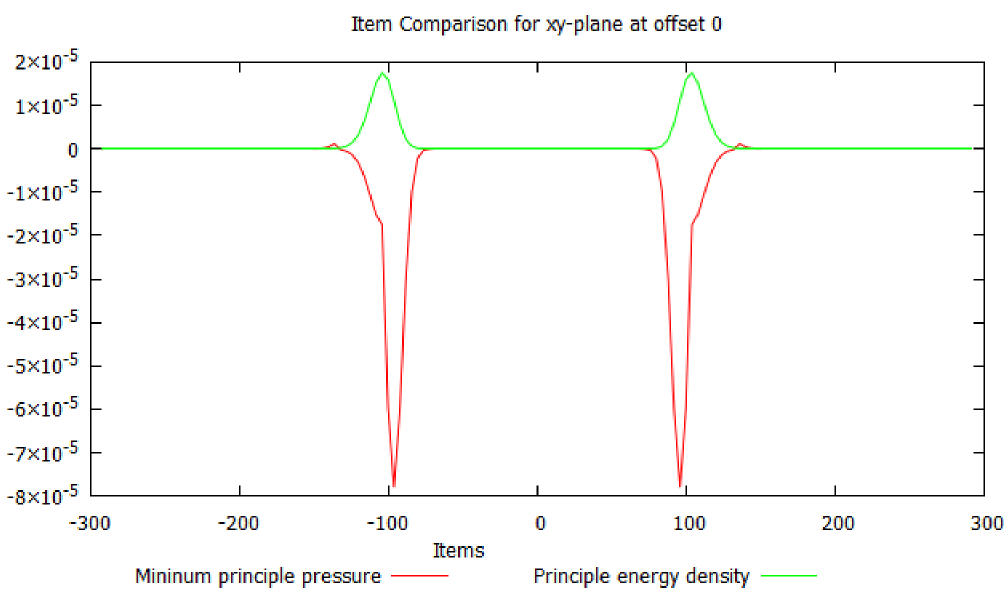


Note that:

- The bulk of the non-zero (negative) pressure resides in the spherical shell at $r = R = 100$.
- There is a faint (negative) secondary shell just outside the first.
- There is an extremely faint (positive) tertiary shell outside the second.
- The region enclosed by the primary shell has essentially zero pressure.
- The maximum magnitude of the minimum principal pressure is about four times that of maximum principal energy density.

Sum of principal energy density and minimum principle pressure



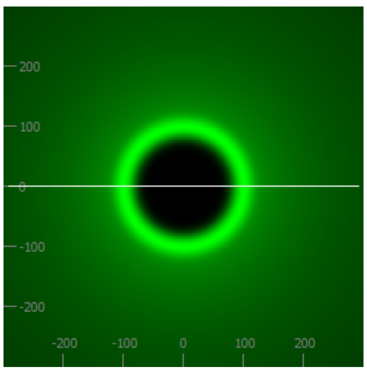


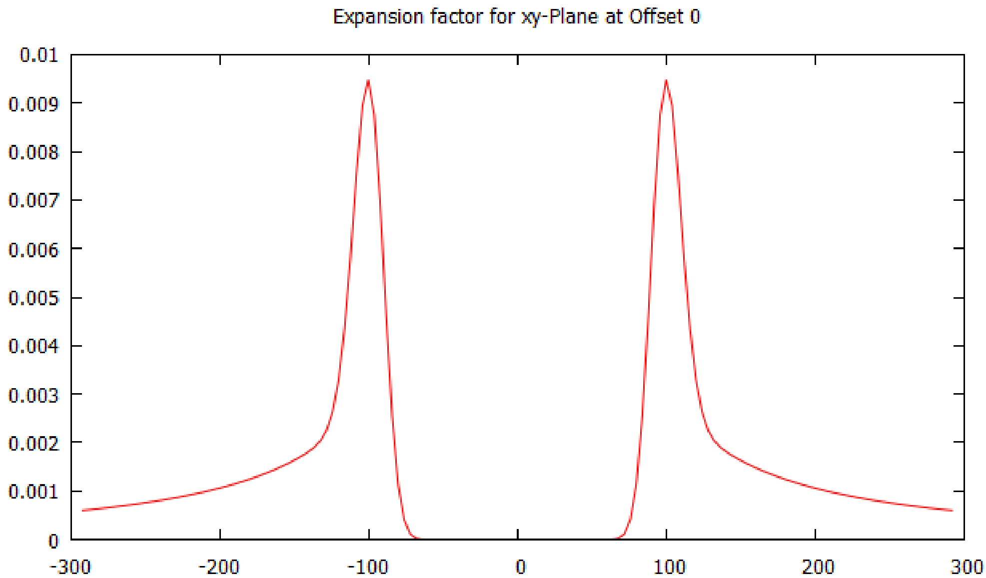
Note that:

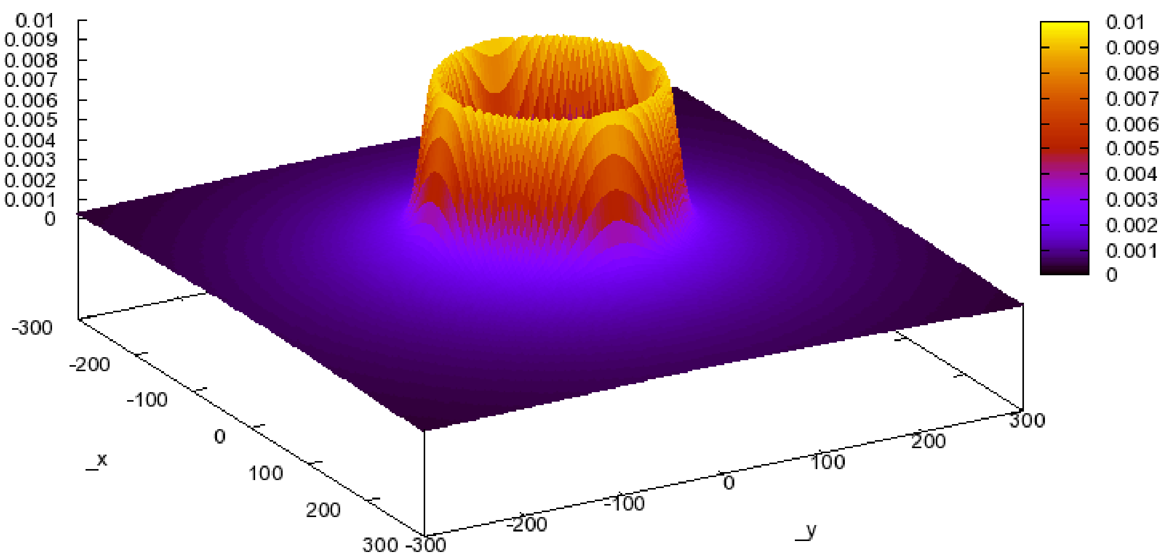
- There is a faint net positive shell at $r > R$.
- There is a net negative shell at $r = R$, i.e. a region where $p + \rho < 0$.
- This means that the WEC is violated (despite the 1st bullit).

1.3.3 Expansion Factor

Expansion factor cross-section at $z = 0$, with $P = 0.25$, $D = 0.5$, $R = 100$







Note that:

- The bulk of the non-zero expansion resides in a spherical shell at $r = R$.
- There is, however, a non-zero expansion outside the shell, that tapers off as $r \to \infty$.
- The expansion is spherically symmetric.
- This means that there is no movement.