The Fell-Heisenberg Drive
(Revisited)

July 20, 2023

1 Fell-Heisenberg Ansatz

In 2021, physicists Shaun Fell and Lavinia Heisenberg considered a time-like foliation of spacetime which satisifies the Alcubierre conditions

- The space-like hypersurfaces are intrinsically flat, i.e. $[h_{**}] = I_{3\times 3}$
- The chosen coordinate system on the space-like hypersurfaces is Cartesian, with coordinates (x, y, z)
- The lapse function is given by $N = 1$

but wherein the shift vector is different.

1.1 The Shift Vector Potential

In fact, the shift vector is not given directly. Instead, we are given

The shift vector potential: a scalar-valued function on the hypersurfaces whose gradient is the shift vector.

F and H actually produced two papers in 2021 (April and August), having two different forms of the shift vector potential, which reduce to the same thing in the spherically symmetric case, but are otherwise distinct, with the August form being considerably more complicated. For this reason, we will discuss the April form, which may be written compactly as follows:

$$\phi(x, y, z) = \frac{1}{2} V\sqrt{\sigma}\left(\Sigma_1 + \sqrt{\pi}\Sigma_2\right) + \frac{1}{4}\sqrt{\pi} V P z\left(\sqrt{\sigma_{\max}} - \sqrt{\sigma_{\min}}\right) \tag{1}$$

where Σ_1 and Σ_2 are sums defined by

$$\Sigma_1 := \exp\left(-q_-^2\right) + \exp\left(-q_+^2\right) - 2\exp\left(-q_0^2\right) \tag{2}$$

$$\Sigma_2 := q_- \mathrm{erf}\left(q_-\right) + q_+ \mathrm{erf}\left(q_+\right) - 2\rho\,\mathrm{erf}\left(q_0\right) \tag{3}$$

and where q_0, q_- and q_+ are quotients defined by

$$q_0 := \frac{\rho}{\sqrt{\sigma}} \tag{4}$$

$$q_\pm \quad := \quad \frac{\rho \pm r^{2P}}{\sqrt{\sigma}} \tag{5}$$

and finally $\sqrt{\sigma}$, ρ, P and V are parameters. It will be seen that, with these parameters in the proper range, (1) through (5) give rise to a family of spacetimes wherein the non-zero energy densities and pressures and therefore the non-zero curvatures and expansions are confined to a relatively thin, spherically symmetric shell (as in the Alcubierre ansatz).

Additional note: At the meeting, it was asked (I believe by Jose), where did this shift vector potential come from? I answered that in was essentially an educated guess (which is what an ansatz is), but as Bill correctly pointed out, there is more to it than that. Firstly, the F-H paper arrived about eight months after a paper by Erik Lentz, who used a shift vector potential that is not a smooth function, and claimed a finite, positive Eulerian energy. It looks like F-H were motivated to achieve the same thing using a smooth potential. In addition, they make an argument that any shift vector that is purely the gradient of a potential must result in finite, positive Eulerian energy. As for the actual form of their expression, it seems they wanted it to be spherically symmetric in the degenerate case ($\sigma = const$), with the idea that a simple change to this parameter would produce an appropriate asymmetry that generates motion.

1.2 Interpretation of Parameters

The purpose here is to give a physically meaningful interpretation of the parameters. We begin by redefining ρ as follows:

$$\rho \quad := \quad R^{2P} \tag{6}$$

where R is a new parameter representing the radius of the shell.

Next, we consider σ, for which F and H discuss two cases:

- σ is constant.
- σ is a Gaussian weighting function over the hypersurfaces.

We redefine σ as follows:

$$\sigma := D^{4P} w \tag{7}$$

where

- w is dimensionless, and is either 1 or a Gaussian weighting function in the range $(0, 1]$.
- D is a new parameter that determines the shell thickness. Specifically, D is half the thickness of the shell, when $w = 1$ and $P = 1/2$. With these values, the shell has a thickness of roughly $2D$, as in the Alcubierre ansatz. For $P <1/2$, D is related to the shell thickness in a complicated way. Testing has shown that for $P = 1/4$, D must decrease by two orders of magnitude, in order to maintain roughly the same shell thickness as in the $P = 1/2$ case.

Using definitions (6) and (7), definitions (3), (4) and (5) can be rewritten as

$$\Sigma_2 := q_- \mathrm{erf}\,(q_-) + q_+ \mathrm{erf}\,(q_+) - 2R^{2P} \mathrm{erf}\,(q_0) \tag{8}$$

$$q_0 := \frac{R^{2P}}{D^{2P}\sqrt{w}} \tag{9}$$

and

$$q_\pm \quad = \quad \frac{R^{2P} \pm r^{2P}}{D^{2P}\sqrt{w}}. \tag{10}$$

From (9) and (10), we see that the quotients are now manifestly dimensionless (as they should be as arguments to the exp and erf functions). Using these modified definitions, (1) may be rewritten as

$$\phi(x, y, z) = \frac{1}{2} V D^{2P} \sqrt{w} \left(\Sigma_1 + \sqrt{\pi}\Sigma_2\right) + \frac{1}{4}\sqrt{\pi} V P D^{2P} z \left(\sqrt{w_{\max}} - \sqrt{w_{\min}}\right) \tag{11}$$

Finally, we turn to the parameter V which is related to the velocity of the Eulerian observers. This can be made explicit, by defining a new parameter

$$S := \frac{1}{4}\sqrt{\pi} V P D^{2P} \tag{12}$$

where the RHS of (12) is the coefficient of z in the second term of (11). With S defined in this way, we obtain

$$V = \frac{4S}{\sqrt{\pi}PD^{2P}} \tag{13}$$

and substituting this in (11), we obtain

$$\phi(x, y, z) = \frac{2S}{P}\sqrt{w}\left(\Sigma_1/\sqrt{\pi} + \Sigma_2\right) + Sz\left(\sqrt{w_{\text{max}}} - \sqrt{w_{\text{min}}}\right) \tag{14}$$

This maintains the relative contributions from the two terms originally found on the RHS of (11). The advantage, as we will see shortly, is that the parameter S is now explicitly the speed of the Eulerian observers, which in (14) is intended to be along the postive z-axis. (This will be explained shortly.)

It will be convienent to give names to the two terms on the RHS of (14), so let us define

$$T_g := \frac{2S}{P}\sqrt{w}\left(\Sigma_1/\sqrt{\pi} + \Sigma_2\right) \tag{15}$$

and

$$T_v := Sz\left(\sqrt{w_{\text{max}}} - \sqrt{w_{\text{min}}}\right). \tag{16}$$

It will be seen that T_g and T_v determine, respectively, the geometry and velocity. Note that the factor $\left(\sqrt{w_{\text{max}}} - \sqrt{w_{\text{min}}}\right)$ in T_v serves as the means be which T_v is zero when $w = 1$ and $T_v = Sz$, otherwise. Thus, when w is a Gaussian weight, T_v adds a constant to the z-component of the shift vector, and zero to the other components. This can be generalized by introducing a new parameter $\boldsymbol{u}$, which is set to zero when $w = 1$, and is otherwise a unit vector given by

$$u_x := \cos(\alpha)\cos(\varepsilon) \tag{17}$$

$$u_y := \sin(\alpha)\cos(\varepsilon) \tag{18}$$

$$u_z := \sin(\varepsilon) \tag{19}$$

where

- α is a new parameter representing an azimuth angle in the xy-plane.
- ε is a new parameter representing the elevation angle, relative to the xy-plane.

Using these new definitions, (16) can be rewritten as

$$T_v := S\left(u_x x + u_y y + u_z z\right) \tag{20}$$

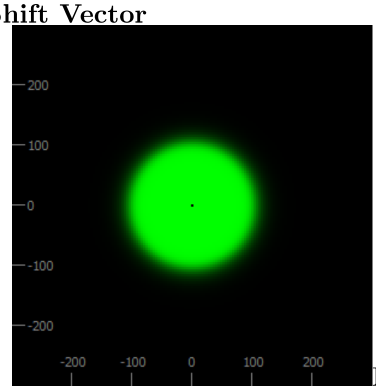
where $u_x = u_y = u_z = 0$ if $w = 1$, or given by (17) through (19) otherwise. In this formulation, S should be thought of as speed only in T_v; in T_g it should be thought of as just scale factor.

2 Results

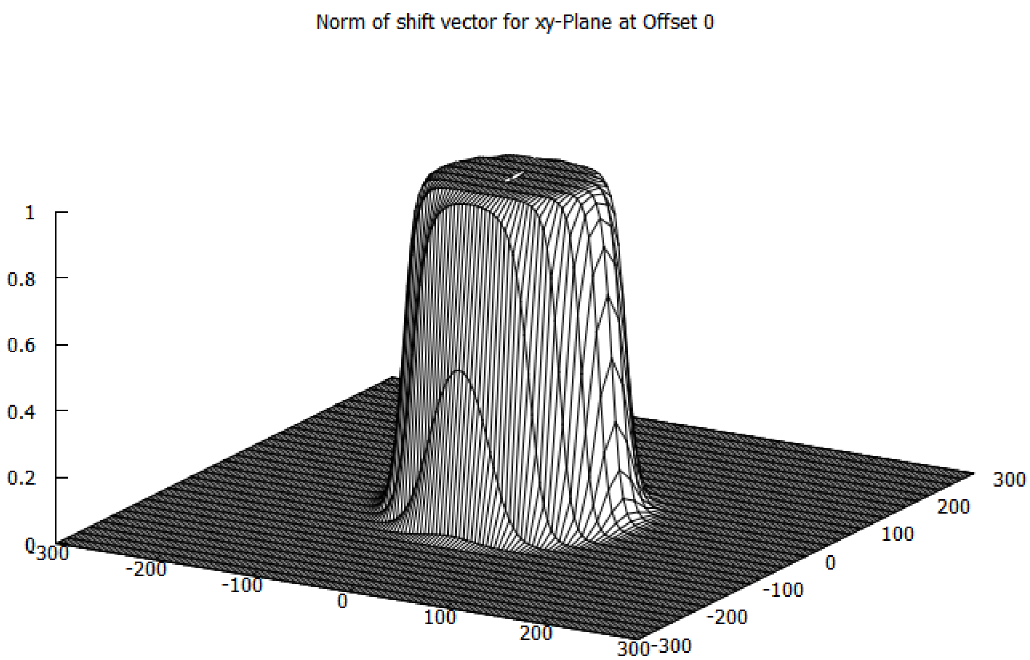
In cross-section images wherein scale if given by color intensity, green, red and black indicate positive, negative and zero (or nearly zero), respectively.

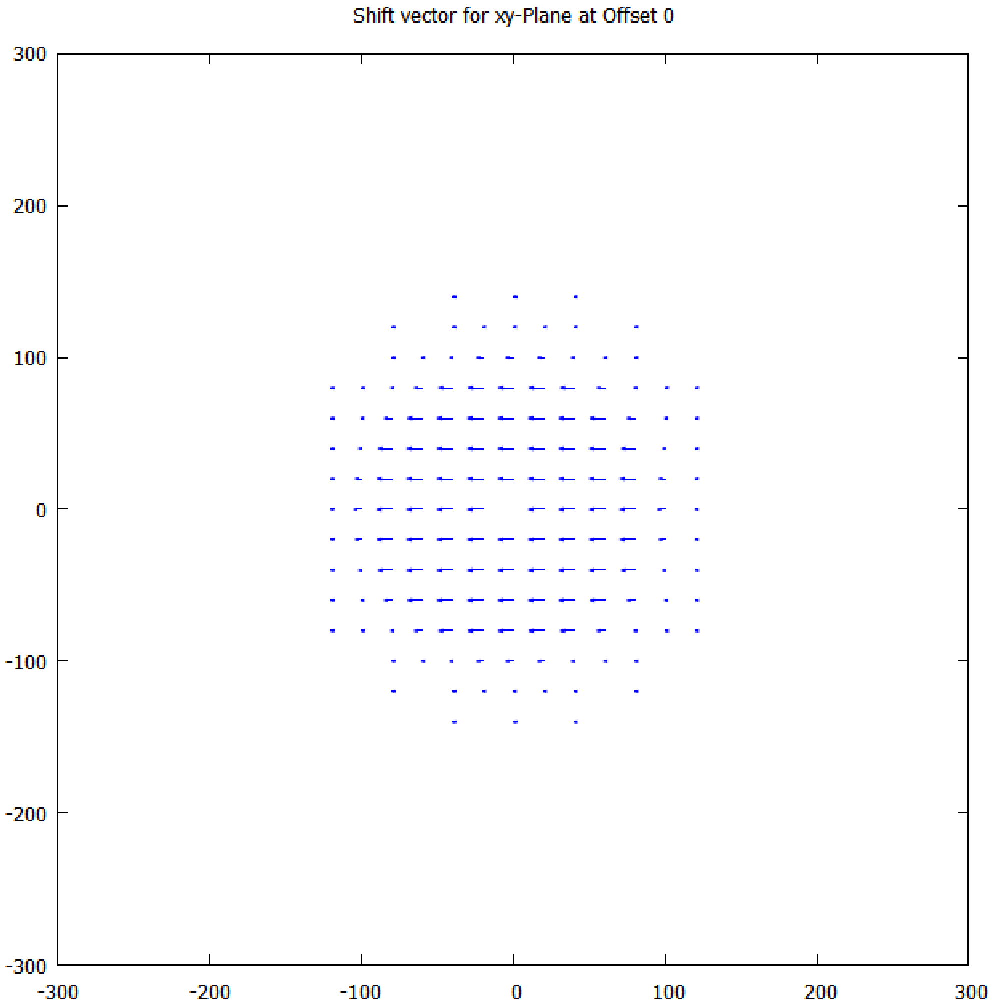
2.1 Review of Alcubierre

Shift Vector

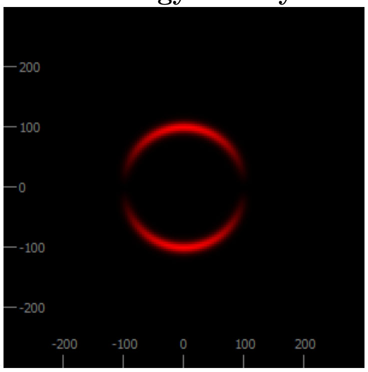


Norm

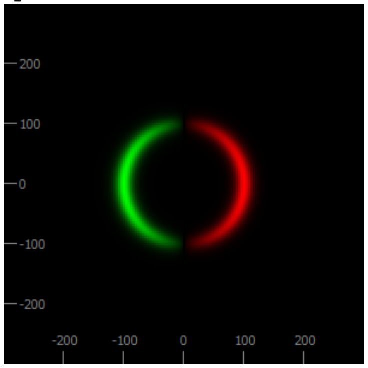




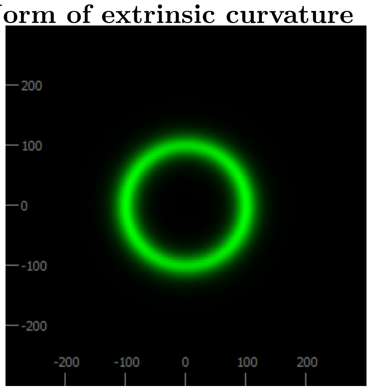
Eulerian energy density



Expansion factor



Norm of extrinsic curvature



2.2 Fell-Heisenburg: Spherically Symmetric Case

In the way of specific results, F and H say the following:

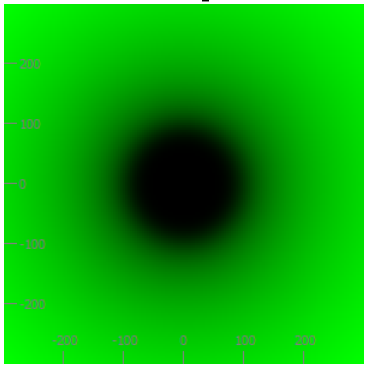
- If σ is constant, the geometry is spherically symmetric.
- "A simple modification to the Gaussian weight parameter results in a configuration possessing superluminal shift in the interior region. By increasing the Gaussian weight parameter in the +z region and decreasing it in -z region, the energy density correspondingly increases in the +z region and decreases in the -z region. This results in the gradient of the ϕ-field ;increasing in the region $r < R$, generating a high and level shift."
  THEY DO NOT PROVIDE THE ACTUAL FORM OF THE GAUSSIAN WEIGHT, just the hint (red).
  What sort of Gaussian weight $w(x, y, z)$ behaves like this?

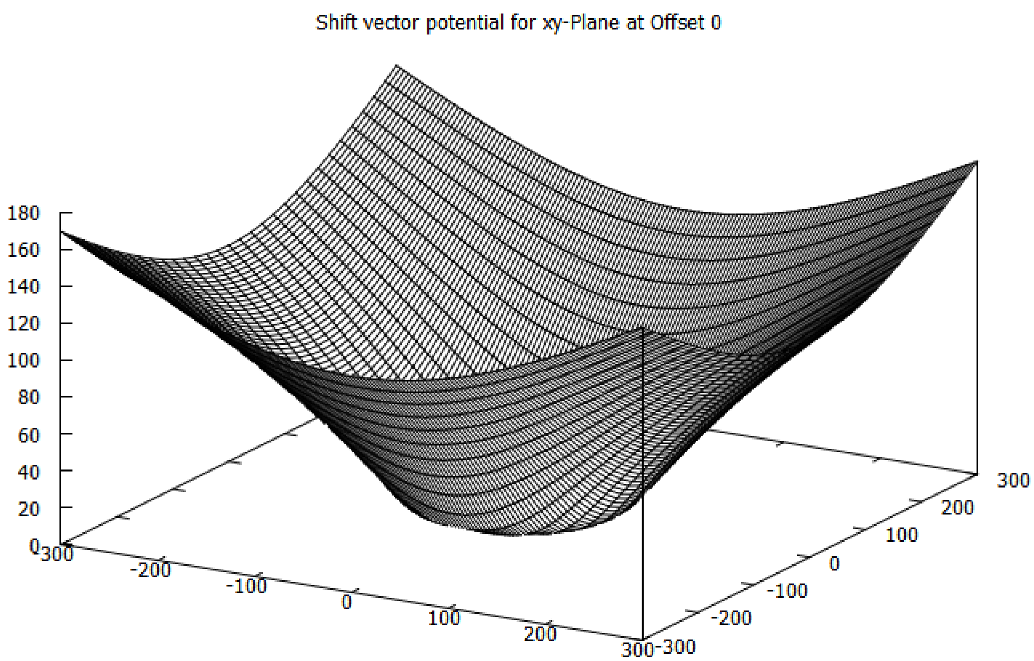
The claim that constant σ produces a spherically symmetric configuration is verified by the results below. All results correspond to the following parameter settings:

- $P = 1/4$
- $D = 1/10$
- $R = 100$
- $S = 1$

2.2.1 Using the Shift Vector Potential as Given by F-H

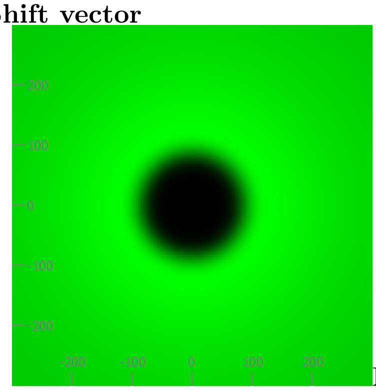
The shift vector potential

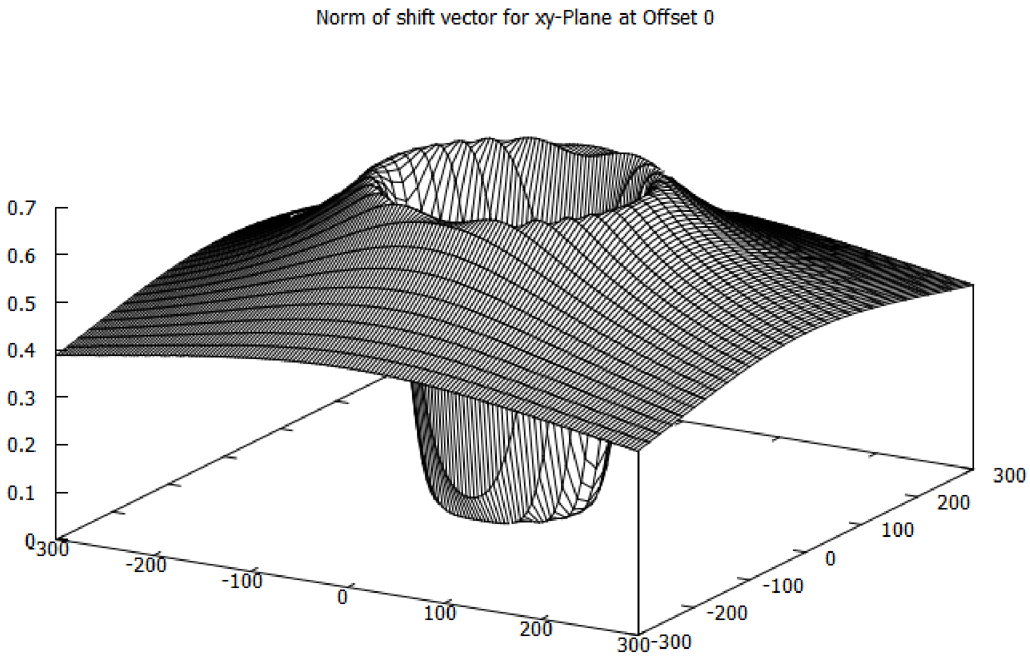




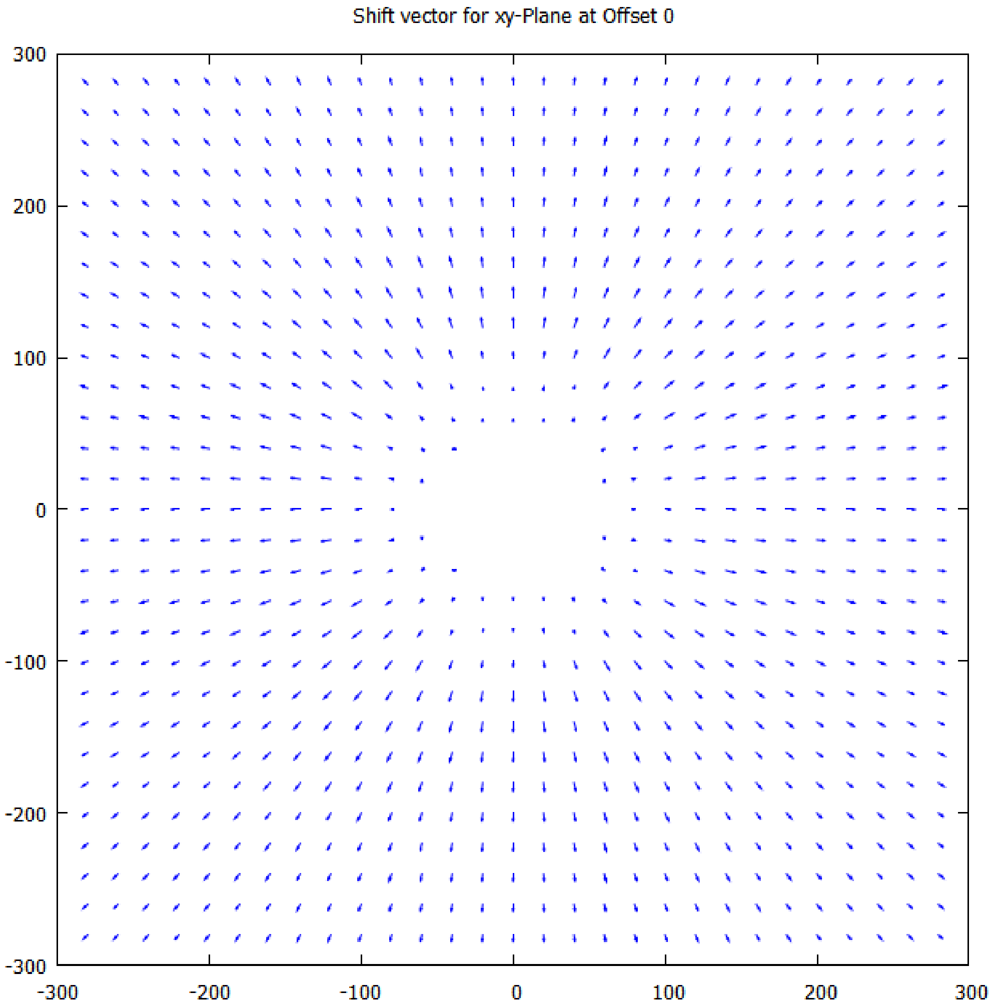
Embedding

Shift vector

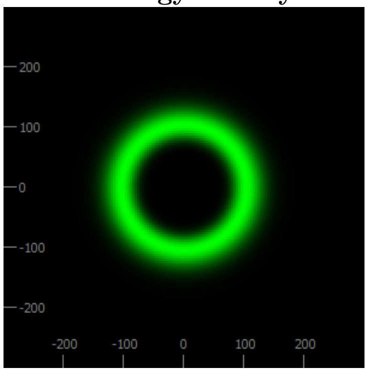
Norm



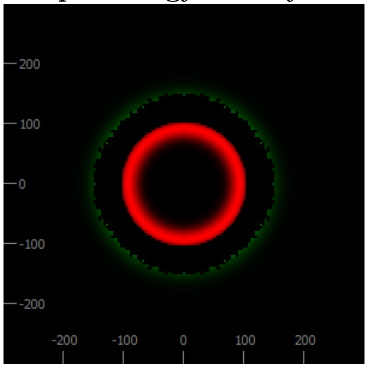
Norm (Embedding)



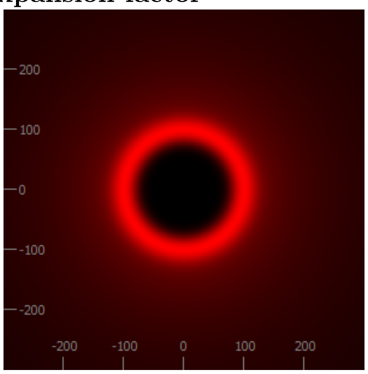
Eulerian energy density



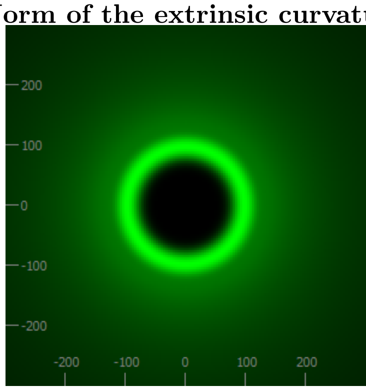
Principle energy density + minimum principle pressure



Expansion factor

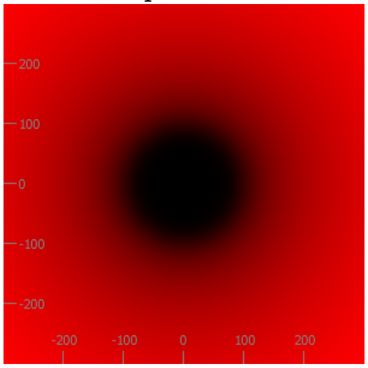


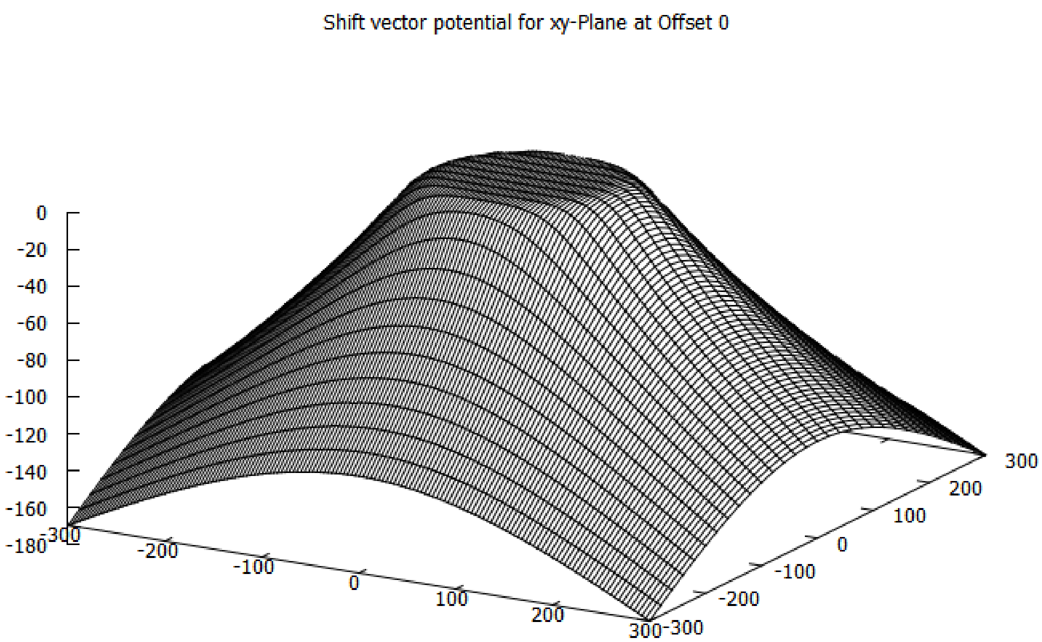
Norm of the extrinsic curvature



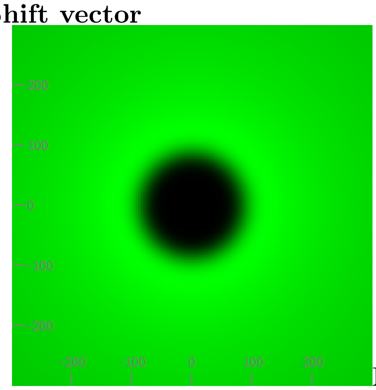
2.2.2 Reversing the Sign on the Shift Vector Potential

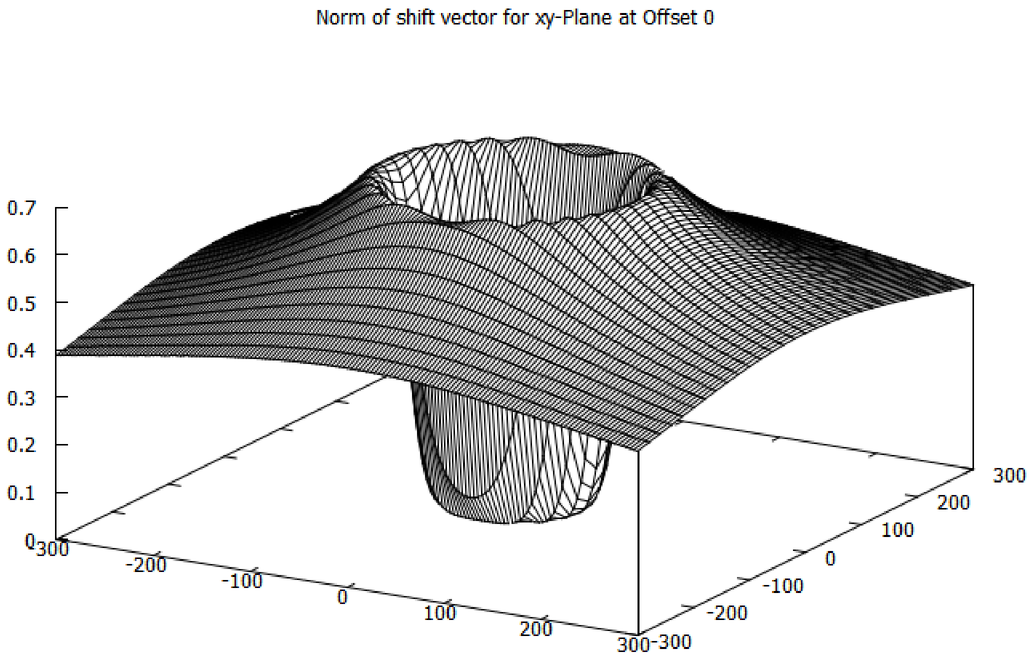
Shift vector potential



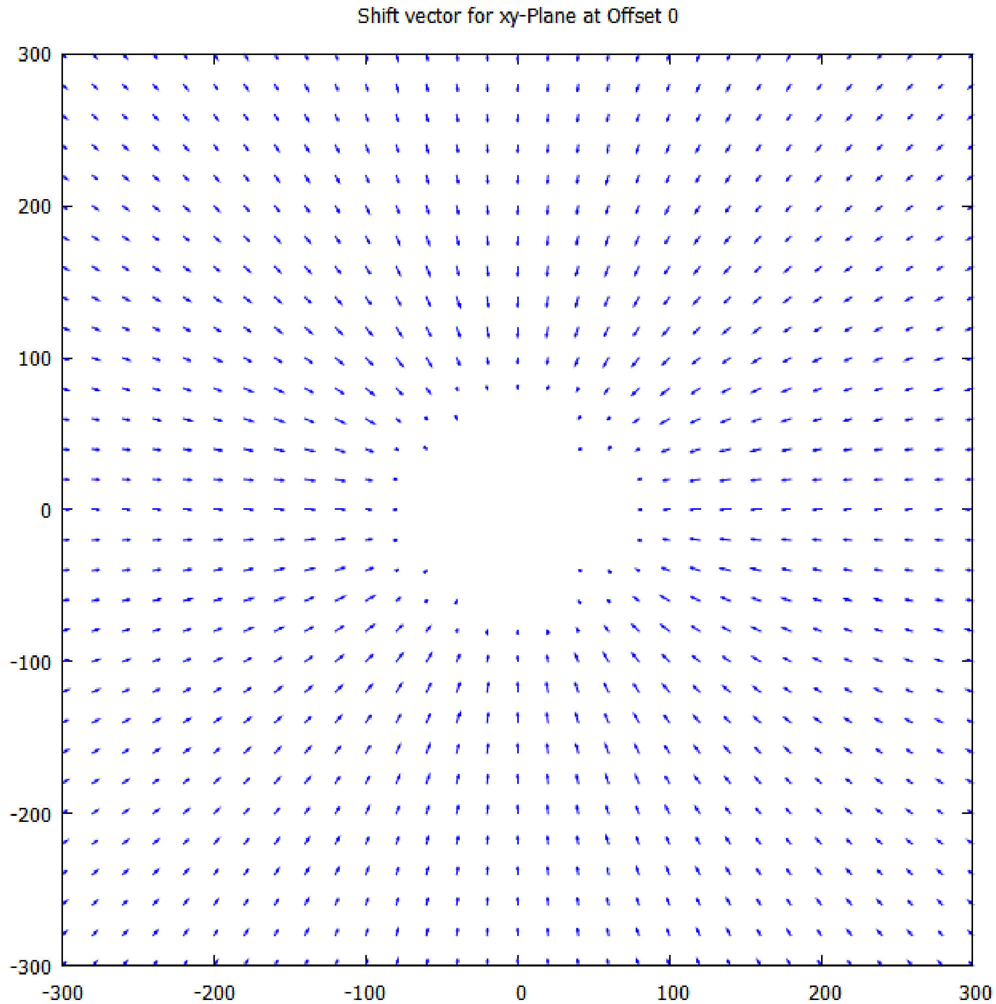


Shift vector


Norm

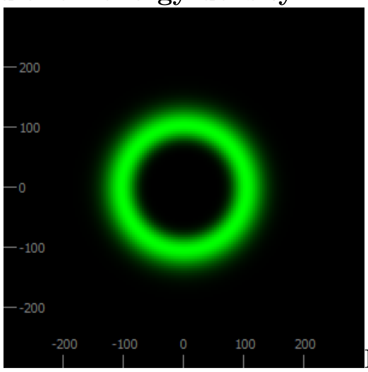


Norm (embedding)



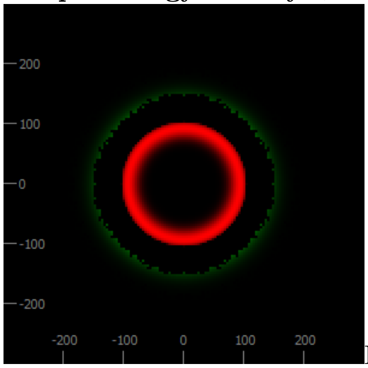
Arrows point outward (rather than inward, as in the previous subsection).

Eulerian energy density



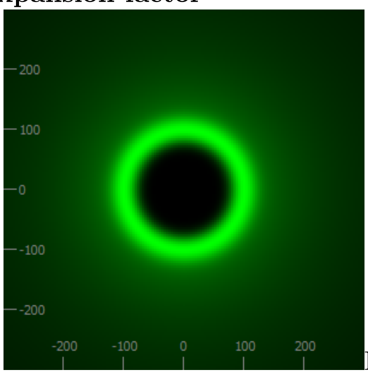
Remains the same after the sign reversal.

Principle energy density + Minimum Principle Pressure



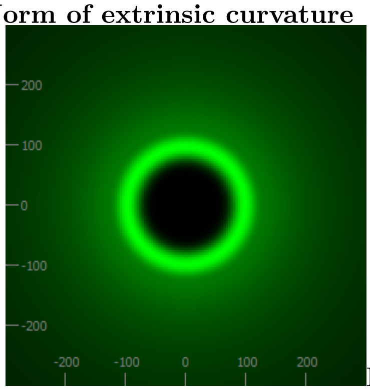
Remains the same after the sign reversal.

Expansion factor



Reverses sign after the sign reversal

Norm of extrinsic curvature

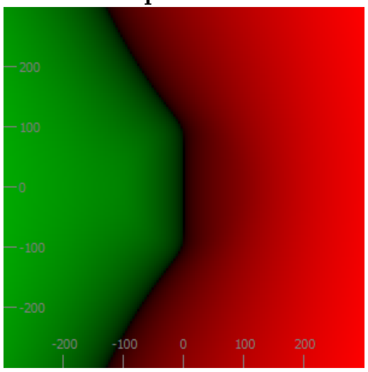


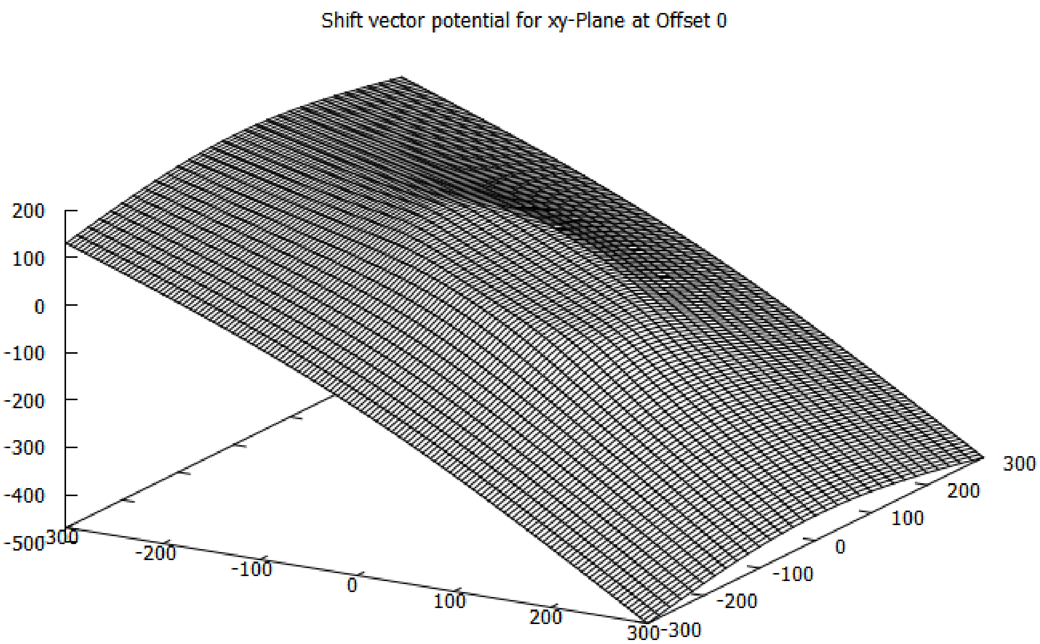
Remains the same after sign reversal.

2.3 Fell-Heisenberg: Quasi-Non-Symmetric Case

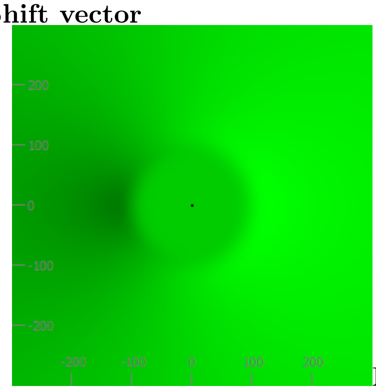
The F-H paper says that the non-symmetric case comes about as a result of the proper choice of Gaussian weight, which is then accompanied by a level shift in the interior region. However, these two things are deliberately coupled in their shift vector potential. If we substitute (20) for (16), with $\boldsymbol{u} = 0$, this is no longer the case. A level shift can then be produced while the geometry remains spherically symmetric.

Shift vector potential

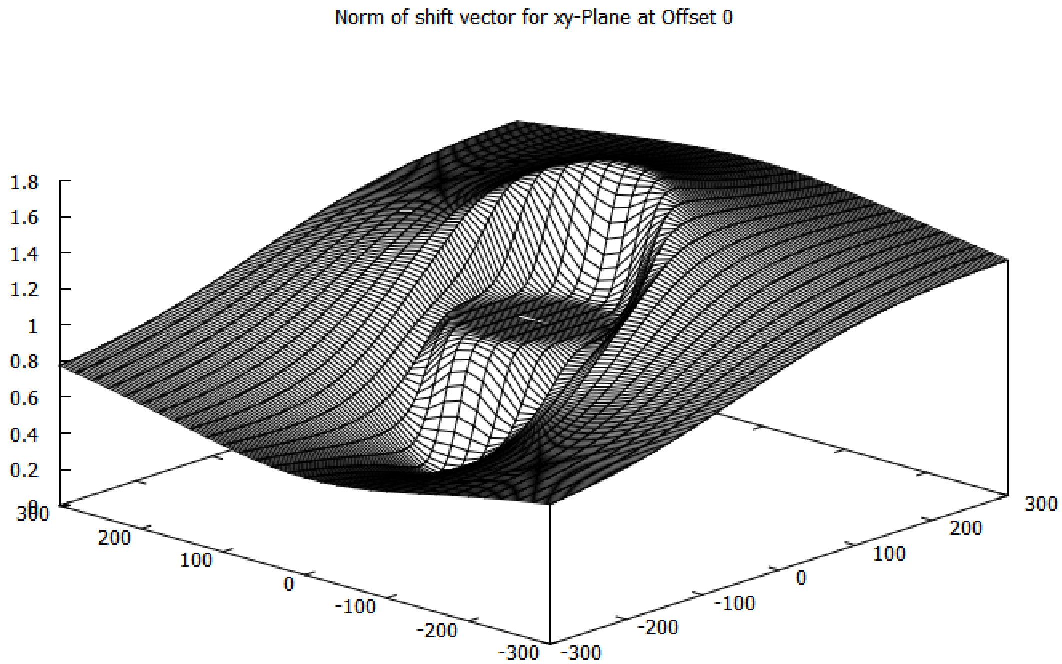




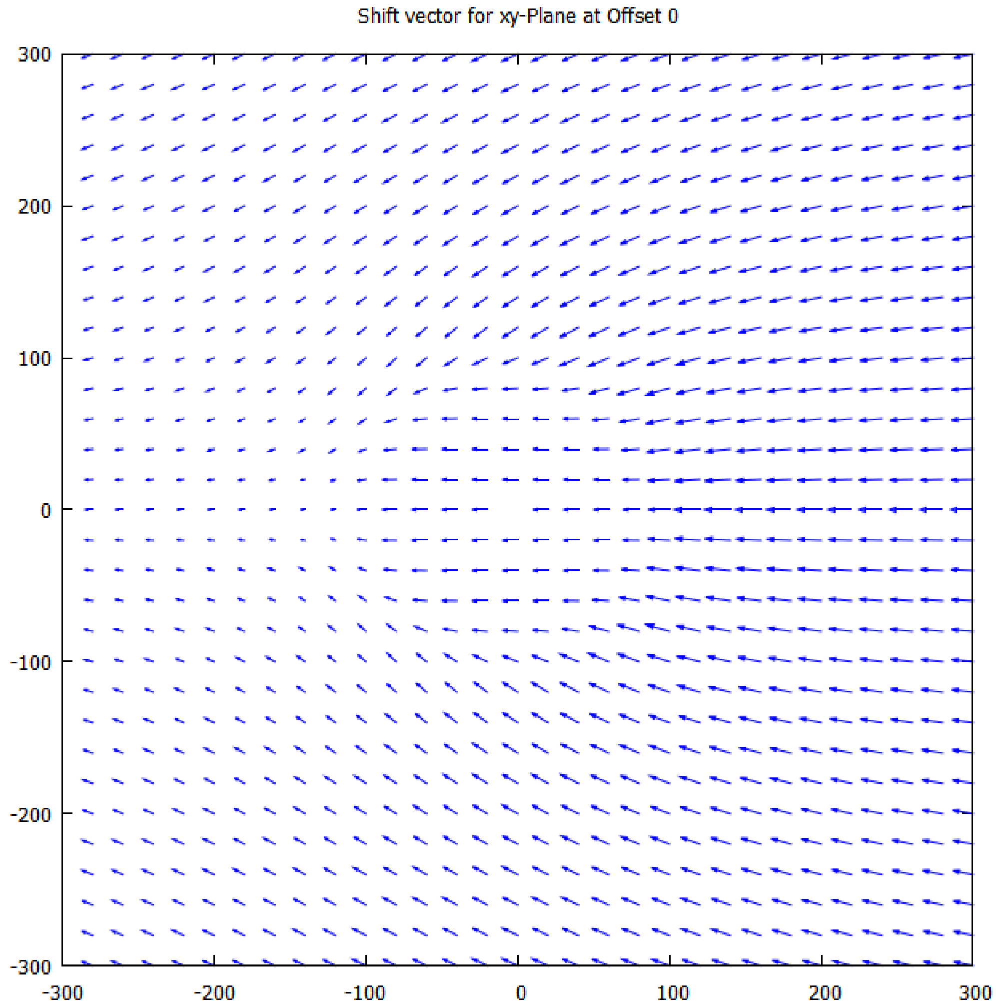
Shift vector



Norm

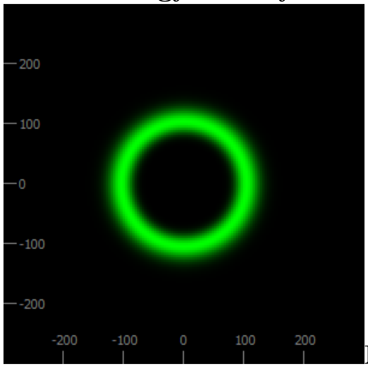


Norm (embedding)



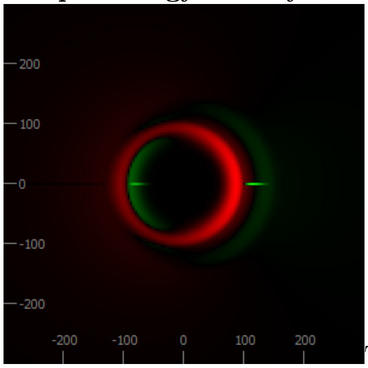
Note the steady shift in the interior region (compared to zero shift in the previous two subsections).

Eulerian energy density



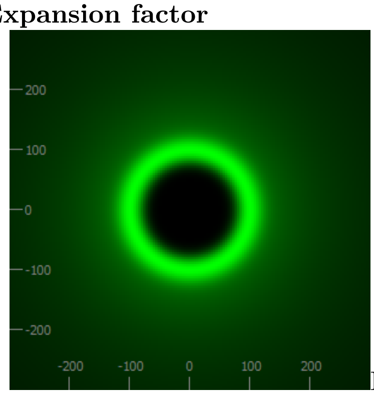
Nevertheless, the Eulerian energy density remains symmetric.

Principle energy density + minimum principle pressure



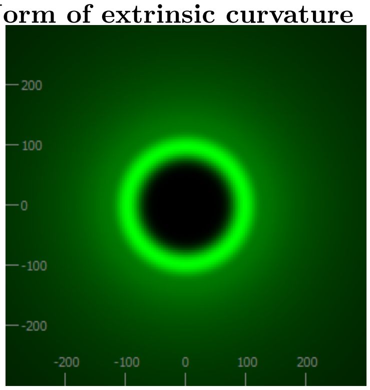
There is now a slight asymmetry int the minimum principle pressure.

Expansion factor



But the expansion factor remains spherically symmetric.

Norm of extrinsic curvature



Gaussian Weighting Function

I do not see how a Gaussian weighting function can provide the desired difference in the expansion factor on opposite sides of the shell. Suppose the Gaussian weighting function has the form

$$w = \exp\left(-\sum_{i=1}^{3}(x_i - U_i)^2/d^2\right) \tag{21}$$

where $U_i,\ i = 1, 2, 3$ is displacement vector and d is a characteristic distance. Then,

$$\sqrt{w} = \exp\left(-\frac{1}{2}\sum_{i=1}^{3}(x_i - U_i)^2/d^2\right). \tag{22}$$

If this is substituted into (15), what effect does it have on the term T_g? If $\boldsymbol{U} = 0$ and d is sufficiently large, say $d > 5R$, the potential (although not zero) will remain approximately level in the interior region. Thus, the T_v term will operate as before, producing a linear dependence on r in the interior, which translates into the desired level shift there. However, this will not introduce an asymmetry into the geometry. On the other hand, $|\boldsymbol{U}| > 0$, there will be an asymmetry, but the potential in the interior will neither be level nor linearly dependent on r. Thus, the desired level shift will not be present.

Despite this problem, F-H has an example (do to some choice of Gaussian weight) such that:

- The shift vector is high and level in the interior region.
- The Eulerian energy density and expansion factor are different on either side of the shell along the line of motion, although the there are no negative values of the expansion, F-H seem to be saying that the difference is enough to generate motion.