Cosmology in General Relativity

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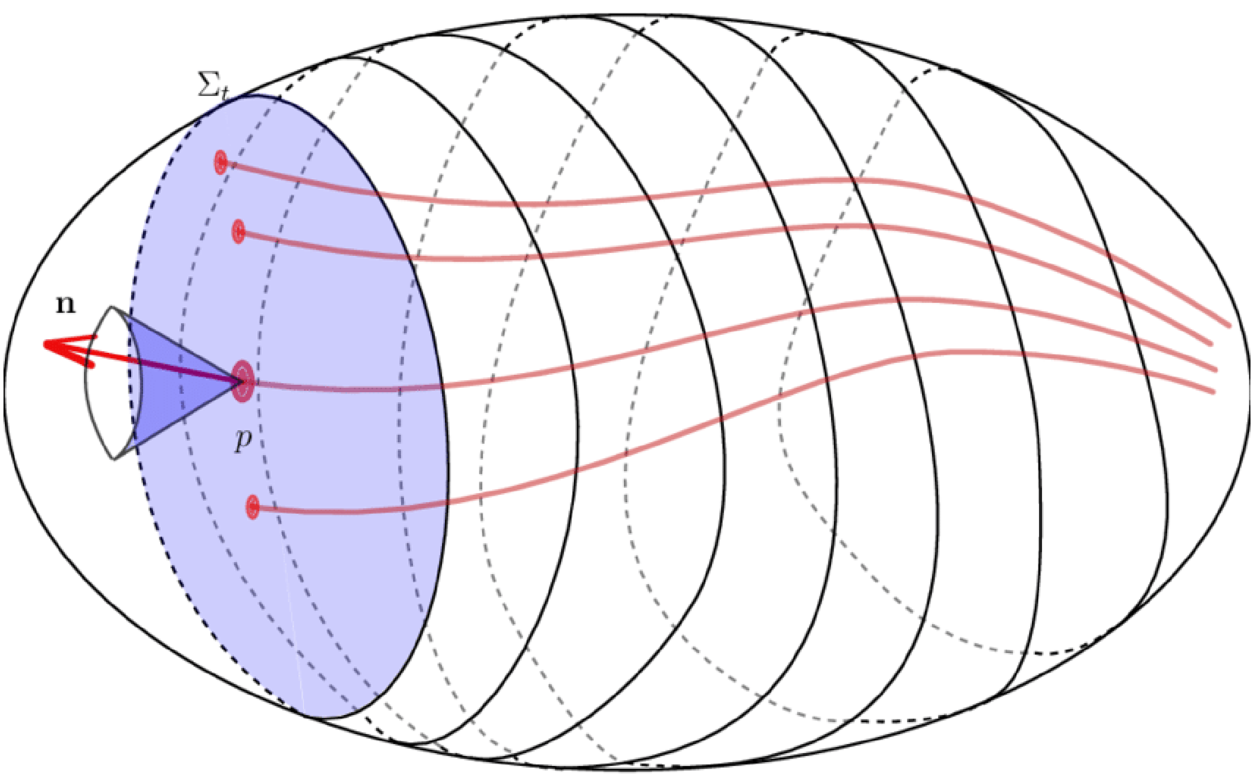
We digress into cosmology for the following reasons:

- Certain facts from cosmology spawned research into warp drive.
- Cosmology provides a relatively simple example of a conventional GR application.
- Such an example provides an excellent opportunity to review previously covered concepts.
- It is possible to put the standard cosmological model into a form that satisfies the Alcubierre constraints.

1 The Robertson-Walker (RW) Cosmological Model

1.1 Time-Like Foliation of the Universe

The starting point for the RW model is a time-like foliation of spactime.



Properties (review)

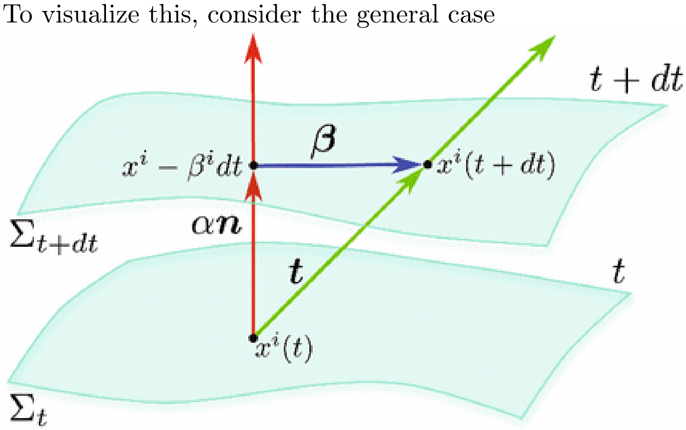
- Coordinate time t increases from right to left.
- At each time t, there is a space-like hypersurface Σ_t (disks: 1 spatial dimension is supressed).

- Each space-like hypersurface is a snapshot of all of space at time t.
- There is a family of time-like curves (red), which never intersect.
- Each time-like curve passes through each hypersurface only once.
- The unit normal vector $\boldsymbol{n}$ to any hypersurface at any point p is tangent to time-like curve passing through p, i.e. the time-like curves are everywhere orthogonal to the hypersurfaces.
- Each unit tangent vector (or normal vector) lies within its local light cone;
  otherwise the red curves would not be time-like.
- The unit tangent vector field is also the 4-velocity of a family of observers moving along the time-like curves;
  In the literature these are called "Eulerian" observers.

1.2 Fundamental Observers (a Special Case)

The RW model considers a special family of observers called "fundamental" observers. We may describe these observers in two ways:

- **Physically**, these are observers at rest with respect to the cosmic microwave background (CMB).
  Observers at galactic centers (if there were any) would be a very good approximation to the fundamental observers.
  Even observers on Earth are a pretty good approximation to the fundamental observers.
- **Mathematically**, the spatial coordinates $x^i, i = 1, 2, 3$ of the fundamental observers do not change with time, i.e. from one hypersurface to the next.
  To visualize this, consider the general case



  and let $x^i(t + dt) = x^i(t)$ for all t. Therefore,
  - The shift vector ($\boldsymbol{\beta}$ in the figure) is zero.
  - The lapse function (α in the figure) is one, i.e. $d\tau/dt$=1
    (because the fundamental observers are at rest).

- It is also important to note that the tangent vector n^μ to the world lines (or equivalently the 4-velocity) of the fundamental observers is given by

$$\left(n^0, n^1, n^2, n^3\right) = (1, 0, 0, 0) \tag{1}$$

  One way to show this is to recall that, in the 3+1 formalism, the tangent vector (or 4-velocity) is given in all generality by

$$\left(n^0, n^1, n^2, n^3\right) = \frac{1}{N}\left(1, -N^1, -N^2, -N^3\right) \tag{2}$$

  where N is the lapse function and $N^i, i = 1, 2, 3$ are the components of the shift vector in the notation originally introduced by Arnowitz, Deser and Misner (ADM). Clearly, if $N = 1$ and $N^i = 0$ for all i, then (2) reduces to (1).

1.3 The Geometry of the Space-Like Hypersurfaces

To fix the geometry of the hypersurfaces, we must settle on a metric for those hypersurfaces. The following is a step-by-step procedure by which we arrive at an appropriate metric.

1.3.1 The Cosmological Principle

The search for the appropriate metric begins with astronomical observations. Over many decades, it has become increasingly clear that on scales larger that about 100 million light years the Universe is:

- **Spatially Homogenious** - conditions (e.g. energy density) are the same at every point in space.
- **Spatially Isotropic** - at every point in space, space looks the same in every direction.

Over the decades, increasingly numerous and increasingly precise observations have elevated these finding from a hypothesis to a principle - *the cosmological principle.*

1.3.2 Mathematical Implications of the Cosmological Principle

The cosmological principle implies that each space-like hypersurface Σ_t is homogeneous and isotropic. Mathematical implications:

- Homogeneity - the metric on Σ_t must be invariant under 3-dimensional translations.
- Isotropy - the metric on Σ_t must be invariant under 3-dimensional rotations.
- Transformations that leave the metric unchanged are called *isometries.*
- Σ_t has 6 isometries: 3 translations and 3 rotations.

1.3.3 Ramifications of Six Isometries on Σ_t

For any (pseudo)Riemannian manifold:

- The maximum possible number $I_{\#,\max}$ of isometries is equal to the number $C_{\#}$ of independent components of any rank-2, symmetric tensor,
  i.e. $C_{\#} = n(n+1)/2$, where n is the dimension of the manifold.
- Thus, for spacetime $I_{\#,\max} = 10$, but for the hypersurfaces $I_{\#,\max} = 6$.
- The actual number of isometries $I_{\#}$ may be anywhere in the range $0 \leq I_{\#} \leq I_{\#,\max}$.
- A manifold is *maximally symmetric* if $I_{\#} = I_{\#,\max}$.
- For the space-like hypersurfaces Σ_t, $I_{\#} = I_{\#,\max} = 6$, so the Σ_t are maximally symmetric.

1.3.4 Ramifications of Maximal Symmetry

There are two important results related to maximally symmetric spaces:

- Theorem:
  - Let M be a manifold of dimension n that can be decomposed into
    - a family U_α of maximally symmetric subspaces of M, of dimension $m < n$, and
    - a family V_α of subspaces orthogoanl to the subspaces U_α of dimension $n - m$.
  - Then the metric on M, i.e.

    $$ds^2 = g_{ab}\left(x^*\right) dx^a dx^b \tag{3}$$

    can be written as the sum of two terms, i.e.

    $$ds^2 = \bar{h}_{ab}\left(v\right) dv^a dv^b + f\left(v\right) h_{ij}\left(u\right) du^i du^j \tag{4}$$

    where

    * the$\bar{h}_{ab}$ are the metric tensor components on the family V_α, functions of the coordinates $v^a, a = 1, ..., n-m$ on V_α.
    * the h_{ij} are the metric tensor components on the family U_α, functions of the coordinates $u^i, i = m+1, ..., n$ on U_α.
    * $f(v)$ is a scalar-valued function of the coordinates v^a.
  - This theorem applies directly to the spacetime described above because
    * The Σ_t are a family maximally symmetric subspaces (like U_α).
    * The time-like curves are a family of subspaces, orthogonal to the family Σ_t (like V_α).

- Simplification of the Riemann tensor
  - If a Riemannian manifold is maximally symmetric, it can be shown that the Riemann tensor simplifies to

$$R_{abcd} = \bar{K}\left(g_{cb}g_{ad} - g_{db}g_{ac}\right) \tag{5}$$

    where $\bar{K}$ is a constant.
    NOTE: in general, the Riemann tensor components depend on the 1st and 2nd partial derivatives of the metric tensor components.
  - It can then be shown that the Ricci scalar simplifies to

$$R = -n(n-1)\bar{K} \tag{6}$$

    where n is the dimension of the manifold.
  - Therefore, for the spacelike hypersurfaces Σ_t,

$$R = -6\bar{K}. \tag{7}$$

    Note: R is actually the trace of the mixed Ricci tensor, which in this case is the following diagonal matrix

$$\begin{bmatrix} -2\bar{K} & 0 & 0 \\ 0 & -2\bar{K} & 0 \\ 0 & 0 & -2\bar{K} \end{bmatrix}$$

    where the diagonal elements are identical sectional curvatures of 3 independent, 2-dimensional submanifolds (surfaces), and $\bar{K}$ is the Gaussian curvature of these surfaces.

- From the preceding discussion, we see that maximally symmetric spaces are *spaces of constant curvature*, and that the hypersurfaces Σ_t, in particular, are 3-dimensional spaces of constant curvature.
  NOTE: in retrospect, the Σ_t could be nothing other than spaces of constant curvature; otherwise, they would not satisfy the original premise that they are homogeneous and isotropic.

1.3.5 Ramifications of Constant Curvature

There are only three possible geometries for spaces of constant curvature, as shown in the table below, for all Σ_t:

$\bar{K}$	Symbol	Description
$+$	$\mathbb{S}^3$	Spherical 3-space
0	$\mathbb{R}^3$	Euclidean 3-space
$-$	$\mathbb{H}^3$	Hyperbolic 3-space

so the Σ_t must have one of these three geometries. The metric for each of these possibilities is fully determined, once an appropriate choice is made for the coordinates system, i.e. 3-dimensional spherical, Cartesian, or hyperbolic coordinates, respectively.

1.4 The Robertson-Walker Metric

Referring to (4) in Section 1.3.4:

- The V_α are 1-dimensional subspaces, so $\bar{h}_{ab}$ collapses to just 1 function of one coordinate, which is the proper time τ, and thus the expression $\bar{h}_{ab}(v)\, dv^a dv^b$ collapses to

$$\bar{h}(\tau)\, d\tau^2 = g_{00} d\tau^2 = -n_0^2 d\tau^2 = -1 d\tau^2 \text{ (in natural units).} \tag{8}$$

- The U_α are identified with the space-like hypersurfaces Σ_t, and therefore the expression $f(v)\, h_{ij}(u)\, du^i du^j$ takes one of three forms

$$f(\tau) \begin{cases} \text{metric for} & \mathbb{S}^3 \text{in spherical coordinates} \\ \text{metric for} & \mathbb{R}^3 \text{in Cartesian coordinates} \\ \text{metric for} & \mathbb{H}^3 \text{in hyperbolic coordinates} \end{cases} \tag{9}$$

- Also, it will be convenient to define

$$a(\tau) := \sqrt{f(\tau)} \tag{10}$$

From this point forward, we will restrict our attention to the Euclidean case $\bar{K} = 0$, for two reasons:

- Observations seem to indicate that this is actually the case for our Universe.
- This case is most easily transformed to an Alcubierre-like spacetime.

Combining (8), (9 for $\bar{K} = 0$) and (10), the FW metric takes the form

$$ds^2 = -d\tau^2 + a^2(\tau) \left\{ dx^2 + dy^2 + dz^2 \right\} \tag{11}$$

and thus the corresponding metric tensor is given in matrix form by

$$[g_{**}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(\tau) & 0 & 0 \\ 0 & 0 & a^2(\tau) & 0 \\ 0 & 0 & 0 & a^2(\tau) \end{bmatrix}. \tag{12}$$

Now that we have the metric tensor we can obtain the form of the cotangent vectors, i.e.

$$[n_*] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(\tau) & 0 & 0 \\ 0 & 0 & a^2(\tau) & 0 \\ 0 & 0 & 0 & a^2(\tau) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

In other words, $n_0 = -n^0$ and all other components of both forms of the vector are zero.

1.5 Stress-Energy Tensor for the Robertson-Walker Model

The most general form of the stress-energy tensor, consistent with the cosmological principle, is given by

$$T_{\mu\nu} = \rho(\tau) n_\mu n_\nu + P(\tau)(g_{\mu\nu} + n_\mu n_\nu). \tag{14}$$

where ρ is energy density and P is radiation pressure. Note that $n_0 n_0 = 1$ and all other components of $n_\mu n_\nu$ are zero, so the matrix form of the stress-energy tensor is

$$[T_{**}] = \begin{bmatrix} \rho(\tau) & 0 & 0 & 0 \\ 0 & a^2(\tau)P(\tau) & 0 & 0 \\ 0 & 0 & a^2(\tau)P(\tau) & 0 \\ 0 & 0 & 0 & a^2(\tau)P(\tau) \end{bmatrix}. \tag{15}$$

1.6 Determining the Unknown Function a

We see from (12) and (15) that, respectively, the RW metric and corresponding stress-energy tensor each contain an unknown function $a(\tau)$. To determine this function, we turn to Einstein's field equations, i.e.

$$G_{\mu\nu} = kT_{\mu\nu} \tag{16}$$

where

$$\begin{aligned} k &= 8\pi \quad \text{(in natural units)} && (17) \\ k &= \frac{8\pi \mathsf{G}}{c^4} \quad \text{(in SI units)} && (18) \end{aligned}$$

where G is Newton's gravitational constant. There is quite a bit of work to do to obtain the LHS of (16), or Maxima can be used to obtain

$$[G_{**}] = \begin{bmatrix} 3\frac{\dot{a}^2}{a^2} & 0 & 0 & 0 \\ 0 & -\dot{a}^2 - 2a\ddot{a} & 0 & 0 \\ 0 & 0 & -\dot{a}^2 - 2a\ddot{a} & 0 \\ 0 & 0 & 0 & -\dot{a}^2 - 2a\ddot{a} \end{bmatrix} \tag{19}$$

where $\dot{a} = da/d\tau$ and $\ddot{a} = d^2a/d\tau^2$, i.e. the 1st and 2nd derivatives of a w.r.t τ, respectively. Now, substituting (19) and (15) into (16), Einstein's equations for the Robertson-Walker model can be written in matrix form, i.e.

$$\begin{bmatrix} 3\frac{\dot{a}^2}{a^2} & 0 & 0 & 0 \\ 0 & -\dot{a}^2 - 2a\ddot{a} & 0 & 0 \\ 0 & 0 & -\dot{a}^2 - 2a\ddot{a} & 0 \\ 0 & 0 & 0 & -\dot{a}^2 - 2a\ddot{a} \end{bmatrix} = k \begin{bmatrix} \rho(\tau) & 0 & 0 & 0 \\ 0 & a^2(\tau)P(\tau) & 0 & 0 \\ 0 & 0 & a^2(\tau)P(\tau) & 0 \\ 0 & 0 & 0 & a^2(\tau)P(\tau) \end{bmatrix} \tag{20}$$

which makes it clear that we have two independent equations.

$$3\frac{\dot{a}^2(\tau)}{a^2(\tau)} = k\rho(\tau) \tag{21}$$

or

$$\rho(\tau) = 3\frac{1}{k}\frac{\dot{a}^2(\tau)}{a^2(\tau)} \tag{22}$$

and

$$-\dot{a}^2 - 2a(\tau)\ddot{a}(\tau) \quad = \quad ka^2(\tau)P(\tau) \tag{23}$$

or

$$P(\tau) = -\frac{1}{k}\left(\frac{\dot{a}^2(\tau)}{a^2(\tau)} + 2\frac{\ddot{a}(\tau)}{a(\tau)}\right) \tag{24}$$

NOTE: the preceding discussion illustrates how it becomes practical to solve the Einstein equations after take advantage of symmetries; the general problem of determining ten functions of all four coordinates has been reduced to determining one function of one coordinate, by solving ODEs, instead a coupled system of PDEs.

Units analysis for (22) In SI units, the temporal component of the normal vector is $n^0 = c$, so $a_0 = -c$. Then according to (8), $g_{00} = -c^2$, rather than -1, and according to (14), $T_{00} = \rho c^2$, rather than just ρ. As a consequence of the modified g_{00}, Maxima yields $G_{00} = \frac{3}{c^2}\frac{\dot{a}^2}{a^2}$.Therefore, with k given by (18), in SI units (22) becomes

$$\rho(\tau)c^2 = \frac{3}{c^2}\frac{c^4}{8\pi \mathsf{G}}\frac{\dot{a}^2(\tau)}{a^2(\tau)} \tag{25}$$

$$\rho(\tau) = \frac{3}{8\pi}\mathsf{G}^{-1}\frac{\dot{a}^2(\tau)}{a^2(\tau)} \tag{26}$$

Now, the SI units of $\mathsf{G}$are

$$\frac{N \cdot m^2}{kg^2} = \frac{(kg \cdot m/s^2)\, m^2}{kg^2} = \frac{m^3}{kg \cdot s^2} \tag{27}$$

and $a(\tau)$ is dimensionless and therefore $\dot{a}(\tau)$ has dimension 1/sec in SI units. Thus, the SI units of (26) are

$$\left(\frac{m^3}{kg \cdot s^2}\right)^{-1} \cdot \frac{1}{s^2} = \frac{kg \cdot s^2}{m^3} \cdot \frac{1}{s^2} = \frac{kg}{m^3}. \tag{28}$$

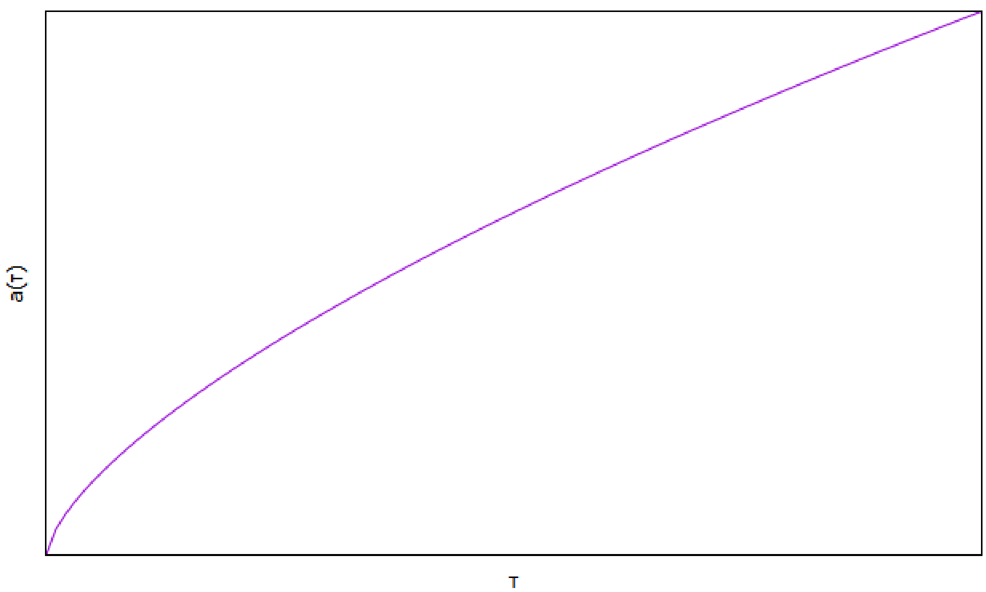
Thus, we have mass density.

1.7 Results

In solving equations (21) and (23), cosmologists make a further simplification for the present-day universe, by assuming that the radiation pressure is negligible. We will not go through the details of finding the solution here, but the result is

$$a(\tau) = C\tau^{2/3} \tag{29}$$

where C is a constant that has to be determined by observations. Here is a plot of (29), assuming the $C = 1$.



We see that the expansion continues forever.

1.7.1 The Hubble Parameter

Let r be the distance between any two points on any space-like hypersurface. Then

$$v = \frac{dr}{d\tau} = \frac{dr}{da}\frac{da}{d\tau} = \frac{r}{a}\frac{da}{d\tau} = \frac{r}{a}\dot{a} = \frac{\dot{a}}{a}r \tag{30}$$

and we define $H(\tau) := \frac{\dot{a}}{a}$ as the Hubble parameter (also called the Hubble constant), and then (30) expresses Hubble's law, i.e.

$$v = H(\tau)r \tag{31}$$

That is, at any given time τ, the recessional velocity v is proportional to the separatioin distance r, where the Hubble parameter $H(\tau)$ at time τ is the proportionality constant. The current best estimate (via WMAP) of the present-day Hubble parameter is

$$\begin{aligned} H_0 &\approx 73.8 \pm 2.4 \text{ (km/s)/Mpc} \\ H_0 &\approx 22 \text{ (km/s) per million light-years} \\ 1 \text{ Mpc} &\approx 3.09 \times 10^{19} \text{ km} \end{aligned}$$

The *Hubble radius* r_H defines a sphere, centered at any fundamental observer, beyond which galaxies are receding faster than light. To find this, we simply set $v = c$ in (31) and solve for r, using the best estimate of H_0, yielding

$$r_H \approx 13.6 \text{ billion light years.}$$

By comparison, it can be shown that (for the flat case $\bar{K} = 0$) the furthest a fundamental observer can see, which is called the *particle horizon* r_p (also called the *cosmological horizon*), has a present-day value given by $r_{p,0} = 3c\tau_0$, where τ_0 is the present-day value of the Hubble time, i.e. $\tau_0 = 1/H_0 \approx 13.3$ billion years. Using the fact that $c = 1\text{Ly/yr}$, we arrive at

$$r_{p,0} \approx 40 \text{ billion light years.}$$

Thus, the majority of galaxies that we can see are receding from us faster than light. This is the inspiration behind the idea of warp drive.

Next Time:

Converting Robertson-Walker spacetime into a spacetime that satisfies the Alcubierre conditions.