Using Robertson-Walker Cosmology to Illustrate Concepts Previously Discussed

August 21, 2023

After reviewing Robsertson-Walker cosmology (as discussed last time), we will discuss the following:

- Status of the energy conditions in RW spacetime (with $\bar{K} = 0$, i.e. flat space-like hypersurfaces).
- Working backwards from RW geometry.
- Transformation to coordinates, such that the Alcubierre conditions are satisfied.

1 Review of RW Cosmology for $\bar{K} = 0$

The Robertson-Walker metric for the flat, space-like hypersurfaces is given by

$$ds^2 = -d\tau^2 + a^2(\tau)\left\{dx^2 + dy^2 + dz^2\right\} \tag{1}$$

where $a(\tau)$ is a scalar-valued function of the cosmic time τ, i.e. time as measured by the fundamental observers. The corresponding metric tensor is therefore given by

$$[g_{**}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(\tau) & 0 & 0 \\ 0 & 0 & a^2(\tau) & 0 \\ 0 & 0 & 0 & a^2(\tau) \end{bmatrix}. \tag{2}$$

Then, using the metric tensor (2) as input to the *standard algorithm* (talk 3), i.e.

$$g_{**}\left\{\begin{array}{c} \stackrel{\text{inv}}{\to} g^{**} \\ \stackrel{\text{dif}}{\to} \partial_* g_{**} \end{array}\right\} \stackrel{(14)}{\to} \Gamma^*{}_{**} \stackrel{\text{dif}}{\to} \partial_*\Gamma^*{}_{**}$$

$$\left\{\begin{array}{c} \Gamma^*{}_{**} \\ \partial_*\Gamma^*{}_{**} \end{array}\right\} \stackrel{(13)}{\to} R_{**}$$

$$\left\{\begin{array}{c} g^{**} \\ R_{**} \end{array}\right\} \stackrel{\text{raise}}{\to} R^*_* \stackrel{\text{tr}}{\to} R$$

$$\left\{\begin{array}{c} R_{**} \\ R \\ g_{**} \end{array}\right\} \stackrel{(2)}{\to} G_{**}$$

we obtain (via Maxima) the Einstein tensor G_{**}. which (in matrix form) turns out to be

$$[G_{**}] = \begin{bmatrix} 3\frac{\dot{a}^2}{a^2} & 0 & 0 & 0 \\ 0 & -\dot{a}^2 - 2a\ddot{a} & 0 & 0 \\ 0 & 0 & -\dot{a}^2 - 2a\ddot{a} & 0 \\ 0 & 0 & 0 & -\dot{a}^2 - 2a\ddot{a} \end{bmatrix} \tag{3}$$

NOTE: the standard algorithm tells us that the Einstein tensor must depend on the functions defined in the metric tensor, along with their 1st and 2nd derivatives; In talk 3, we used the algorithm to obtain the Einstein tensor for a fully-defined metric tensor (no unknown functions), and thus obtained a fully-defined Einstein tensor; however, here we have an unknown function in the metric tensor, so the algorithm produces an Einstein tensor in which the 1st and 2nd derivatives of that unknown function are also unknown.

Independent of all the preceding considerations, the most general form of the stress-energy tensor, consistent with the cosmological principle is given by

$$T_{\mu\nu} = \rho(\tau) n_\mu n_\nu + P(\tau)(g_{\mu\nu} + n_\mu n_\nu). \tag{4}$$

where $\rho(\tau)$ and $P(\tau)$ are, respectively, the matter density and radiation pressure at time τ and n_μ is the covariant form of the normal vector to the hypersurfaces (or equivalently the tangent vector to the world lines of the fundamental observers. The corresponding matrix form of is given by

$$[T_{**}] = \begin{bmatrix} \rho(\tau) & 0 & 0 & 0 \\ 0 & a^2(\tau)P(\tau) & 0 & 0 \\ 0 & 0 & a^2(\tau)P(\tau) & 0 \\ 0 & 0 & 0 & a^2(\tau)P(\tau) \end{bmatrix}. \tag{5}$$

Then, from Einstein's field equation

$$[G_{**}] = k\,[T_{**}] \tag{6}$$

or specifically

$$\begin{bmatrix} 3\frac{\dot{a}^2}{a^2} & 0 & 0 & 0 \\ 0 & -\dot{a}^2 - 2a\ddot{a} & 0 & 0 \\ 0 & 0 & -\dot{a}^2 - 2a\ddot{a} & 0 \\ 0 & 0 & 0 & -\dot{a}^2 - 2a\ddot{a} \end{bmatrix} = k \begin{bmatrix} \rho(\tau) & 0 & 0 & 0 \\ 0 & a^2(\tau)P(\tau) & 0 & 0 \\ 0 & 0 & a^2(\tau)P(\tau) & 0 \\ 0 & 0 & 0 & a^2(\tau)P(\tau) \end{bmatrix} \tag{7}$$

we have two ordinary differential equations, i.e.

$$3\frac{\dot{a}^2}{a^2} = k\rho \tag{8}$$

and

$$-\dot{a}^2 - 2a\ddot{a} = ka^2P. \tag{9}$$

A solution to these equations (presented last time), applicable to the present-day universe (under the assumption $P = 0$), is given by

$$a(\tau) = C_1\tau^{2/3}. \tag{10}$$

We will also make use of a solution (not presented last time), applicable to the very early universe (under the assumption $P = \rho/3$), given by

$$a(\tau) = C_2\tau^{1/2}. \tag{11}$$

respectively.

2 Energy Conditions in RW Cosmology

2.1 Review of Energy Conditions for Hawking-Ellis Type I

It will be seen shortly that the mixed energy tensor for the RW cosmology is H-E Type I, which means the matrix $[T^*_*]$ has

- Four, mutually-independent eigenvectors: 1 time-like and 3 space-like.
- Four real eigenvalues with the following physical interpretation:

- The principal energy density is given by $\hat{\rho} = -\lambda_0$,
  where λ_0 is the eigenvalue belonging to the time-like eigenvector.
- The principal pressures are given by $\hat{P}_i = \lambda_i,\ i = 1,2,3$,
  where $\lambda_i,\ i = 1,2,3$ are the eigenvalues belonging to the space-like eigenvectors.

The table below provides the requirements to satisfy the energy condtions for H-E Type I, in terms of the eigenvalues of $[T^*_*]$ and also the principal, physical quantities those eigenvalues represent, as defined above.

Table I: Energy Conditions for H-E Type I

Energy Conditions	Requirements (in terms of eigenvalues)	Requirements (in terms of Principals)
Null Energy Condition (NEC)	$-\lambda_0 + \lambda_i \geq 0,\ i = 1,2,3$	$\hat{\rho} + \hat{P}_i \geq 0,\ i = 1,2,3$
Weak Energy Condition (WEC)	NEC satisfied and $-\lambda_0 \geq 0,\ i = 1,2,3$	NEC satisified and $\hat{\rho} \geq 0,\ i = 1,2,3$
Strong Energy Condition (SEC)	NEC satisfied and $-\lambda_0 + \sum_{i=1}^{3} \lambda_i \geq 0$	NEC satisfied and $\hat{\rho} + \sum_{i=1}^{3} \hat{P}_i \geq 0$

2.2 Application to RW Cosmology

In general, a mixed, rank-2 tensor is obtained by raising an index of a fully-covariant form. (Recall from talk 1 that raising an index is done using the contravariant form of the metric tensor.) Applying this concept to the stress-energy tensor, we have

$$T^{\mu}_{v} = g^{\mu a} T_{a\nu} \tag{12}$$

or in matrix form (12) is written as

$$[T^*_*] = [g_{**}]^{-1}\,[T_{**}]\,. \tag{13}$$

(Recall from talk 1 that the contravariant metric tensor is the matrix inverse of the covariant metric tensor, i.e. $[g^{**}] := [g_{**}]^{-1}$.) Now, we apply this to the RW stress-energy tensor, by substituting (2) and (5) into (13), yielding

$$\begin{aligned}
[T^*_*] &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(\tau) & 0 & 0 \\ 0 & 0 & a^2(\tau) & 0 \\ 0 & 0 & 0 & a^2(\tau) \end{bmatrix}^{-1} \begin{bmatrix} \rho(\tau) & 0 & 0 & 0 \\ 0 & a^2(\tau)P(\tau) & 0 & 0 \\ 0 & 0 & a^2(\tau)P(\tau) & 0 \\ 0 & 0 & 0 & a^2(\tau)P(\tau) \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^{-2}(\tau) & 0 & 0 \\ 0 & 0 & a^{-2}(\tau) & 0 \\ 0 & 0 & 0 & a^{-2}(\tau) \end{bmatrix} \begin{bmatrix} \rho(\tau) & 0 & 0 & 0 \\ 0 & a^2(\tau)P(\tau) & 0 & 0 \\ 0 & 0 & a^2(\tau)P(\tau) & 0 \\ 0 & 0 & 0 & a^2(\tau)P(\tau) \end{bmatrix} \\
&= \begin{bmatrix} -\rho(\tau) & 0 & 0 & 0 \\ 0 & P(\tau) & 0 & 0 \\ 0 & 0 & P(\tau) & 0 \\ 0 & 0 & 0 & P(\tau) \end{bmatrix}.
\end{aligned} \tag{14}$$

Now, the fact that $[T^*_*]$ is diagonal implies that

- The diagonal elements are the eigenvalues, and they are obviously real.
- The matrix whose columns are the eigenvectors is the 4×4 identity matrix,
  i.e. 4 mutually independent eigenvectors.

Therefore, $[T^*_*]$ is H-E Type I, which means we can apply Table I. Before doing so, note that:

- The principle quantities are, respectively, the same as those seen by the fundamental observers.
- Fundamental observers (such as ourselves) see only non-negative matter density and radiation pressure.

Thus, according to Table I, the energy conditions are satisfied.

Important note: In general, it is not true that the principle quantities are the same as those corresponding to the original frame in which the stress-energy tensor is calculated.

3 Working Backwards

3.1 Working Backwards in Part

Suppose we have the Einstein tensor, as derived from the RW metric, containing the unknown functions a, $\dot{a}$ and $\ddot{a}$, but no model for the stress-energy tensor. We can still determine the status of the energy conditions. First, we obtain the mixed Einstein tensor, in a manner analogous to (13), i.e.

$$[G^*_*] = [g_{**}]^{-1} [G_{**}] . \tag{15}$$

Substituting (2) and (3) into (15) yields

$$\begin{aligned}
[G^*_*] &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(\tau) & 0 & 0 \\ 0 & 0 & a^2(\tau) & 0 \\ 0 & 0 & 0 & a^2(\tau) \end{bmatrix}^{-1} \begin{bmatrix} 3\frac{\dot{a}^2}{a^2} & 0 & 0 & 0 \\ 0 & -\dot{a}^2 - 2a\ddot{a} & 0 & 0 \\ 0 & 0 & -\dot{a}^2 - 2a\ddot{a} & 0 \\ 0 & 0 & 0 & -\dot{a}^2 - 2a\ddot{a} \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^{-2}(\tau) & 0 & 0 \\ 0 & 0 & a^{-2}(\tau) & 0 \\ 0 & 0 & 0 & a^{-2}(\tau) \end{bmatrix} \begin{bmatrix} 3\frac{\dot{a}^2}{a^2} & 0 & 0 & 0 \\ 0 & -\dot{a}^2 - 2a\ddot{a} & 0 & 0 \\ 0 & 0 & -\dot{a}^2 - 2a\ddot{a} & 0 \\ 0 & 0 & 0 & -\dot{a}^2 - 2a\ddot{a} \end{bmatrix} \\
&= \begin{bmatrix} -3\frac{\dot{a}^2}{a^2} & 0 & 0 & 0 \\ 0 & -\frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} & 0 & 0 \\ 0 & 0 & -\frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} & 0 \\ 0 & 0 & 0 & -\frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} \end{bmatrix} .
\end{aligned} \tag{16}$$

Once again, we have a mixed tensor that is diagonal, so the diagonal elements are the eigenvalues. Then, from Einstein's field equations, we have

$$\begin{aligned}
[T^*_*] &= \frac{1}{k} [G^*_*] \\
&= \frac{1}{k} \begin{bmatrix} -3\frac{\dot{a}^2}{a^2} & 0 & 0 & 0 \\ 0 & -\frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} & 0 & 0 \\ 0 & 0 & -\frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} & 0 \\ 0 & 0 & 0 & -\frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} \end{bmatrix} .
\end{aligned} \tag{17}$$

so the principal quantities are given by

$$\hat{\rho} = \frac{1}{k} 3 \frac{\dot{a}^2}{a^2} \tag{18}$$

and

$$\hat{P}_i = -\frac{1}{k} \left(\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} \right), \ i = 1, 2, 3. \tag{19}$$

We can now check the energy conditions.

Checking the NEC According to Table I, 3rd column, the NEC is satisfied if

$$\hat{\rho} + \hat{P}_i \geq 0, \; i = 1, 2, 3.$$

According to (18) and (19),

$$\begin{aligned}
\hat{\rho} + \hat{P}_i &= \frac{1}{k} 3 \frac{\dot{a}^2}{a^2} - \frac{1}{k}\left(\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a}\right) \\
&= \frac{1}{k} 2 \frac{\dot{a}^2}{a^2} - \frac{1}{k} 2 \frac{\ddot{a}}{a} \\
&= \frac{2}{k}\left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right) \qquad (20)
\end{aligned}$$

Clearly, the 1st term in parentheses is positive, but what about the second term. Well

- a is a scale factor, which positive by defnition.
- In RW cosmology, it is assumed that the mutual gravitational attraction of all the mass/energy in the Universe has always acted to slow the expansion, so $\ddot{a}$ is negative.

Therefore, the term $-\ddot{a}/a$ is positive, so the RHS of (20) is creater than zero, so the NEC is satisfied.

Checking the WEC According to Table I, the WEC is satisfied if

- The NEC is satisfied, which has already been established above, and
- The condition $\hat{\rho} \geq 0$ is satisfied, which is established by (18).

Therefore, the WEC is satisfied.

Checking the SEC According to Table I, the SEC is satisifed if

- The NEC is satisifed, which (again) has already been established above, and
- The condition $\hat{\rho} + \sum_{i=1}^{3} \hat{P}_i \geq 0$ is satisfied, which we now establish.

$$\begin{aligned}
\hat{\rho} + \sum_{i=1}^{3} \hat{P}_i &= \frac{1}{k} 3 \frac{\dot{a}^2}{a^2} - \frac{3}{k}\left(\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a}\right) \\
&= 0 - \frac{6}{k}\frac{\ddot{a}}{a} \qquad (21)
\end{aligned}$$

where we have already established $-\ddot{a}/a$ is positive.

Therefore, the SEC is satisfied.

3.2 Working Backwards in Full

Suppose now that we just assume (for whatever reason) that the function a is given by (11). Then, RW metric tensor is given by

$$[g_{**}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \left(C_2 \tau^{1/2}\right)^2 & 0 & 0 \\ 0 & 0 & \left(C_2 \tau^{1/2}\right)^2 & 0 \\ 0 & 0 & 0 & \left(C_2 \tau^{1/2}\right)^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & C_2^2 \tau & 0 & 0 \\ 0 & 0 & C_2^2 \tau & 0 \\ 0 & 0 & 0 & C_2^2 \tau \end{bmatrix} \qquad (22)$$

That is, there are no unknown functions. Then, using this metric as input to the standard algorithm, one obtains

$$[G_{**}] = \begin{bmatrix} \frac{3}{4\tau^2} & 0 & 0 & 0 \\ 0 & \frac{C_2^2}{4\tau} & 0 & 0 \\ 0 & 0 & \frac{C_2^2}{4\tau} & 0 \\ 0 & 0 & 0 & \frac{C_2^2}{4\tau} \end{bmatrix} \tag{23}$$

and converting to the mixed Einstein tensor, we obtain

$$\begin{aligned} [G^{*}_{\ *}] &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & C_2^2\tau & 0 & 0 \\ 0 & 0 & C_2^2\tau & 0 \\ 0 & 0 & 0 & C_2^2\tau \end{bmatrix}^{-1} \begin{bmatrix} \frac{3}{4\tau^2} & 0 & 0 & 0 \\ 0 & \frac{C_2^2}{4\tau} & 0 & 0 \\ 0 & 0 & \frac{C_2^2}{4\tau} & 0 \\ 0 & 0 & 0 & \frac{C_2^2}{4\tau} \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{C_2^2\tau} & 0 & 0 \\ 0 & 0 & \frac{1}{C_2^2\tau} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2^2\tau} \end{bmatrix} \begin{bmatrix} \frac{3}{4\tau^2} & 0 & 0 & 0 \\ 0 & \frac{C_2}{4\tau} & 0 & 0 \\ 0 & 0 & \frac{C_2}{4\tau} & 0 \\ 0 & 0 & 0 & \frac{C_2}{4\tau} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{4\tau^2} & 0 & 0 & 0 \\ 0 & \frac{1}{4\tau^2} & 0 & 0 \\ 0 & 0 & \frac{1}{4\tau^2} & 0 \\ 0 & 0 & 0 & \frac{1}{4\tau^2} \end{bmatrix}. \end{aligned} \tag{24}$$

From the discussion in the preceding section, we know that the principle are $\hat{\rho} = -\frac{1}{k}\lambda_0 = \frac{3}{4k}\tau^2$ and $P_i = \frac{1}{k}\frac{1}{4k}\tau^2$. Thus, the principal quantities are manifestly non-negative and, therefore, the energy conditions are satisfied (without the need for any assumptions about energy content of the Universe).

For completeness, we follow the same procedure for the function a as given by (10). Then the RW metric tensor is given by

$$[g_{**}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \left(C_1\tau^{2/3}\right)^2 & 0 & 0 \\ 0 & 0 & \left(C_1\tau^{2/3}\right)^2 & 0 \\ 0 & 0 & 0 & \left(C_1\tau^{2/3}\right)^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & C_1^2\tau^{4/3} & 0 & 0 \\ 0 & 0 & C_1^2\tau^{4/3} & 0 \\ 0 & 0 & 0 & C_1^2\tau^{4/3} \end{bmatrix} \tag{25}$$

and the mixed Einstein tensor turns out to be

$$[G^{*}_{\ *}] = \begin{bmatrix} -\frac{3}{4\tau^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{26}$$

Once again, we see that the principal quantities are non-negative, so the energy conditions are satisfied. (Notice that the principal energy density is the same as in the preceding case and the principal pressures are all zero. The latter is not surprising, since $P = 0$ was assumed in obtaining the solution (10)).

4 Satisfying the Alcubierre Conditions

The Alcubierre conditions may be expressed as follows:

- The lapse function is given by $N = 1$.
- The space-like hypersurfaces are intrinsically flat.
- The metric tensor on the space-like hypersurfaces is given by $h_{ij} = \delta_{ij},\ i, j = 1, 2, 3$ or in matrix notation $[h_{**}] = I_{3\times3}$.

We have seen that the first two bullits are satisfied by the RW model, with $\bar{K} = 0$. However, the third bullit is not satsified, since $[h_{**}] = a^2 I_{3\times3}$. Can we find a coordinates system, such that the third bullit holds? The answer is yes.

4.1 The Coordinate Transformation

Let $\mathcal{T} : x^\mu \to \bar{x}^\mu,\ \mu = 0, 1, 2, 3$, be a transformation from the original coordinates

$$\boldsymbol{x} = (x^0, x^1, x^2, x^3) = (\tau, x, y, z) \tag{27}$$

to new coordinates

$$\bar{\boldsymbol{x}} = (\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3) = (\bar{\tau}, \bar{x}, \bar{y}, \bar{z}) \tag{28}$$

defined by

$$\bar{\tau} = \tau \tag{29}$$
$$\bar{x} = a(\tau)x \tag{30}$$
$$\bar{y} = a(\tau)y \tag{31}$$
$$\bar{z} = a(\tau)z \tag{32}$$

Then, the inverse transformation $\mathcal{T}^{-1} : \bar{x}^\mu \to x^\mu$ is given by

$$\tau = \bar{\tau} \tag{33}$$
$$x = a^{-1}(\bar{\tau})\bar{x} \tag{34}$$
$$y = a^{-1}(\bar{\tau})\bar{y} \tag{35}$$
$$z = a^{-1}(\bar{\tau})\bar{z} \tag{36}$$

4.2 Direct Transformation of the RW Metric ($\bar{K} = 0$)

Using the inverse transformation $\mathcal{T}^{-1}$, one may obtain expressions for the dx^μ in terms of the $d\bar{x}^\mu$. Specifically, from (33) we have

$$d\tau = d\bar{\tau} \tag{37}$$

and from (34) through (36) we have for $i = 1, 2, 3$

$$\begin{aligned} \frac{dx^i}{d\tau} &= -a^{-2}(\tau)\bar{x}^i \frac{da}{d\tau} + a^{-1}(\tau)\frac{d\bar{x}^i}{d\tau} \\ &= -a^{-2}(\tau)\bar{x}^i \dot{a} + a^{-1}(\tau)\frac{d\bar{x}^i}{d\tau}. \end{aligned} \tag{38}$$

Now, multipying both sides of (38) by $d\tau$ yields

$$dx^i = -a^{-2}(\tau)\bar{x}^i \dot{a}(\tau)d\tau + a^{-1}(\tau)d\bar{x}^i \tag{39}$$

and substituting (37) yields

$$dx^i = -a^{-2}(\tau)\bar{x}^i \dot{a}(\tau)d\bar{\tau} + a^{-1}(\tau)d\bar{x}^i. \tag{40}$$

Then, squaring both sides, the RHS expands into four terms, i.e.

$$\begin{aligned} \left(dx^i\right)^2 &= \left(-a^{-2}(\tau)\bar{x}^i \dot{a}(\tau)d\bar{\tau} + a^{-1}(\tau)d\bar{x}^i\right)^2 \\ &= a^{-2}(\tau)\left(d\bar{x}^i\right)^2 - a^{-3}(\tau)\bar{x}^i \dot{a}(\tau)d\bar{\tau}d\bar{x}^i - a^{-3}(\tau)\bar{x}^i \dot{a}(\tau)d\bar{x}^i d\bar{\tau} + a^{-4}(\tau)\left(\bar{x}^i\right)^2 \dot{a}^2(\tau)d\bar{\tau}^2. \end{aligned} \tag{41}$$

Next, substituting (37) and (41) into the RW metric (1), repeated immediately below, for convenience

$$ds^2 = -d\tau^2 + a^2(\tau)\left\{dx^2 + dy^2 + dz^2\right\},$$

we obtain

$$ds^2 = -d\bar{\tau}^2 + a^2(\tau)\sum_{i=1}^{3}\left\{a^{-2}(\tau)\left(d\bar{x}^i\right)^2 - a^{-3}(\tau)\bar{x}^i\dot{a}(\tau)d\bar{\tau}d\bar{x}^i - a^{-3}(\tau)\bar{x}^i\dot{a}(\tau)d\bar{x}^i d\bar{\tau} + a^{-4}(\tau)\left(\bar{x}^i\right)^2\dot{a}^2(\tau)d\bar{\tau}^2\right\} \tag{42}$$

and carrying through the multiplication by $a^2(\tau)$ yields

$$ds^2 = -d\bar{\tau}^2 + \sum_{i=1}^{3}\left\{\left(d\bar{x}^i\right)^2 - \frac{\dot{a}(\tau)}{a(\tau)}\bar{x}^i d\bar{\tau}d\bar{x}^i - \frac{\dot{a}(\tau)}{a(\tau)}\bar{x}^i d\bar{x}^i d\bar{\tau} + \frac{\dot{a}^2(\tau)}{a^2(\tau)}\left(\bar{x}^i\right)^2 d\bar{\tau}^2\right\}. \tag{43}$$

Now, collecting terms corresponding to each combination of differentials, we have

$$\begin{aligned} ds^2 &= \left(\frac{\dot{a}^2(\tau)}{a^2(\tau)}\sum_{i=1}^{3}\left(\bar{x}^i\right)^2 - 1\right)d\bar{\tau}^2 \\ &- \sum_{i=1}^{3}\frac{\dot{a}(\tau)}{a(\tau)}\bar{x}^i d\bar{\tau}d\bar{x}^i \\ &- \sum_{i=1}^{3}\frac{\dot{a}(\tau)}{a(\tau)}\bar{x}^i d\bar{x}^i d\bar{\tau} \\ &+ \sum_{i=1}^{3}\left(d\bar{x}^i\right)^2 \end{aligned} \tag{44}$$

which may be rewritten as

$$\begin{aligned} ds^2 &= \left(\frac{\dot{a}^2(\tau)}{a^2(\tau)}r^2 - 1\right)d\bar{\tau}^2 \\ &- \frac{\dot{a}(\tau)}{a(\tau)}\bar{x}d\bar{\tau}d\bar{x} - \frac{\dot{a}(\tau)}{a(\tau)}\bar{y}d\bar{\tau}d\bar{y} - \frac{\dot{a}(\tau)}{a(\tau)}\bar{z}d\bar{\tau}d\bar{z} \\ &- \frac{\dot{a}(\tau)}{a(\tau)}\bar{x}d\bar{x}d\bar{\tau} - \frac{\dot{a}(\tau)}{a(\tau)}\bar{y}d\bar{y}d\bar{\tau} - \frac{\dot{a}(\tau)}{a(\tau)}\bar{z}d\bar{z}d\bar{\tau} \\ &+ d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 \end{aligned} \tag{45}$$

where r is Euclidean distance on any of the (intrinsically flat) space-like hypersurface. We can now, by inspection of (45), pick out the coefficients of the various products of differentials, and write down the metric tensor in the new coordinates, i.e.

$$[\bar{g}_{**}] = \begin{bmatrix} \frac{\dot{a}^2(\tau)}{a^2(\tau)}r^2 - 1 & -\frac{\dot{a}(\tau)}{a(\tau)}\bar{x} & -\frac{\dot{a}(\tau)}{a(\tau)}\bar{y} & -\frac{\dot{a}(\tau)}{a(\tau)}\bar{z} \\ -\frac{\dot{a}(\tau)}{a(\tau)}\bar{x} & 1 & 0 & 0 \\ -\frac{\dot{a}(\tau)}{a(\tau)}\bar{y} & 0 & 1 & 0 \\ -\frac{\dot{a}(\tau)}{a(\tau)}\bar{z} & 0 & 0 & 1 \end{bmatrix}. \tag{46}$$

Now, using metric tensor $[\bar{g}_{**}]$ as input to the standard algorithm, we may obtain (via Maxima) the Einstein tensor $\left[\bar{G}_{**}\right]$ and also the mixed Einstein tensor $\left[\bar{G}^*_*\right] = [\bar{g}_{**}]^{-1}\left[\bar{G}_{**}\right]$. The latter result is given in matrix form by

$$\left[\bar{G}^*_*\right] = \begin{bmatrix} -3\frac{\dot{a}^2}{a^2} & 2\left(\frac{\dot{a}\ddot{a}}{a^2} - \frac{\dot{a}^3}{a^3}\right)\bar{x} & 2\left(\frac{\dot{a}\ddot{a}}{a^2} - \frac{\dot{a}^3}{a^3}\right)\bar{y} & 2\left(\frac{\dot{a}\ddot{a}}{a^2} - \frac{\dot{a}^3}{a^3}\right)\bar{z} \\ 0 & -\frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} & 0 & 0 \\ 0 & 0 & -\frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} & 0 \\ 0 & 0 & 0 & -\frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} \end{bmatrix}. \tag{47}$$

It is a standard result from linear algebra that a lower or upper triangular matrix has its eigenvalues along the diagonal. Comparing the diagonal elements of (47) with those of (16), we see that the eigenvalues are the same. Thus, the entire analysis following (16) in Section 3.1 is the same, which means that the stress energy tensor in the new coordinates, i.e.

$$\left[\bar{T}^{*}_{*}\right] = \frac{1}{k}\left[\bar{G}^{*}_{*}\right]$$

satisfies the energy condtions.

4.3 General Coordinate Transformations

It turns out that there is a general, systematic approach to dealing with coordinate transformations, and the consequent effect on tensor components. This general, systmatic approach is called the *tensor transformation law*, as discussed below.

4.3.1 Coordinate Transformations and the Tensor Transformation Law

The tensor transformation law, defined immediately below, applies to all tensors of all ranks on any smooth manifold of any dimension. This law may be stated as follows: Let

- $A^{*\cdots}{}_{*\ldots}$ be a tensor with an arbitrary number of upper and lower indices, with components given in terms of coordinates x^{μ}.
- $\mathcal{T} : x^{\mu} \to \bar{x}^{\mu}$ be an invertable transformation, i.e. the inverse inverse $\mathcal{T}^{-1}$ exists.

Then, the components $\bar{A}^{*\cdots}{}_{*\ldots}$ relative to the new coordinate system $\bar{x}^{\mu}$ are determined by:

- Applying the Jacobian matrix $\bar{J}_{\mathcal{T}}$ of transformation $\mathcal{T}$ once for each contravariant (upper) index, and
- Applying the Jacobian matrix $\bar{J}_{\mathcal{T}^{-1}}$ of the inverse transformation $\mathcal{T}^{-1}$ once for each covariant (lower) index,

where the bar over J indicates the the elements of these matrices are written in terms of th new coordinates $\bar{x}^{\mu}$. Note that the following is always true:

$$\bar{J}_{\mathcal{T}^{-1}} = \bar{J}_{\mathcal{T}}^{-1}. \tag{48}$$

That is, the Jacobian matrix of the inverse transformation is equal to the matrix inverse of the original transformation. Thus, we can drop the labels, and just refer to $\bar{J}$ and $\bar{J}^{-1}$ being applied to the contravariant and covariant indices, respectively.

4.3.2 Application of the Tensor Transformation Law to Rank-2 Tensors

Here we make explicit the meaning of tensor transformation law, as applied to an arbitrary, rank-2 tensor. Although the explicit formulae are usually written in index notation, we will write them down in matrix notaton.

For a rank-2 tensor there are only three posibilities:

- A fully contravariant form A^{**},
- A fully covariant form A_{**}, and
- A mixed form A^{*}_{*}.

According to the transformation law, these transform as follows:

$$\left[\bar{A}^{**}\right] = \bar{J}^{T}\left[A^{**}\right]\bar{J}, \tag{49}$$

$$\left[\bar{A}_{**}\right] = \left(\bar{J}^{-1}\right)^{T}\left[A_{**}\right]\bar{J}^{-1} \tag{50}$$

and

$$\left[\bar{A}^{*}_{*}\right] = \bar{J}^{-1}\left[A^{*}_{*}\right]\bar{J}. \tag{51}$$

Note that (51) is a similarity transformation, whereas in general (49) and (50) are not. Thus, any *mixed*, rank-2 tensor has the same eigenvalues in any coordinate system. In particular, this is true for T^{*}_{*} (or G^{*}_{*}), which is another way of seeing that the eigenvalues of the mixed tensors T^{*}_{*} (or $\frac{1}{k}G^{*}_{*}$) determine the status of the energy conditions in all coordinate systems, i.e. for all observers.

4.3.3 Application to the Robertson-Walker Model

Let us apply the transformation of Section 4.1, repeated here for convenience, i.e. $\mathcal{T}: x^\mu \to \bar{x}^\mu$ is defined by

$$\begin{aligned} \bar{\tau} &= \tau \\ \bar{x} &= a(\tau)x \\ \bar{y} &= a(\tau)y \\ \bar{z} &= a(\tau)z \end{aligned}$$

and $\mathcal{T}^{-1}: \bar{x}^\mu \to x^\mu$ is then necessarily given by

$$\begin{aligned} \tau &= \bar{\tau} \\ x &= a^{-1}(\bar{\tau})\bar{x} \\ y &= a^{-1}(\bar{\tau})\bar{y} \\ z &= a^{-1}(\bar{\tau})\bar{z} \end{aligned}$$

From the transformation $\mathcal{T}$, we calculate

$$J = \begin{bmatrix} \frac{\partial\bar{\tau}}{\partial\tau} & \frac{\partial\bar{\tau}}{\partial x} & \frac{\partial\bar{\tau}}{\partial y} & \frac{\partial\bar{\tau}}{\partial z} \\ \frac{\partial\bar{x}}{\partial\tau} & \frac{\partial\bar{x}}{\partial x} & \frac{\partial\bar{x}}{\partial y} & \frac{\partial\bar{x}}{\partial z} \\ \frac{\partial\bar{y}}{\partial\tau} & \frac{\partial\bar{y}}{\partial x} & \frac{\partial\bar{y}}{\partial y} & \frac{\partial\bar{y}}{\partial z} \\ \frac{\partial\bar{z}}{\partial\tau} & \frac{\partial\bar{z}}{\partial x} & \frac{\partial\bar{z}}{\partial y} & \frac{\partial\bar{z}}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \dot{a}(\tau)x & a(\tau) & 0 & 0 \\ \dot{a}(\tau)y & 0 & a(\tau) & 0 \\ \dot{a}(\tau)z & 0 & 0 & a(\tau) \end{bmatrix}. \tag{52}$$

However, because the result is in terms of the original coordinates $x^\mu$and we are transforming to the new coordinates, we want to write the result in terms of the new coordinates $\bar{x}^\mu$. To accomplish this, we substitute the expressions for the original coordinates in terms of the new coordinates, as given by the inverse transfromation $\mathcal{T}^{-1}$, which yields

$$\bar{J} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{\dot{a}(\tau)}{a^2(\tau)}\bar{x} & a(\bar{\tau}) & 0 & 0 \\ \frac{\dot{a}(\tau)}{a^2(\tau)}\bar{y} & 0 & a(\bar{\tau}) & 0 \\ \frac{\dot{a}(\tau)}{a^2(\tau)}\bar{z} & 0 & 0 & a(\bar{\tau}) \end{bmatrix}. \tag{53}$$

From the inverste transformation $\mathcal{T}^{-1}$, we calculate

$$\bar{J}^{-1} = \begin{bmatrix} \frac{\partial\tau}{\partial\bar{\tau}} & \frac{\partial\tau}{\partial\bar{x}} & \frac{\partial\tau}{\partial\bar{y}} & \frac{\partial\tau}{\partial\bar{z}} \\ \frac{\partial x}{\partial\bar{\tau}} & \frac{\partial x}{\partial\bar{x}} & \frac{\partial x}{\partial\bar{y}} & \frac{\partial x}{\partial\bar{z}} \\ \frac{\partial y}{\partial\bar{\tau}} & \frac{\partial y}{\partial\bar{x}} & \frac{\partial y}{\partial\bar{y}} & \frac{\partial y}{\partial\bar{z}} \\ \frac{\partial z}{\partial\bar{\tau}} & \frac{\partial z}{\partial\bar{x}} & \frac{\partial z}{\partial\bar{y}} & \frac{\partial z}{\partial\bar{z}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{\dot{a}(\bar{\tau})}{a^2(\bar{\tau})}\bar{x} & \frac{1}{a(\bar{\tau})} & 0 & 0 \\ -\frac{\dot{a}(\bar{\tau})}{a^2(\bar{\tau})}\bar{y} & 0 & \frac{1}{a(\bar{\tau})} & 0 \\ -\frac{\dot{a}(\bar{\tau})}{a^2(\bar{\tau})}\bar{z} & 0 & 0 & \frac{1}{a(\bar{\tau})} \end{bmatrix} \tag{54}$$

which is already written in terms of the new coordinates.

Now, $\bar{J}$ and $\bar{J}^{-1}$, as given by (53) and (54), can be used to obtain the components of any tensor in the new coordinates $\bar{x}^\mu$. For example, we can obtain the new metric tensor $[\bar{g}_{**}]$ by substituting the original metric tensor $[g_{**}]$ for $[A_{**}]$ in (50). Similarly, we can obtain the new mixed Einstein tensor $\left[\bar{G}^*_*\right]$ by substituting the original mixed Einstein tensor tensor $[G^*_*]$ for $[A^*_*]$ in (51). Simply carrying out the required matrix multiplication shows that these two calculations produce the same result as the the rather lengthy calculation leading to (46), followed by application of the standard algorithm to obtain (47).

4.4 Relation of Transformed RW to the 3+1 Formalism

Recall (talk 3) that in the 3+1 formalism the metric tensor is given, in all generality, by

$$[g_{\mu\nu}] = \begin{bmatrix} N_i N^i - N^2 & N_1 & N_2 & N_3 \\ N_1 & h_{11} & h_{12} & h_{13} \\ N_2 & h_{21} & h_{22} & h_{23} \\ N_3 & h_{31} & h_{32} & h_{33} \end{bmatrix} \tag{55}$$

where N is the lapse function and $N_i = h_{ij} N^j$, where N^j is the shift vector. We will compare form of the transformed metric tensor $[\bar{g}_{**}]$, as given by (46), with that given by (55), but first we rewrite the former in a more compact form, by recalling the definition of the Hubble parameter, i.e.

$$H(\tau) := \frac{\dot{a}(\tau)}{a(\tau)} \tag{56}$$

and substituting into (46), yielding

$$[\bar{g}_{**}] = \begin{bmatrix} H^2(\tau)r^2 - 1 & -H(\tau)\bar{x} & -H(\tau)\bar{y} & -H(\tau)\bar{z} \\ -H(\tau)\bar{x} & 1 & 0 & 0 \\ -H(\tau)\bar{y} & 0 & 1 & 0 \\ -H(\tau)\bar{z} & 0 & 0 & 1 \end{bmatrix}. \tag{57}$$

Now, suppose we define the spatial metric, lapse function and shift vector is given by

$$h_{ij} := \delta_{ij} \tag{58}$$

where δ_{ij} is the Kronecker delta,

$$N := 1 \tag{59}$$

and

$$N^i := -H(\tau)\bar{x}^i, \ i = 1, 2, 3. \tag{60}$$

respectively. Then,

$$N_i = h_{ij}N^j = \delta_{ij}N^j = N^i = -H(\tau)\bar{x}^i \tag{61}$$

and therefore the squared length of the shift vector is given by

$$N_iN^i = \sum_{i=1}^{3} \left(-H(\tau)\bar{x}^i\right)^2 = H^2(\tau)\sum_{i=1}^{3} \left(\bar{x}^i\right)^2 = H^2(\tau)r^2 \tag{62}$$

where

$$r = \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}. \tag{63}$$

Thus, with these definitions, the metric tensor (57) has the form of (55). That is, (57) not only conforms to the 3+1 formalism, but satisfies the Alcubierre conditions (as listed at the top of Section 4).

4.4.1 Properties of the Shift Vector in the Tramsformed RW Cosmology

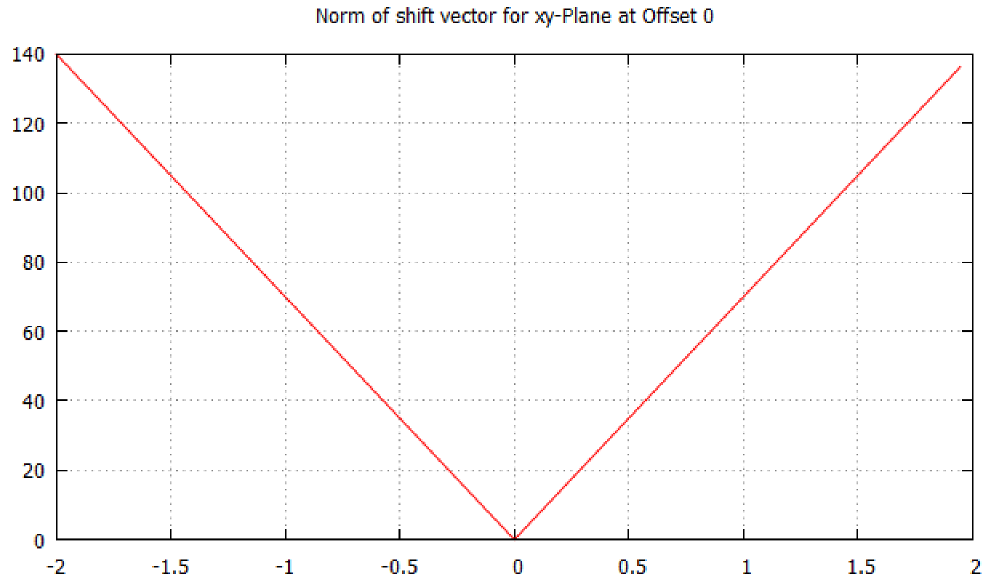
Note also that, from (62), the magnitude of the shift vector is given by

$$|N^*| = \sqrt{N_iN^i} = H(\tau)r. \tag{64}$$

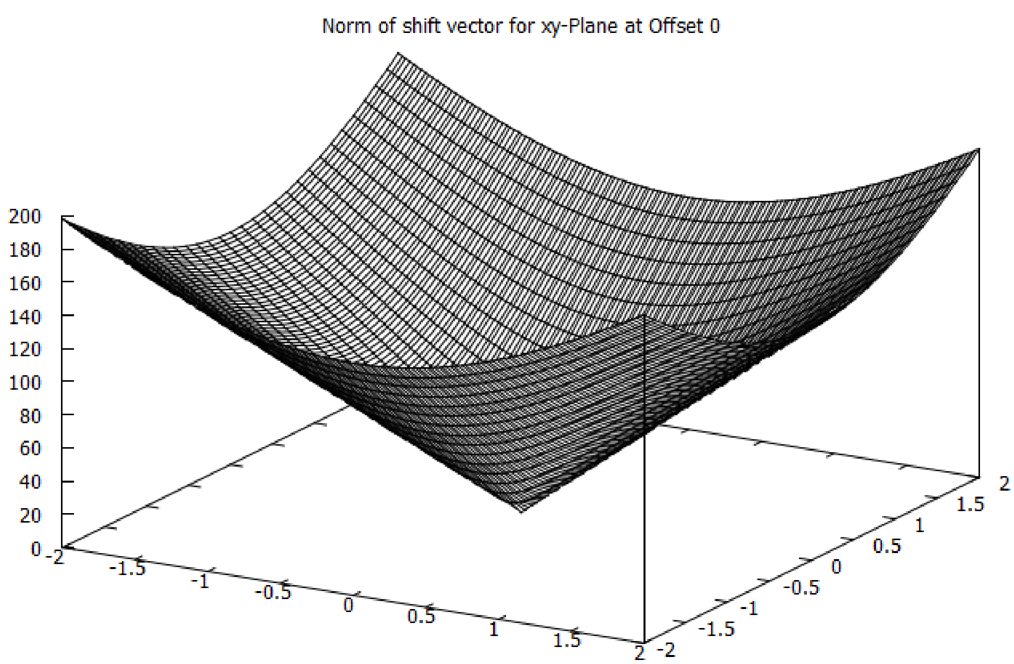
This is Hubbles law. In otherwords, the magnitude of the shift vector is the recessional velocity at radial distant r from the observer. For a fixed hypersurface Σ_0, where the subscript zero indicates the present day hypersurface, we have the linear relation

$$|N^*| = H_0 r. \tag{65}$$

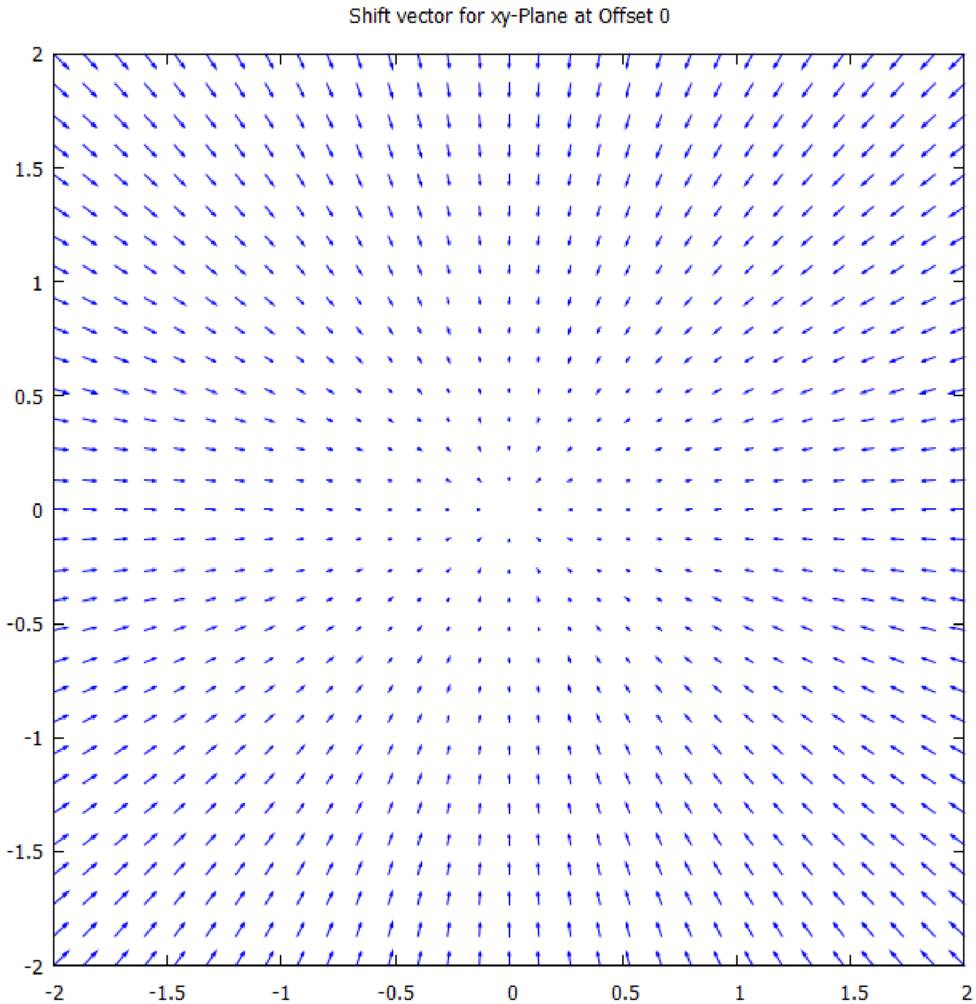
It should therefore come as no surprise that the cross-section of $|N^*|$ along the $\bar{x}$-axis has the following graph:



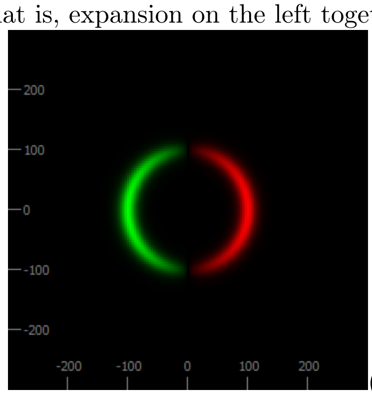
Here, the units on the horizontal and vertical axes are Mpc and km/s, respectively, and $H_0 = 70$ km/s/Mpc was used for the calculation (This is the modern value rounded to the nearest 10 units.) It is also not surpising that the field $|N^*|$ is spherically symmetric about the origin, as seen in the following embedding diagram:



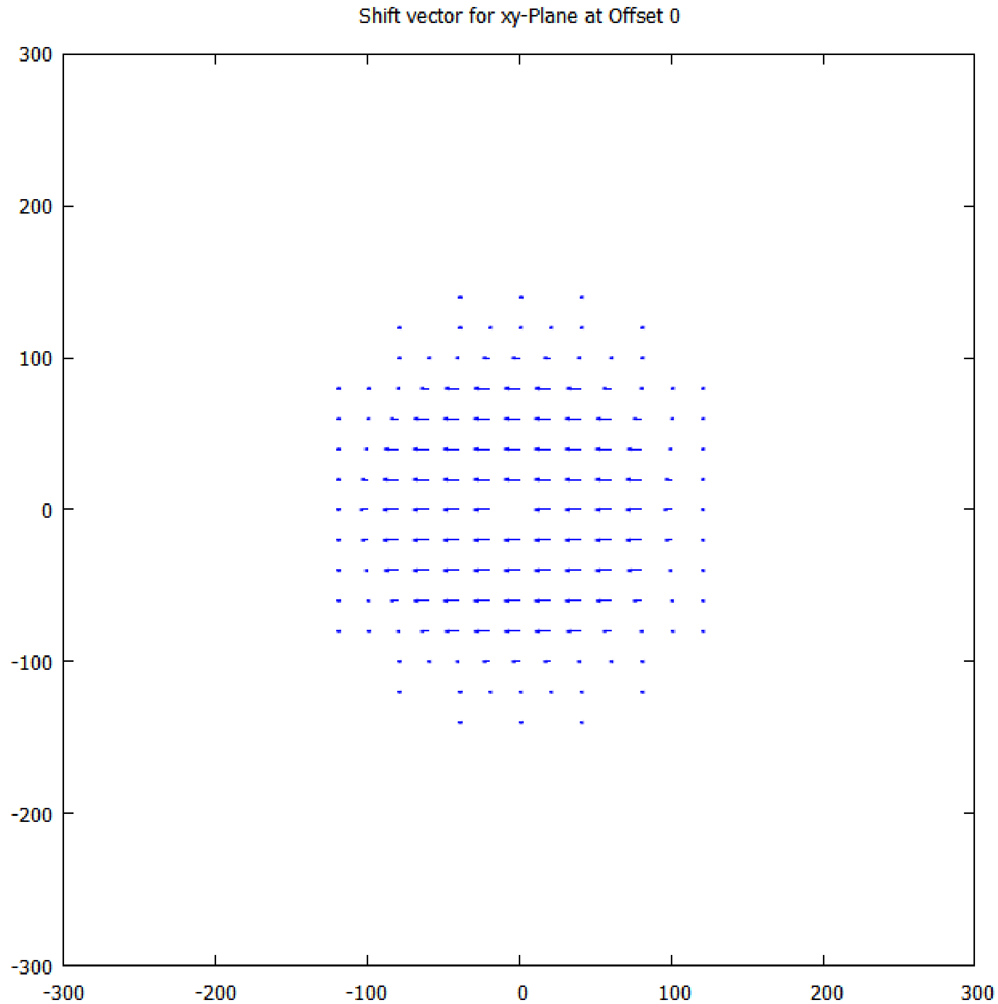
What may surprising is that the direction of the shift vector field is radially inward, as shown in the following vector diagram:



despite the fact that we know that the expansion of space is pushing things outwards. However, this is consistent with the Alcubierre results, wherein the expansion of space is pushing things to the right, but the shift vector field points to the left. That is, expansion on the left together with contraction on the right, as shown below.



(Reminder: green and red indicate expansion and contraction, respectively.)
moves things to right, but the shift vector field points to the left, as shown below.



4.4.2 Properties of the Extrinsic Curvature Tensor in the Transformed RW Cosmology

Recall (Talk 4) that the extrinsic curvature tensor is given, in all generality, by

$$K_{ij} = \frac{1}{2N}\left(\nabla_i^{(3)}N_j + \nabla_j^{(3)}N_i - \frac{\partial h_{ij}}{\partial t}\right),\ i,j = 1,2,3 \tag{66}$$

where $\nabla_j^{(3)}$ is the covariant derivative operator on the hypersurfaces, and that (66) reduces to

$$K_{ij} = \frac{1}{2}\left(\partial_i N_j + \partial_j N_i\right) \tag{67}$$

under the Alcubierre conditions. Substituting into (67) the shift vector defined in (61), yields (in matrix form)

$$[K_{**}] = \begin{bmatrix} -H(\tau) & 0 & 0 \\ 0 & -H(\tau) & 0 \\ 0 & 0 & -H(\tau) \end{bmatrix}. \tag{68}$$

Now, recall (from Talk 4) that the expansion factor is given in all generality by

$$\theta := -N\mathrm{tr}\left[K_*^*\right] = -N\mathrm{tr}\left\{[h_{**}]^{-1}[K_{**}]\right\}. \tag{69}$$

However, under the Alcubierre conditions, specifically that $N = 1$ and $h_{ij} = \delta_{ij}$, (69) reduces to

$$\theta = -\mathrm{tr}\left[K_{**}\right] \tag{70}$$

and substituting (68) yields

$$\theta = 3H(\tau). \tag{71}$$

Thus, the expansion factor is positive and constant over any specific hypersurface (which is consistent with an expanding, homogeneous universe).

Finally, the energy density, as seen by the Eulerian observers, is given in all generality by

$$\rho = \frac{1}{2k}\left(R^{(3)} + \theta^2 - K_{ij}K^{ij}\right) \tag{72}$$

where $R^{(3)}$ is the Ricci scalar on the hypersurfaces. However, under the Alcubierre conditions, (72) reduces to

$$\rho = \frac{1}{2k}\left(\theta^2 - \sum_{i=1}^{3}\sum_{j=1}^{3}K_{ij}^2\right) \tag{73}$$

and substituting (68) and (71) yields

$$\rho = \frac{1}{2k}\left(9H^2 - 3H^2\right) = \frac{1}{k}3H^2. \tag{74}$$

Now, replacing H with its defnition, as given by (56), yields

$$\rho = \frac{1}{k}3\frac{\dot{a}^2(\tau)}{a^2(\tau)} \tag{75}$$

which is the same as the result obtained in (18). That is, the energy density, as seen by the Eulerian observers (in the new coordinates), is the same as the principal energy density, just as it is for the fundamental observers (in the original coordinates).

5 A Cosmology Update

Since the discovery of the fact that the universal expansion is accelerating, it has become necessary to reinstate the cosmological term in Einstein's field equations, i.e.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = kT_{\mu\nu} \tag{76}$$

where Λ is the cosmological constant. It is natural to ask: what does the addition of the new term do to the character of the the field equations? Here is a quote from Wald (P. 99) that answers this question:

> "It can be shown [Lovelock 1972] that a linear combination of $G_{\mu\nu}$ and $g_{\mu\nu}$ is the most general two index symmetric tensor which is divergence free and can be constructed locally from the metric and its derivatives up to second order, so [(76)] gives the most general modification which does not grossly alter the basic properties of Einstein's equation."

In practice, it makes more sense to move the new term to the RHS of (76), where it becomes an additional source term, while the Einstein tensor and the original stress-energy tensor are still determined by the Robertson-Walker model, as before.

With the addition of the new source term, the total energy density is taken to be the sum of three contributions (rather than two), and so expressed as

$$\rho(\tau) = \rho_m(\tau) + \rho_r(\tau) + \rho_\Lambda\rho(\tau) \tag{77}$$

where

- ρ_m is the matter density, including baryonic and dark matter.
- ρ_r is the radiation density, which is given by $\rho_r = 3P$ for an isotropic source (the CMB).

- ρ_Λ is the vacuum energy density, which is given by

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi \mathtt{G}} \tag{78}$$

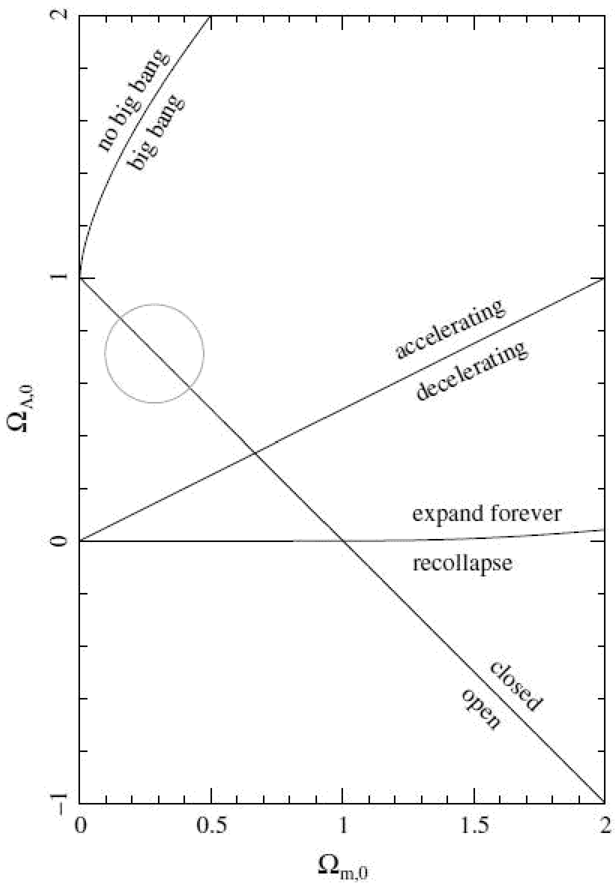
where Λ is the cosmological constant and $\mathtt{G}$ is Newton's gravitational constant.

It has become standard practice in cosmology to work with dimensionless parameters, which correspond to the three density contributions as follows:

$$\Omega_\alpha(\tau) = \frac{8\pi \mathtt{G}}{3H^2(\tau)}\rho_\alpha(\tau) \tag{79}$$

where H is the Hubble parameter and the subscript α is a label that denotes m, r or Λ. The present day value of these parameters, denoted by $\Omega_{\alpha,0}$, is given by (79), by substituting the present-value H_0 of the Hubble parameter and the present-day values $\rho_{\alpha,0}$ of the energy densities.

It is known that, in the present-day universe, the contribution of radiation density $\rho_{r,0}$ is negligible. A plot of the remaining 2-parameter space is shown below.



In the plot:

- The horizontal axis is the dimensionless matter density.
- The vertical axis is the dimensional vacuum energy density.
- The 2-parameter space is divided into several regions, the boundaries of which are determined by the field equations.
- The circle indicates where astronomical observations place the two parameters $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$.

Thus, according to the latest observations:

- There was indeed a big bang.
- The Universe will expand forever.
- The expansion is accelerating.
- The dividing line between an open (hyperbolic) universe and a closed (spherical) universe cuts the circle in half, so this question is not completely settled, but the dividing line itself is statistically the most likely case. That is, the universe is most likely flat.

NOTE: the above plot was taken from the book
"General Relativity: an Introduction for Physicists", 5th edition, 2012
by Hobson, Efstathiou and Lasenby