Photon polarization equalities

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$$
\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}\left(|x\rangle_I \otimes |x\rangle_{II} + |y\rangle_I \otimes |y\rangle_{II}\right) \\
&\equiv \frac{1}{\sqrt{2}}\left(|xx\rangle + |yy\rangle\right)
\end{aligned}
\tag{1}
$$

The second line is a notational simplification to be used later.

Polarizer I

$$
\begin{aligned}
|x\rangle_I &= \cos\alpha|+\rangle_I + \sin\alpha|-\rangle_I \\
|y\rangle_I &= -\sin\alpha|+\rangle_I + \cos\alpha|-\rangle_I
\end{aligned}
\tag{2}
$$

A photon with polarization in the x direction has a probability amplitude $\cos\alpha$ of ending up (after passing through polarizer I) with a polarization in the direction of $\underline{\mathbf{a}}$ (i.e., the + direction) which is at angle α. Thus the probability of + is $\cos^2\alpha$.

This gives us the well-known Malus law which tells us that a polarizer at angle α to the polarization of incident light, allows the light to penetrate the polarizer with a relative intensity of $\cos^2\alpha$.

Polarizer II

$$
\begin{aligned}
|x\rangle_{II} &= \cos\beta|+\rangle_{II} + \sin\beta|-\rangle_{II} \\
|y\rangle_{II} &= -\sin\beta|+\rangle_{II} + \cos\beta|-\rangle_{II}
\end{aligned}
\tag{3}
$$

Then expand $|\psi\rangle$.

$$\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}[(\cos\alpha|+\rangle_I + \sin\alpha|-\rangle_I)\otimes(\cos\beta|+\rangle_{II} + \sin\beta|-\rangle_{II})]+ \\
&\qquad (-\sin\alpha|+\rangle)_I + \cos\alpha|-\rangle_I)\otimes(-\sin\beta|+\rangle_{II} + \cos\beta|-\rangle_{II})] \\
&= \frac{1}{\sqrt{2}}[(\cos\alpha\cos\beta + \sin\alpha\sin\beta)|+\rangle_I\otimes|+\rangle_{II}+ \\
&\qquad (\cos\alpha\cos\beta + \sin\alpha\sin\beta)|-\rangle_I\otimes|-\rangle_{II})+ \\
&\qquad (\cos\alpha\sin\beta - \sin\alpha\cos\beta)|+\rangle_I\otimes|-\rangle_{II})+ \\
&\qquad (-\cos\alpha\sin\beta + \sin\alpha\cos\beta)|-\rangle_I\otimes|+\rangle_{II})]
\end{aligned} \tag{4}$$

The probability amplitude of both photons ending up with polarization + is $\langle++|\psi\rangle$ which projects the $|+\rangle_I\otimes|+\rangle_{II}$ component of the RHS of the above equation. Namely,

$$\begin{aligned}
\langle++|\psi\rangle &= \frac{1}{\sqrt{2}}(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\
&= \frac{1}{\sqrt{2}}\cos(\alpha-\beta),
\end{aligned} \tag{5}$$

so the probability $P_{++}(\alpha,\beta)$ is

$$\begin{aligned}
P_{++}(\alpha,\beta) &= |\langle++|\psi\rangle|^2 \\
&= \frac{1}{2}\cos^2(\alpha-\beta).
\end{aligned} \tag{6}$$

Similarly, $P_{--}(\alpha,\beta) = \frac{1}{2}\cos^2(\alpha-\beta)$.

The probability of photon I ending up with polarization + and photon II with polarization − is $\langle-+|\psi\rangle$ which projects the $|-\rangle_I\otimes|+\rangle_{II}$ component of the RHS of eq. (4) . Namely,

$$\begin{aligned}
\langle-+|\psi\rangle &= \frac{1}{\sqrt{2}}(\cos\alpha\sin\beta - \sin\alpha\cos\beta) \\
&= \frac{1}{\sqrt{2}}\sin(\alpha-\beta),
\end{aligned} \tag{7}$$

so the probability $P_{-+}(\alpha,\beta)$ is

$$\begin{aligned}
P_{-+}(\alpha,\beta) &= |\langle-+|\psi\rangle|^2 \\
&= \frac{1}{2}\sin^2(\alpha-\beta).
\end{aligned} \tag{8}$$

Similarly, $P_{+-}(\alpha,\beta) = \frac{1}{2}\sin^2(\alpha-\beta)$.