# A GENTLE INTRODUCTION TO NEUTRINO OSCILLATIONS

## Lecture 2: Neutrino oscillations

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# Contents

# Chapter 1

# TWO-LEVEL QUANTUM SYSTEM

In the previous chapter we learned about the "fifth force" which manifests itself in permanent oscillations between three neutrino flavors. Next, we would like to build a theoretical model of such oscillations.

Assumption: **Limit our description to only two neutrino flavors: $\mu$-neutrino and $\tau$-neutrino.**

Assumption: **Neutrinos have zero momentum, their spins are ignored.**

Conclusion: **Neutrino states can be described by vectors in a two dimensional Hilbert space $\mathcal{H}$.**

The full interacting Hamiltonian acting in this space will be denoted $H$. We should be able to find its eigenvalues $E_2$, $E_3$ and eigenvectors $|2\rangle$, $|3\rangle$:[1]

$$H|2\rangle = E_2|2\rangle, \tag{1.1}$$
$$H|3\rangle = E_3|3\rangle. \tag{1.2}$$

Observation: **Hamiltonian $H$ is Hermitian, therefore its two eigenvalues $E_2$, $E_3$ are real (we will also assume they are positive) and eigenvectors $|2\rangle$, $|3\rangle$ form an orthonormal basis in the Hilbert space $\mathcal{H}$.** See Fig. 1.1.

[1] We use labels 2 and 3 for historical reasons, in order to be consistent with conventional notation.

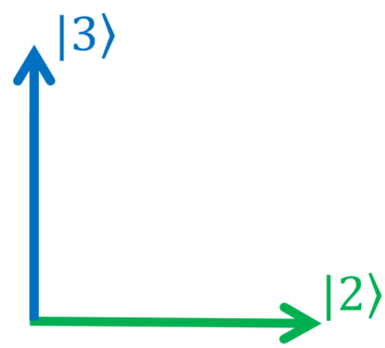


Figure 1.1: 2D Hilbert space of neutrino states with the orthonormal basis of Hamiltonian (or energy) eigenstates $|2\rangle, |3\rangle$.

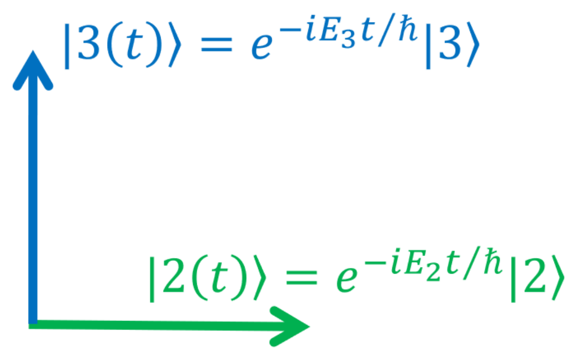


Figure 1.2: Time evolution of energy basis vectors is trivial: Phase factors do not change the physical nature of states.

Observation: **The time evolution of these two vectors is trivial: they are multiplied by time-dependent phase factors, which do not change the physical state.** See Fig. 1.2.

$$e^{-iHt/\hbar}|2\rangle = e^{-iE_2 t/\hbar}|2\rangle \tag{1.3}$$
$$e^{-iHt/\hbar}|3\rangle = e^{-iE_3 t/\hbar}|3\rangle \tag{1.4}$$

Conclusion: **Energy basis vectors $|2\rangle$, $|3\rangle$ cannot be associated with $\mu$-neutrino and $\tau$-neutrino, because these particles experience non-trivial time evolution (oscillations).**

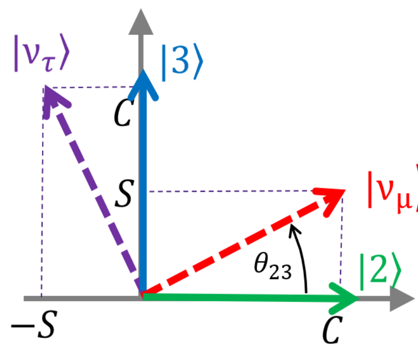


Figure 1.3: 2D Hilbert space of neutrino states with two orthonormal bases: $|\nu_\mu\rangle, |\nu_\tau\rangle$ are flavor eigenstates, $|2\rangle, |3\rangle$ are mass eigenstates.

Conclusion: **The state of $\mu$-neutrino is different from $|2\rangle$ and $|3\rangle$. It may be represented, for example, by vector $|\nu_\mu\rangle$ shown by broken red arrow in Fig. 1.3.**

Conclusion: **State vector of $\tau$-neutrino should be orthogonal to $|\nu_\mu\rangle$, because the two neutrino flavors are mutually exclusive and they should have zero overlap. Definite $\tau$-neutrino flavor state is shown by broken purple arrow in Fig. 1.3.**

Observation: **Energy basis $|2\rangle$, $|3\rangle$ and flavor basis $|\nu_\mu\rangle$, $|\nu_\tau\rangle$ are related to each other by a rotation through angle $\theta_{23}$.**

Notation: **If $|\Psi\rangle$ is an arbitrary vector in the Hilbert space, then its components in the flavor basis will be denoted by square brackets**

$$|\Psi\rangle = \begin{bmatrix} \Psi_\mu \\ \Psi_\tau \end{bmatrix} = \Psi_\mu|\nu_\mu\rangle + \Psi_\tau|\nu_\tau\rangle \tag{1.5}$$

Obviously, $|\Psi_\mu|^2$ is the probability of finding $\mu$-neutrino in the state $|\Psi\rangle$, $|\Psi_\tau|^2$ is the probability of finding $\tau$-neutrino in the state $|\Psi\rangle$.

$$|\Psi_\mu|^2 + |\Psi_\tau|^2 = 1. \tag{1.6}$$

So, vector components in the flavor basis are useful for analyzing the flavor content of the state.

Notation: **Vector components in the energy basis will be denoted by round parentheses**

$$|\Psi\rangle = \begin{pmatrix} \Psi_2 \\ \Psi_3 \end{pmatrix} = \Psi_2|2\rangle + \Psi_3|3\rangle \tag{1.7}$$

These components are useful for analyzing time evolution of the state (see eqs. (1.3) - (1.4)):

$$|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi\rangle = \Psi_2 e^{-iHt/\hbar}|2\rangle + \Psi_3 e^{-iHt/\hbar}|3\rangle \tag{1.8}$$

$$= \Psi_2 e^{-iE_2t/\hbar}|2\rangle + \Psi_3 e^{-iE_3t/\hbar}|3\rangle \tag{1.9}$$

To study time-dependent flavor oscillations we need both flavor and energy bases and we have to learn how to switch between two sets of components. This is accomplished by applying rotation matrices. If

$$\begin{pmatrix} \Psi_2 \\ \Psi_3 \end{pmatrix}$$

is a state vector in the energy basis, then its components in the flavor basis are obtained by a unitary (or orthogonal) transformation

$$\begin{bmatrix} \Psi_\mu \\ \Psi_\tau \end{bmatrix} = \begin{pmatrix} C & S \\ -S & C \end{pmatrix} \begin{pmatrix} \Psi_2 \\ \Psi_3 \end{pmatrix}. \tag{1.10}$$

Transformation from the flavor basis to the energy basis is given by the inverse matrix

$$\begin{pmatrix} \Psi_2 \\ \Psi_3 \end{pmatrix} = \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} \Psi_\mu \\ \Psi_\tau \end{bmatrix}, \tag{1.11}$$

where

$$C \equiv \cos\theta_{23} \tag{1.12}$$

$$S \equiv \sin\theta_{23} \tag{1.13}$$

Side note, to be useful later: **Energy basis vectors $|2\rangle$, $|3\rangle$ are eigenstates of the full Hamiltonian $H$. Likewise, we can assume that there exists a non-interacting Hamiltonian $H_0$ whose eigenvectors are flavor states $|\nu_\mu\rangle$, $|\nu_\tau\rangle$. $H_0$ has all 4 major interactions (strong, weak, electromagnetic and gravitational), and $H = H_0 + V$, where $V$ is the flavor mixing potential energy operator.**

## 1.1 Calculation of oscillations

Let us now evaluate the time evolution of an initially prepared $\mu$-neutrino. This state is easy to write down in the flavor basis

$$|\Psi(0)\rangle = |\nu_\mu\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{1.14}$$

Then we switch to the energy basis using (1.11),

$$|\Psi(0)\rangle = \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{pmatrix} C \\ S \end{pmatrix} = C|2\rangle + S|3\rangle. \tag{1.15}$$

At time $t$ this state vector evolves to

$$|\Psi(t)\rangle = Ce^{-iE_2t/\hbar}|2\rangle + Se^{-iE_3t/\hbar}|3\rangle = \begin{pmatrix} Ce^{-iE_2t/\hbar} \\ Se^{-iE_3t/\hbar} \end{pmatrix}. \tag{1.16}$$

To check the flavor content of this state we have to go back to the flavor basis using transformation (1.10)

$$|\Psi(t)\rangle = \begin{pmatrix} C & S \\ -S & C \end{pmatrix} \begin{pmatrix} Ce^{-iE_2t/\hbar} \\ Se^{-iE_3t/\hbar} \end{pmatrix} = \begin{bmatrix} C^2e^{-iE_2t/\hbar} + S^2e^{-iE_3t/\hbar} \\ CS(e^{-iE_3t/\hbar} - e^{-iE_2t/\hbar}) \end{bmatrix}. \tag{1.17}$$

The probability of finding $\mu$-neutrino in this state is given by the squared modulus of the first coefficient

$$\begin{aligned} \rho_\mu(t) &= |C^2e^{-iE_2t/\hbar} + S^2e^{-iE_3t/\hbar}|^2 \\ &= (C^2e^{iE_2t/\hbar} + S^2e^{iE_3t/\hbar})(C^2e^{-iE_2t/\hbar} + S^2e^{-iE_3t/\hbar}) \\ &= C^4 + S^4 + 2C^2S^2\cos(\gamma t/\hbar), \end{aligned} \tag{1.18}$$

where we denoted

$$\gamma = E_3 - E_2 \tag{1.19}$$

the gap between energy eigenvalues.

Probability (1.18) has an oscillating component. It is customary to refer to the argument of the cosine as the *oscillation phase*

$$\Delta\phi = \frac{\gamma t}{\hbar} = \frac{(E_3 - E_2)t}{\hbar}. \tag{1.20}$$

The flavor content of the instantaneous neutrino state depends on this phase. When

$$\Delta\phi = 0,\ 2\pi,\ 4\pi \ldots \tag{1.21}$$

the cosine factor is equal to 1, and the probability of finding $\mu$-neutrino is at its maximum

$$\rho_\mu(t) = C^4 + S^4 + 2C^2S^2 = (C^2 + S^2)^2 = 1 \tag{1.22}$$

When

$$\Delta\phi = \pi,\ 3\pi,\ 5\pi \ldots \tag{1.23}$$

the cosine factor is -1, and the probability of finding $\mu$-neutrino is at its minimum

$$\rho_\mu(t) = C^4 + S^4 - 2C^2S^2 = (C^2 - S^2)^2 \tag{1.24}$$

Using trigonometrical identities, it is not difficult to transform (1.18) to the conventional notation

$$\rho_\mu(t) = 1 - \sin^2(2\theta_{23})\sin^2(\Delta\phi/2) = 1 - \sin^2(2\theta_{23})\sin^2(\gamma t/2\hbar). \tag{1.25}$$

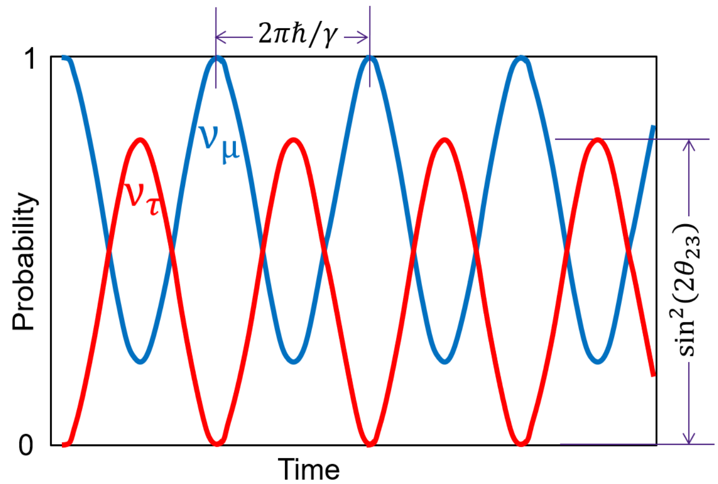


Figure 1.4: Plots of time dependent flavor probabilities (1.25) and (1.26). Experimental studies of these functions may reveal two oscillation parameters: period $2\pi\hbar/\gamma$ and amplitude $\sin^2(2\theta_{23})$.

Correspondingly, the probability to observe $\tau$-neutrino is

$$\rho_\tau(t) = 1 - \rho_\mu(t) = \sin^2(2\theta_{23})\sin^2(\Delta\phi/2) = \sin^2(2\theta_{23})\sin^2(\gamma t/2\hbar). \quad (1.26)$$

These probabilities are plotted in Fig. 1.4.

Conclusion: **Our simple two-level model provides a qualitatively correct description of neutrino oscillations.**

# Chapter 2

# OSCILLATIONS OF MOVING NEUTRINOS

## 2.1 Quantum mechanics of moving neutrinos

In the previous chapter we considered neutrinos at rest. But this was an oversimplifications. Neutrinos are moving, moreover they are ultra-relativistic.

Assumption: **For simplicity, we will consider neutrinos in 1D space.**

The Hilbert space of neutrino states has few important operators-observables. The operator of momentum $P$ has continuous spectrum of eigenvalues $-\infty < p < \infty$. The operator of mass $M$ has discrete spectrum with two eigenvalues $m_2$ and $m_3$. We assume for definiteness that $m_3 > m_2$. In a relativistic theory, the Hamiltonian is $H = \sqrt{M^2c^4 + P^2c^2}$.[1] The three operators commute with each other

$$[H, P] = [H, M] = [M, P] = 0 \tag{2.1}$$

Therefore, we can define a full orthonormal basis of their common eigenvectors $|2, p\rangle$, $|3, p\rangle$.

[1]Properties of observables will be derived by using Wigner-Dirac relativistic quantum mechanics in chapter xx.

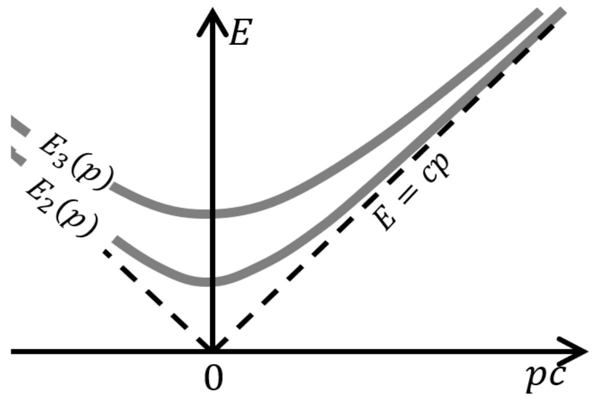


Figure 2.1: Momentum-energy eigenvalues (2.8) - (2.9) plotted as hyperbolas on the $pc - E$ plane.

$$P|2,p\rangle = p|2,p\rangle, \tag{2.2}$$
$$P|3,p\rangle = p|3,p\rangle, \tag{2.3}$$
$$M|2,p\rangle = m_2|2,p\rangle, \tag{2.4}$$
$$M|3,p\rangle = m_3|3,p\rangle, \tag{2.5}$$
$$H|2,p\rangle = E_2(p)|2,p\rangle, \tag{2.6}$$
$$H|3,p\rangle = E_3(p)|3,p\rangle. \tag{2.7}$$

where

$$E_3(p) = \sqrt{m_3^2c^4 + p^2c^2}, \tag{2.8}$$
$$E_2(p) = \sqrt{m_2^2c^4 + p^2c^2}. \tag{2.9}$$

Each eigenvector can be labeled by a pair of numbers $(p, E)$ and these pairs are plotted in Fig. 2.1. Each point on the lower hyperbola indicates eigenstate $(p, E_2(p))$ with mass $m_2$. Each point on the upper hyperbola indicates eigenstate $(p, E_3(p))$ with mass $m_3$.

Besides $M, H, P$ there are other important operators-observables. One of them is the operator of position $R$. It has a continuous spectrum of eigenvalues $-\infty < r < \infty$ and Heisenberg's commutator with momentum

$$[R, P] = i\hbar \tag{2.10}$$

Position commutes with mass

$$[R, M] = 0 \tag{2.11}$$

Therefore, we can define a mass-position basis $|2, r\rangle$, $|3, r\rangle$ with following properties

$$R|2, r\rangle = r|2, r\rangle, \tag{2.12}$$
$$R|3, r\rangle = r|3, r\rangle, \tag{2.13}$$
$$M|2, r\rangle = m_2|2, r\rangle, \tag{2.14}$$
$$M|3, r\rangle = m_3|3, r\rangle, \tag{2.15}$$

We may choose to represent state vectors by wave functions $\psi(m, p)$ in the momentum basis $|m, p\rangle$ or by functions $\psi(m, r)$ in the position basis $|m, r\rangle$. The momentum and position wave functions are related to each other by Fourier transform.

If $\psi(2, r)$ is a position wave function with mass $m_2$, then its momentum-space counterpart is

$$\psi(2, p) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(2, r) e^{-ipr/\hbar} dr \tag{2.16}$$

Conversely, position wave function corresponding to $\psi(2, p)$ is

$$\psi(2, r) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(2, p) e^{ipr/\hbar} dp \tag{2.17}$$

For example, if our state is an eigenstate of momentum with eigenvalue $p_0$

$$\psi(2, p) = \delta(p - p_0) \tag{2.18}$$

then its position counterpart is a plane wave that expands through entire space[2]

$$\psi_{p_0}(2,r) = \frac{1}{\sqrt{2\pi\hbar}} \int \delta(p-p_0) e^{ipr/\hbar} dp = \frac{1}{\sqrt{2\pi\hbar}} e^{ip_0 r/\hbar} \tag{2.20}$$

State (2.20) is an eigenstate of the energy operator, so its time dependence is given by a phase factor

$$\psi_{p_0}(2,r,t) = \frac{1}{\sqrt{2\pi\hbar}} e^{-i(E_2(p_0)t - p_0 r)/\hbar} \tag{2.21}$$

Similarly, the time-dependent position-space wave function of the mass state $m_3$ with definite momentum $p_0$ is

$$\psi_{p_0}(3,r,t) = \frac{1}{\sqrt{2\pi\hbar}} e^{-i(E_3(p_0)t - p_0 r)/\hbar} \tag{2.22}$$

## 2.2 Momentum-energy composition of a moving neutrino

In the previous chapter we discussed zero-momentum $\mu$-neutrino. Its mass components also had zero momentum. So, on the energy-momentum plot 2.2, these components can be indicated by circles $A$ and $B$. They have energies $E_2(0) = m_2c^2$ and $E_3(0) = m_3c^2$ and oscillation frequency

$$f = \frac{E_3 - E_2}{\hbar} = \frac{(m_3 - m_2)c^2}{\hbar} \tag{2.23}$$

But what about moving neutrinos? They also have two mass components.

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[2]This is an example of the Heisenberg uncertainty relationship

$$\Delta p \cdot \Delta r \approx \hbar \tag{2.19}$$

$\Delta p = 0$ for the delta function $\delta(p - p_0)$ in (2.18) and $\Delta r = \infty$ for the plane wave (2.20). The product of these uncertainties is finite (2.19).

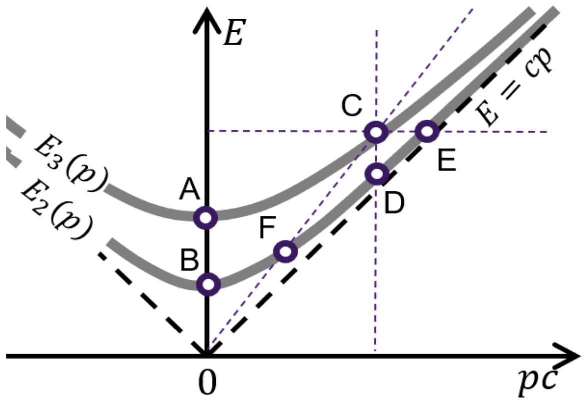


Figure 2.2: If neutrino is at rest then the two mass components $(A - B)$ of its wave packet have equal (zero) momenta. What about mass components of a moving neutrino? Do they have equal momenta $(C - D)$ or equal energies $(C - E)$ or equal velocities $(C - F)$?

*Do these components have equal momentum [1], or equal energy [2], or equal velocity [3]?*

If we fix the $m_3$ mass component of the moving neutrino to point $C$ in Fig. 2.2. Then the question is, which point on the lower hyperbola should be selected to represent the $m_2$ component: $D$ (equal momentum with $C$), $E$ (equal energy) or $F$ (equal velocity[3])? For many years, researchers continue publishing papers arguing about these possibilities, but still there is no consensus and clear understanding about the structure of high velocity/momentum/energy neutrino states.

[3]Recall that relativistic velocity is defined as $v = pc^2/E$. So all points on the line $0 - F - C$ have the same $p/E$ ratio and the same velocity. All points on the broken line $E = pc$ have the same velocity $v = c$.

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