A GENTLE INTRODUCTION TO NEUTRINO OSCILLATIONS

Lecture 3: Discussion of Thomson's section 13.4

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Contents

Chapter 1

OSCILLATIONS OF MOVING NEUTRINOS

1.1 Discussion of section 13.4 in Thomson [1]

In the previous chapter we discussed zero-momentum μ-neutrino. Its mass components also had zero momentum. So, on the energy-momentum plot 1.1, these components can be indicated by circles A and B. They have energies $E_2(0) = m_2c^2$ and $E_3(0) = m_3c^2$ and oscillation frequency

$$f = \frac{E_3 - E_2}{\hbar} = \frac{(m_3 - m_2)c^2}{\hbar} \tag{1.1}$$

But what about moving neutrinos? They also have two mass components.

Do these components have equal momentum [2], or equal energy [3], or equal velocity [4]?

If we fix the m_3 mass component of the moving neutrino to point C in Fig. 1.1. Then the question is, which point on the lower hyperbola should be selected to represent the m_2 component: D (equal momentum with C), E (equal energy) or F (equal velocity[1])? For many years, researchers continue publishing papers arguing about these possibilities, but still there is no consensus and clear understanding about the structure of high velocity/momentum/energy neutrino states.

[1]Recall that relativistic velocity is defined as $v = pc^2/E$. So all points on the line $0 - F - C$ have the same p/E ratio and the same velocity. All points on the broken line $E = pc$ have the same velocity $v = c$.

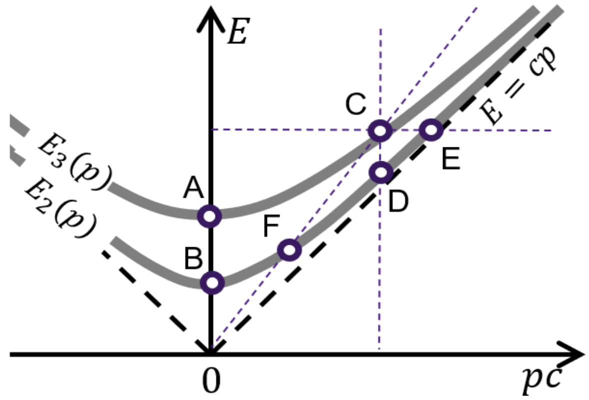

Figure 1.1: If neutrino is at rest then the two mass components $(A-B)$ of its wave packet have equal (zero) momenta. What about mass components of a moving neutrino? Do they have equal momenta $(C-D)$ or equal energies $(C-E)$ or equal velocities $(C-F)$?

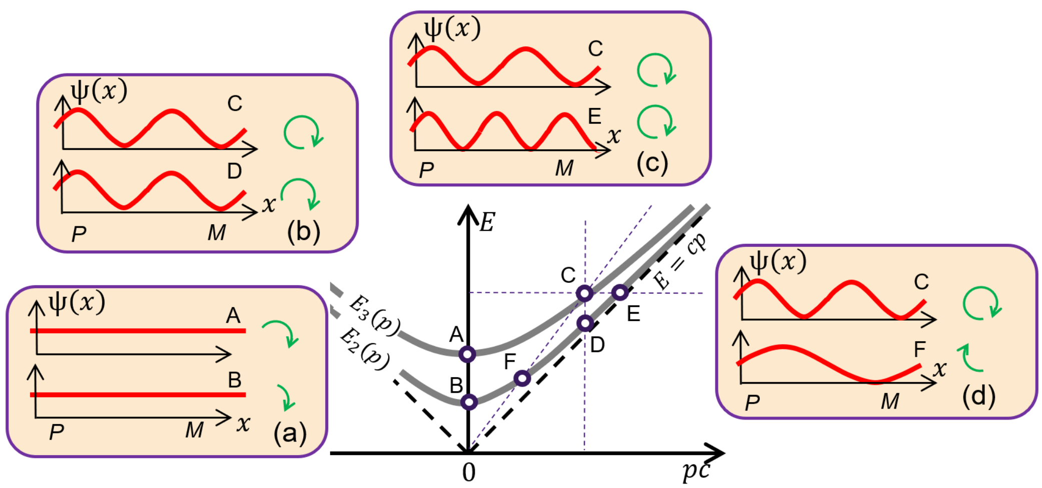

Figure 1.2: To discussion of section 13.4 in [1]. Red waves show spatial parts $e^{ipr/\hbar}$ of wave functions of the two mass components. Green arc arrows indicate time phase factors $e^{-iEr/\hbar}$.

One school of thought [1] says that it doesn't matter, and that oscillation formula comes out the same in all conditions. Let me explore this argument.[2] We consider neutrino states that are eigenvectors of momentum, mass and energy (**??**) - (**??**). Correspondingly, these are plane waves in the position space (**??**) - (**??**)

$$\psi(2,r,t) = \frac{1}{\sqrt{2\pi\hbar}} e^{-i(E_2 t - p_2 r)/\hbar} = \frac{1}{\sqrt{2\pi\hbar}} e^{-i\phi_2(r,t)} \tag{1.2}$$

$$\psi(3,r,t) = \frac{1}{\sqrt{2\pi\hbar}} e^{-i(E_3 t - p_3 r)/\hbar} = \frac{1}{\sqrt{2\pi\hbar}} e^{-i\phi_3(r,t)} \tag{1.3}$$

Each such plane wave is characterized by its own phase (which depends on time and position)

$$\phi_2(r,t) = (E_2 t - p_2 r)/\hbar \tag{1.4}$$

$$\phi_3(r,t) = (E_3 t - p_3 r)/\hbar \tag{1.5}$$

The flavor composition at a given space-time point depends on the phase difference. If $\phi_3 - \phi_2 = 0,\ 2\pi \ldots$, then we have a μ-neutrino. If $\phi_3 - \phi_2 = \pi,\ 3\pi, \ldots$, we have the maximum probability for τ-neutrino.

Zero momentum states $\boldsymbol{A - B}$. Let's go back to the pair of states with zero momentum A and B in Fig. 1.2. In the inset 1.2(a) this situation is shown in the position representation. We choose our time-coordinate system so that neutrino is emitted or produced at point $t = 0, r = 0$ (point P in Fig. 1.2(a)). The phase difference is zero at this point $\Delta\phi = \phi_3 - \phi_2 = 0$, so we have an initial μ-neutrino. Neutrino measurement is performed at point M, which is located at distance L from the production point. Moreover, the measurement is performed T seconds later than the emission. Both parameters L and T affect the flavor composition of the measured neutrino. Setting $t = T, r = L$ in our phase formulas (1.4) - (1.5), we get the measured phase difference

$$\begin{aligned}\Delta\phi &= (E_3 T - p_3 L)/\hbar - (E_2 T - p_2 L)/\hbar \\ &= T(E_3 - E_2)/\hbar - L(p_3 - p_2)/\hbar \end{aligned}\tag{1.6}$$

[2] I am in a difficult position, because I don't regard this argument as valid, but still have to explain it.

This is a general expression that is valid for any combination of momentum-energy pairs (p_2, E_2) and (p_3, E_3). Let us now calculate the phase difference for the zero-momentum case $A - B$.

Position-space wave functions of the two neutrino components A and B are plane waves with zero wave vector, i.e., they are just constants, as shown by red lines in the inset 1.2(a). In this case $p_2 = p_3 = 0$, so the $L(p_3 - p_2)$ contribution to the phase shift (1.6) is zero. However, the two states have different energies and different time dependent phase factors $e^{-iE_2t/\hbar}$ and $e^{-iE_3t/\hbar}$.[3] Then the phase difference depends on time as $\Delta\phi = T(E_3 - E_2)/\hbar$. μ-neutrino changes to τ-neutrino and back simultaneously in the entire space.

Equal momentum states $\boldsymbol{C - D}$. Now, look at inset 1.2(b) where I showed position-space plane waves corresponding to the pair of states $C - D$ having equal momentum. These two plane waves have the same wave vector $p_2 = p_3 = p$. Just like in the case of $A - B$, the $L(p_3 - p_2)$ contribution to the phase shift is zero. Only the energy part $T(E_3 - E_2)$ contributes, and oscillations proceed synchronously at all positions. Obviously, the energy difference $E_3 - E_2$ is smaller than in the zero-momentum case $A - B$. This means that oscillations become slower as neutrino energy increases.

Taking into account that neutrinos are ultra-relativistic ($p \gg m_2c,\ m_3c$), we may approximate

$$
\begin{aligned}
E_3(p) &= \sqrt{m_3^2c^4 + p^2c^2} = pc\sqrt{1 + \frac{m_3^2c^4}{p^2c^2}} \approx pc\left(1 + \frac{m_3^2c^4}{2p^2c^2}\right) = pc + \frac{m_3^2c^3}{2p} \\
E_2(p) &\approx pc + \frac{m_2^2c^3}{2p} \\
\gamma(p) &= E_3(p) - E_2(p) \approx \frac{(m_3^2 - m_2^2)c^3}{2p} \qquad (1.7)
\end{aligned}
$$

Then the oscillation phase is

$$\Delta\phi = \frac{(E_3 - E_2)T}{\hbar} = \frac{(m_3^2 - m_2^2)c^3T}{2\hbar p} \qquad (1.8)$$

In experiments, oscillations are measured not as functions of time, but as functions of L - the distance between neutrino source and detector. For our

[3] This difference is indicated by different sizes of green arc arrows in 1.2(a).

ultra-relativistic particles $T \approx L/c$ and $p \approx E/c$, where E is neutrino energy. This leads to oscillation phase

$$\Delta\phi = \frac{(m_3^2 - m_2^2)c^3 L}{2\hbar E} \tag{1.9}$$

Substituting this expression in (**??**), we obtain the standard result of neutrino theory

$$\rho_\mu(E, L) = 1 - \sin^2(2\theta_{23}) \sin^2 \frac{(m_3^2 - m_2^2)c^3 L}{4\hbar E}. \tag{1.10}$$

We see that oscillations depend on two parameters: the mixing angle θ_{23} and the difference of squared masses $m_3^2 - m_2^2$. The first parameter controls the oscillations amplitude $\sin^2(2\theta_{23})$. The second parameter is related to the spatial period of oscillations. Values of these parameters can be extracted from observations[4]

$$\sin^2(2\theta_{23}) > 0.9 \tag{1.11}$$

$$m_3^2 - m_2^2 = 23.2 \times 10^{-4}\ eV^2/c^4 \tag{1.12}$$

Unfortunately, oscillation studies cannot provide neutrino mass eigenvalues m_2 and m_3.[5]

Equal energy states $C - E$. Now, let us assume that the two mass components of the neutrino have equal energies as in the pair of states $C - E$. The energy part $T(E_3 - E_2)/\hbar$ does not contribute to the phase shift (1.6). The two plane waves shown in the inset 1.2(c) have the same time-dependent phase factors.[6] The phase difference between points P and M comes from the difference in position-space wave functions

$$\Delta\phi = -L(p_3 - p_2)/\hbar \tag{1.13}$$

[4]In general, the mixing angle θ_{23} may depend on neutrino momentum/energy. However, experiments indicate that this parameter is constant within wide range of energies. We will also make this assumption in our calculations.

[5]As we mentioned previously, experimental data indicate that these values do not exceed 1 eV/c^2.

[6]This is indicated in Fig. 1.2(c) by equal sizes of green arc arrows.

In the ultra-relativistic case,

$$p_3 - p_2 \approx \frac{E_3 - E_2}{c} \approx \frac{(m_3^2 - m_2^2)c^3}{2pc} \approx \frac{(m_3^2 - m_2^2)c^3}{2E} \tag{1.14}$$

and we return to our previous result (1.9)

$$\Delta\phi = \frac{(m_3^2 - m_2^2)c^3 L}{2\hbar E} \tag{1.15}$$

One can get a better idea of why such different physical assumptions (equal momentum and equal energy) lead to the same oscillation phase by considering the following. In the general formula (1.6), we can substitute $T = L/c$, then

$$\Delta\phi = \frac{L[(E_2 - E_3) - (p_2 - p_3)c]}{\hbar c} \tag{1.16}$$

Let us now keep point $C = (p_3, E_3)$ on the upper hyperbola and move the lower hyperbola point (p_2, E_2) along the path $F - D - E$. This path roughly coincides with the "light cone" $E = pc$, therefore $p_2 \approx E_2/c$, and the phase shift can be approximated

$$\Delta\phi \approx L(E_2 - E_3 - E_2 + p_3 c)/(\hbar c) = L(p_3 c - E_3)/(\hbar c). \tag{1.17}$$

This expression remains constant, i.e., independent on the choice of the point (p_2, E_2) on the lower hyperbola. This "proves" that oscillation pattern depends on the energy/momentum composition of the oscillating neutrino only weakly.

What's wrong with this theory? It is very unrealistic to represent neutrinos by plane waves having infinite extension in space. In OPERA experiment, the size of the position-space wave function has to be more than 732 km! Such a representation disregards the fact that neutrino propagates from the point of its creation to the detector and covers the distance of L in the course of its propagation.

A better theory should represent neutrinos by wave packets, which are localized in both position and momentum spaces. At time $t = 0$ the localized wave packet is created near the source. Then the center of the wave packet moves through space with velocity $v \approx c$ and after time $T = L/v$ it reaches the detector. In the next section, we will review a quantum-mechanical approach to wave packets and their propagation in space.

1.2 Wave packets and their trajectories

For simplicity, in this section we will consider quantum mechanics of one free particle in one-dimensional space.

In section **??** we already established that position-space wavefunction $\psi(r)$ and momentum-space wavefunction $\psi(p)$ of the same state are related to each other by direct and inverse Fourier transforms

$$\psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(r) e^{-ipr/\hbar} dr \tag{1.18}$$

$$\psi(r) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(p) e^{ipr/\hbar} dp \tag{1.19}$$

Let us now explore this connection in some more detail.

First look at the upper panel (a) in Fig. 1.3. On the right hand side I showed wave function $\psi(r)$ of a particle localized close to the origin $r = 0$. Inserting this function in the Fourier integral (1.18), we obtain its momentum-space counterpart on the left hand side of 1.3(a). The position wave packet $\psi(r)$ has no sinusoidal components, so its Fourier frequency profile $\psi(p)$ should be localized around zero momentum $p = 0$, which indicates that our state is not moving.

On the right hand side of Fig. 1.3(b), the position-space wave packet is shifted to the right by distance a. The relevant Fourier integral can be calculated by shifting integration variable $r - a = s$ in (1.18)

$$\begin{aligned}
\psi'(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int \psi(r-a) e^{-ipr/\hbar} dr \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int \psi(s) e^{-ip(s+a)/\hbar} ds \\
&= e^{-ipa/\hbar} \frac{1}{\sqrt{2\pi\hbar}} \int \psi(s) e^{-ips/\hbar} ds \\
&= e^{-ipa/\hbar} \psi(p)
\end{aligned}$$

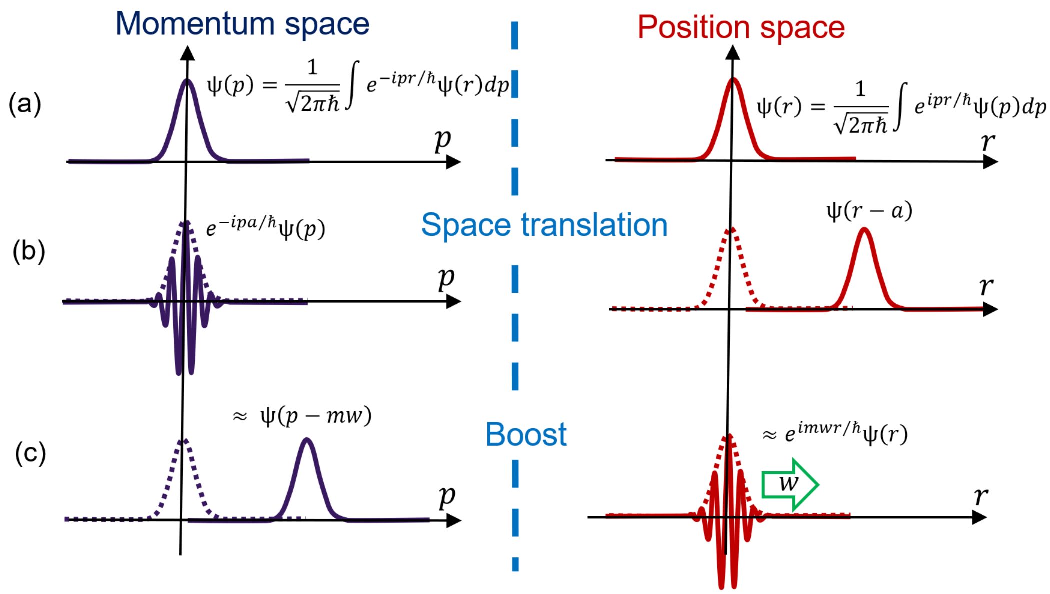

Figure 1.3: Localized particle states described by momentum and position wave packets.

Evidently, the momentum-space Fourier transform of a displaced state acquires a p-dependent phase factor $e^{-ipa/\hbar}$, as shown on the left hand side of 1.3(b). Function $\psi(p)$ is still localized near $p = 0$, because the position-space wave packet is not moving.

In order to describe a moving particle, we will apply a boost transformation to the momentum wave packet in 1.3(a). The result is shown on the left panel of Fig. 1.3(c). The true relativistic boost transformation is rather complicated,[7] but in a reasonable approximation it may be represented as a simple function shift $\psi(p) \to \psi(p - mw)$, where m is particle's mass and w is boost velocity. Then, the position-space counterpart acquires a phase factor $e^{imwr/\hbar}$; its center moves in the position space with velocity w.

In order to see the actual movement of the position-space wave packet, note that the time evolution of the momentum wave function is given by simple phase factor

[7] see section xx

$$\psi(p,t) = e^{-iE(p)t/\hbar}\psi(p) \tag{1.20}$$

Then the corresponding position-space wave function at time t is

$$\psi(r,t) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{ipr/\hbar} e^{-iE(p)t/\hbar}\psi(p)dp. \tag{1.21}$$

For approximate calculation of this integral, we expand $E(p)$ in a Taylor series around the center of the momentum-space wave packet $p_0 = mw$

$$E(p) = E(p_0) + \frac{dE(p)}{dp}|_{p=p_0} q + \dots \tag{1.22}$$

$$= E(p_0) + vq + \dots \tag{1.23}$$

Here we took into account that momentum derivative of particle energy coincides with particle's velocity (see Fig. 1.4)

$$\frac{dE(p)}{dp}\Big|_{p=p_0} = v. \tag{1.24}$$

We also shift the integration variable

$$p = mw + q$$
$$\Psi(q) = \psi(mw + q)$$

Then

$$\begin{aligned}
\psi(r,t) &\approx \frac{1}{\sqrt{2\pi\hbar}} e^{ip_0r/\hbar} e^{-iE(p_0)t/\hbar} \int e^{iqr/\hbar} e^{-ivqt/\hbar}\Psi(q)dq \\
&= e^{ip_0r/\hbar} e^{-iE(p_0)t/\hbar} \left(\frac{1}{\sqrt{2\pi\hbar}} \int e^{iq(r-vt)/\hbar}\Psi(q)dq \right) \\
&= e^{ip_0r/\hbar} e^{-iE(p_0)t/\hbar}\psi(r - vt, 0)
\end{aligned}$$

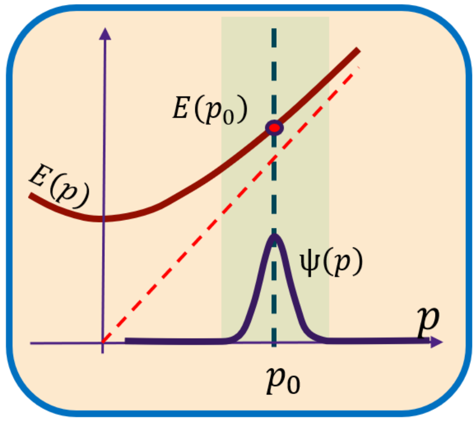

Figure 1.4: Position-space wave packet moves with velocity, which is momentum derivative of energy in the center of the momentum-space wave packet: $v = dE(p)/dp|_{p=p_0}$.

This result means that apart from being multiplied by a unimodular phase factor $e^{ip_0 r/\hbar}e^{-iE(p_0)t/\hbar}$, the position-space wave packet remains undisturbed and simply moves in space with constant velocity v.[8]

In this quasi-classical approach, our particle may be approximated as a point (p_0, r_0) in the momentum-position phase space, so that particle trajectory can be studied by classical mechanics.

Another conclusion: We assumed that sizes of our wave packets are rather small, much smaller than the distance between neutrino source P and neutrino detector M. This means that derivations of the oscillation formula in Thomson's section 13.4 [1] can no longer be trusted. In the next chapter we will build a theory of neutrino oscillation, which will take into account both the ultra-relativistic character of these particles and the localized nature of their wave packets.

[8]Of course, in order to obtain this simple formula we used the linear approximation (1.23). If non-linear terms in this Taylor expansion are taken into account, then the wave packet will spread out in addition to the uniform movement of its center.

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