A GENTLE INTRODUCTION TO NEUTRINO OSCILLATIONS

Lecture 5: Dirac's forms of dynamics

Eugene Stefanovich
eugene_stefanovich@usa.net

San Jose
California
December 11, 2023

Contents

Chapter 1

DIRAC'S FORMS OF DYNAMICS

In the previous chapter we discussed quantum mechanics of a single free particle. Our goal is to reach a relativistic description of neutrino, which is a momentum-dependent two-level system with mixing interaction. Wigner's theory has to be enhanced to take the interaction into account. Extension of Wigner's theory to interacting systems was achieved by Dirac in 1949 [1].

1.1 Non-interacting representation of the Poincaré Lie algebra

Wigner's relativistic quantum mechanics tells us that each quantum system has a Hilbert space where three Hermitian generators P, H, K act and their actions satisfy commutation relations (**??**) - (**??**). This is true for the oscillating (interacting) neutrino system as well. But this should remain true also if we turn off the interaction responsible for neutrino mixing. Let us mark generators of this non-interacting neutrino by the subscript "0": P_0, H_0, K_0. Non-interacting generators must satisfy standard Poincaré commutators

$$[H_0, P_0] = 0, \tag{1.1}$$

$$[K_0, P_0] = -\frac{i\hbar}{c^2} H_0, \tag{1.2}$$

$$[K_0, H_0] = -i\hbar P_0. \tag{1.3}$$

Let us now build in our Hilbert space the orthonormal basis $|\mu, p\rangle, |\tau, p\rangle$ of common eigenvectors of the three commuting operators: total momentum P_0, total energy H_0 and mass $M_0 = c^{-2}\sqrt{H_0^2 - P_0^2 c^2}$.

$$P_0|\mu, p\rangle = p|\mu, p\rangle, \tag{1.4}$$
$$P_0|\tau, p\rangle = p|\tau, p\rangle, \tag{1.5}$$
$$M_0|\mu, p\rangle = m_\mu|\mu, p\rangle, \tag{1.6}$$
$$M_0|\tau, p\rangle = m_\tau|\tau, p\rangle, \tag{1.7}$$
$$H_0|\mu, p\rangle = E_\mu(p)|\mu, p\rangle, \tag{1.8}$$
$$H_0|\tau, p\rangle = E_\tau(p)|\tau, p\rangle. \tag{1.9}$$

where[1]

$$E_\mu(p) = \sqrt{m_\mu^2 c^4 + p^2 c^2}, \tag{1.10}$$
$$E_\tau(p) = \sqrt{m_\tau^2 c^4 + p^2 c^2}. \tag{1.11}$$

It is convenient to represent states $|\Psi\rangle \in \mathcal{H}$ as superpositions (integrals) of basis vectors $|\mu, p\rangle, |\tau, p\rangle$

$$|\Psi\rangle = \int \Psi_\mu(p)|\mu, p\rangle dp + \int \Psi_\tau(p)|\tau, p\rangle dp.$$

Coefficients of these superpositions $\Psi_\mu(p)$ and $\Psi_\tau(p)$ are complex momentum-space wave functions satisfying the normalization condition

$$\int \left(|\Psi_\mu(p)|^2 + |\Psi_\tau(p)|^2\right) dp = 1.$$

Obviously, we can interpret two parts of this integral as probabilities of finding μ-neutrino and τ-neutrino in our state $|\Psi\rangle$

$$\rho_\mu = \int |\Psi_\mu(p)|^2 dp,$$
$$\rho_\tau = \int |\Psi_\tau(p)|^2 dp.$$

[1] In imaginary world where neutrino mixing interaction is turned off, m_μ and m_τ would be neutrino masses.

It will be convenient to put the two wave functions into one 2-component momentum-dependent vector[2]

$$|\Psi\rangle = \left[\begin{array}{c} \Psi_\mu(p) \\ \Psi_\tau(p) \end{array} \right].$$

As there is no mixing between the flavors, each particle ν_μ and ν_τ lives independently in its own subspace, so all relevant operators can be represented by diagonal 2×2 matrices in the flavor basis.

$$M_0 = \left[\begin{array}{cc} m_\mu & 0 \\ 0 & m_\tau \end{array} \right], \tag{1.12}$$

$$P_0 = \left[\begin{array}{cc} p & 0 \\ 0 & p \end{array} \right], \tag{1.13}$$

$$H_0 = \left[\begin{array}{cc} E_\mu(p) & 0 \\ 0 & E_\tau(p) \end{array} \right]. \tag{1.14}$$

$$K_0 = -i\hbar \left[\begin{array}{cc} \frac{E_\mu(p)}{c^2}\frac{d}{dp} + \frac{p}{2E_\mu(p)} & 0 \\ 0 & \frac{E_\tau(p)}{c^2}\frac{d}{dp} + \frac{p}{2E_\tau(p)} \end{array} \right], \tag{1.15}$$

$$R_0 = i\hbar \left[\begin{array}{cc} \frac{d}{dp} & 0 \\ 0 & \frac{d}{dp} \end{array} \right], \tag{1.16}$$

It is easy to prove the Poincaré commutators between generators (1.13) - (1.15). This completes construction of the non-interacting representation of the Poincaré Lie algebra in the neutrino Hilbert space $\mathcal{H}$.

1.2 Dirac forms of dynamics

Our next goal is to repeat the above construction for interacting representation (P, H, K) of the Poincaré Lie algebra.

Naively, we might assume that the difference between non-interacting and interacting system resides only in the Hamiltonian, meaning that generators of space translations and boosts remain the same in both systems

[2]Recall that we use square brackets to indicate components of vectors and matrices written in the basis of non-interacting (flavor) eigenstates.

$$H = H_0 + V, \tag{1.17}$$
$$P = P_0, \tag{1.18}$$
$$K = K_0. \tag{1.19}$$

This is how interacting theories were usually formulated in non-relativistic classical and quantum mechanics. But in 1949 Dirac discovered [1] that this assumption violates Wigner's principle of relativistic invariance. Indeed, from (**??**) and (1.18) - (1.19) we can write

$$-\frac{i\hbar}{c^2}H = [K, P] = [K_0, P_0].$$

According to (1.2), the last commutator is equal to $-(i\hbar/c^2)H_0$, which results in the absurd equality

$$H = H_0.$$

The way out of this contradiction is to accept that relativistic interaction should be constructed by adding some interaction "potentials" to the generator of space shifts P_0 or to the generator of boosts K_0 or to both these generators. There are different ways to choose these interaction potentials, which are called Dirac's "forms of dynamics". In this work, we will focus only on two such solutions: "instant form" and "front form".

In Dirac's *instant form* of relativistic dynamics, an interaction term Z is added to the boost momentum generator K_0, while the linear momentum P_0 remains interaction-free:

$$H = H_0 + V,$$
$$P = P_0,$$
$$K = K_0 + Z.$$

In addition to the familiar "potential energy" V we have to consider "potential boost" Z.

In the *point form* of Dirac's dynamics, interaction modifies the total momentum, while the boost operator remains non-interacting:

$$\begin{aligned} H &= H_0 + V, \\ P &= P_0 + U, \\ K &= K_0. \end{aligned} \tag{1.20}$$

Here we meet the new notion of "potential momentum" U.

Our goal is to apply Dirac's ideas about relativistic interactions to neutrinos and see how our choice of relativistic dynamics affects the oscillation formula.

1.3 Physical meaning of symmetry generators. Non-interacting case. (A side note)

Now, let us deviate for a moment from the main thread of the story about neutrinos and spend some time on discussing physical relevance of the forms of dynamics. How should we understand the new concepts of "potential boost" Z and "potential momentum" U. Can we measure effects of these operators in experiments? What are observable differences between the instant and point forms of dynamics?[3]

For our discussion, instead of exotic oscillating neutrinos, it will be more convenient to use as our example a system of N particles described in all textbooks on classical and quantum mechanics. Let us begin with the non-interacting set of symmetry generators P_0, H_0, K_0. We claim that they describe a system of free particles. What does it mean exactly?

The non-interacting generators are sums of one-particle operators. For example, the operator of total momentum is a sum of one-particle momenta:

$$P_0 = p_1 + p_2 + \dots \tag{1.21}$$

In order to transform state vectors and observables to the reference frame shifted by distance a we have to apply the unitary operator of space translation

[3]Note that discussion below will deviate from the mainstream, because the consensus opinion is that different forms of dynamics are physically equivalent [2, 3].

$$e^{iP_0 a/\hbar}.$$

Due to the additive character of the generator (1.21) and commutativity of p_i, the space translation operator splits into a product of one-particle operators

$$e^{iP_0 a/\hbar} = e^{i(p_1+p_2+\ldots)a/\hbar} = e^{ip_1 a/\hbar} e^{ip_2 a/\hbar} \ldots \tag{1.22}$$

This means that shifts apply individually and independently to each particle in the system. The action of space shift on each particle does not depend on the presence of other particles.

The non-interacting Hamiltonian is also additive. In non-relativistic mechanics (classical or quantum) we have

$$H_0 = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \ldots \tag{1.23}$$

In relativistic physics we use a more precise expression

$$\begin{aligned} H_0 &= \sqrt{m_1^2c^4 + p_1^2c^2} + \sqrt{m_2^2c^4 + p_2^2c^2} + \ldots \\ &= m_1c^2 + m_2c^2 + \ldots + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \ldots \end{aligned} \tag{1.24}$$

Time evolution operator is an exponent of this generator. Energy operators of different particles commute, therefore the time evolution operator splits into a product of one-particle operators

$$e^{-iH_0t/\hbar} = e^{-i\left(\sqrt{m_1^2c^4+p_1^2c^2}+\sqrt{m_2^2c^4+p_2^2c^2}+\ldots\right)t/\hbar} = e^{-i\sqrt{m_1^2c^4+p_1^2c^2}t/\hbar} e^{-i\sqrt{m_2^2c^4+p_2^2c^2}t/\hbar} \ldots \tag{1.25}$$

The physical interpretation of this product is that individual particles in our system evolve in time independently on the presence of other particles. This is what we mean when we say that particles do not interact. In classical mechanics, the absence of interaction means that each particle has its own linear trajectory

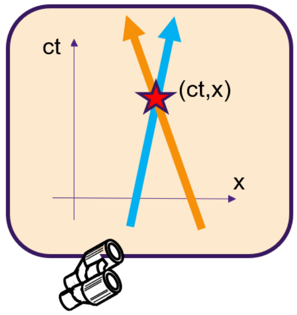

Figure 1.1: Linear trajectories of two non-interacting particles in space-time coordinates $ct - x$.

$$r_i(t) = r_{i0} + v_{i0}t,$$

which depends only on initial conditions r_{i0}, v_{i0} of this particle and nothing else. See Fig. 1.1.

The non-interacting boost generator is also a sum of one-particle terms

$$K_0 = k_1 + k_2 + \dots \tag{1.26}$$

where single-particle boost operators are (compare with Eq. (**??**))

$$k_i = -i\hbar \left(\frac{h_i}{c^2} \frac{d}{dp_i} + \frac{p_i}{2h_i} \right).$$

Boost momenta of different particles commute with each other, therefore the total boost transformation is

$$e^{iK_0 c\theta/\hbar} = e^{i(k_1+k_2+\dots)c\theta/\hbar} = e^{ik_1 c\theta/\hbar} e^{ik_2 c\theta/\hbar} \dots \tag{1.27}$$

Just like space and time translations, non-interacting boost acts independently on each particle.

In the classical approximation $\hbar \to 0$, boosts transform linear trajectories to other linear trajectories, as in Fig. 1.2. Interestingly, in this classical world

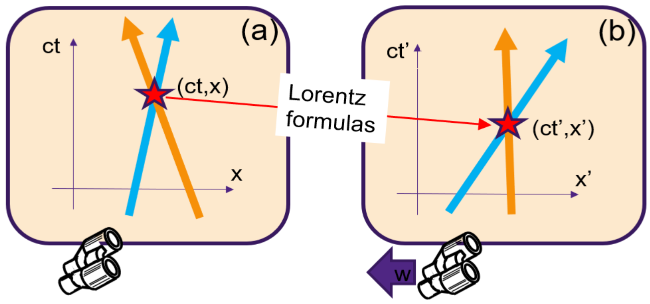

Figure 1.2: Transformation of particle trajectories under non-interacting boost. Coordinates of events (defined as intersections of trajectories) transform by Lorentz formulas.

one can define "events" localized in both space and time as intersections of particle trajectories (or world lines) and assign space-time coordinates (ct, x) to such events. Then one can prove [4] that space-time coordinates of events transform under boosts (1.27) by Lorentz formulas (**??**) - (**??**). Thus, we may conclude that the non-interacting boost generator (1.26) is a quantum-mechanical expression of universal Lorentz transformations of special relativity. They may be called "universal", because they apply equally to all particles and do not depend on whether other particles are present in the system. Curiously, we have derived Lorentz transformations without ever assuming the unification of space and time into one (in our case 2D) Minkowski space-time. Our primary assumption was the absence of interactions between particles.

1.4 Physical meaning of symmetry generators. Interacting case. (A side note)

Now, let us take interaction into account. Traditionally, interaction is introduced in a theory by adding "potential energy" operator V to the free Hamiltonian H_0. For example, the Hamiltonian of a system of two charges - electron and proton - may look like this[4]

[4] The approximate equality sign $\approx$ is used here, because we have not established yet whether this Hamiltonian is relativistic or not, i.e., whether it can satisfy Poincaré commutators with properly selected generators P, K. It can be shown that the true relativis-

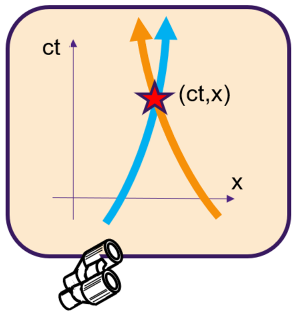

Figure 1.3: Curved trajectories of two interacting particles in space-time coordinates $x - ct$.

$$H = H_0 + V \approx \sqrt{m_e^2c^4 + p_e^2c^2} + \sqrt{m_p^2c^4 + p_p^2c^2} - \frac{e^2}{4\pi|r_e - r_p|}. \tag{1.28}$$

Operators H_0 and V are called "kinetic energy" and "potential energy", respectively, in non-relativistic classical and quantum mechanics. The Coulomb potential energy term mixes position variables of the two particles, so the time evolution operator $e^{-iHt/\hbar}$ no longer splits into nice single-particle factors as in (1.25). Now, trajectory of the electron becomes dependent on the presence of the proton nearby and on the state of the proton. Likewise, the proton "feels" the presence of the electron. This is what we call interaction. See Fig. 1.3.

For further analysis, we have to decide which form of dynamics we would like to explore. Let us first start with the instant form in which the space translation generator remains the same as in the non-interacting case

$$P = P_0 = p_e + p_p. \tag{1.29}$$

Just as in the non-interacting case, this form means that space translations act universally and trivially: the unitary operator of space translation splits into a product of one-particle factors (1.22); each particle shifts in space by the same distance a independent on what other particles are doing and how

tic electron-proton 2-particle Hamiltonian must have relativistic corrections added to the Coulomb term on the right hand side of (1.28).

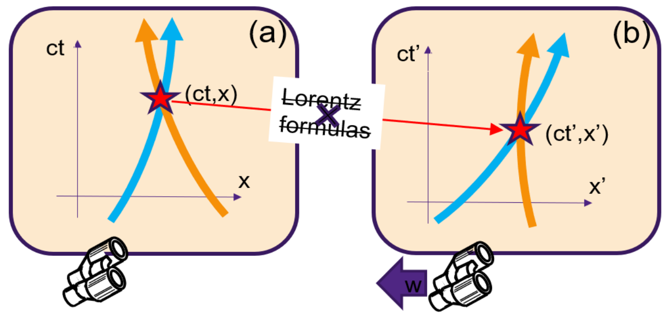

Figure 1.4: Transformation of particle trajectories under interacting boost $e^{-i(K_0+Z)c\theta/\hbar}$. Coordinates of events *do not transform by Lorentz formulas.*

our particles interact. This is a well-known and non-controversial property of space translations, which is confirmed by experiments.

Things become more interesting with boosts. In Dirac's instant form of dynamics, the generator of boosts K has the form "free boost momentum K_0 plus *potential boost* Z".

$$K = K_0 + Z = k_e + k_p + Z.$$

We are not ready to discuss the explicit form of the potential boost operator Z, but in analogy with potential energy V, it seems reasonable to assume that Z mixes variables of different particles. This means that transformations of particle observables to the moving reference frame become non-trivial: they depend on the presence of other particles and on the strength of interaction between them. This in turn means that transformations to the moving frame cannot be described by simple universal Lorentz formulas (**??**) - (**??**) [5]. See Fig. 1.4.

Our derivation of non-Lorentz boost transformations could create an impression that relativistic invariance was sacrificed somewhere along the way. This is not the case. Here we would like to make our terminology a bit more precise [6]. One has to distinguish two ideas: the *relativistic invariance* and the *manifest covariance.*

By "relativistic invariance" we mean Wigner's principle, which was explained earlier: *a quantum description of a physical system is relativistically invariant if generators of inertial transformations are represented in system's Hilbert space by Hermitian operators satisfying commutators of the Poincaré*

Lie algebra. If we managed to choose our interaction operators V and Z to obey these commutators, then we can be sure that our theory is relativistically invariant.

The "manifest covariance" requires that physical observables form 4-scalars, 4-vectors, 4-tensors, etc. with respect to specific linear space-time transformations encoded in Lorentz formulas (**??**) - (**??**). It is important to note that the two ideas are not equivalent. As we established above, Lorentz transformation formulas follow from Poincaré commutators only in the limit of vanishing interaction strength. On the other hand, Lorentz transformations do not necessarily imply relativistic invariance, because in order to satisfy Wigner's condition we also have to understand how the system is transformed with respect to space and time translations and make sure that all these transformations form the Poincaré group.

Our instant-form interacting theory does not satisfy the requirement of manifest covariance. However, one should not regard this as a deadly defect, because physical basis for the "manifest covariance" does not appear solid.

One might have a "bright" idea to reconcile the two conflicting principles by using the point form of Dirac's dynamics. Indeed, in this case the three symmetry generators satisfy Poincaré commutators, so the theory is relativistically invariant. On the other hand, the point form boost operator is non-interacting $K = K_0$, therefore boosts (like familiar Einstein-Lorentz transformations) transform each particle independently on the presence of other particles in the system.

However, this approach creates more problems than it solves. Point-form dynamics requires that interaction term is present in the space translation generator

$$P = P_0 + U = p_e + p_p + U.$$

This means that space translations are interaction-dependent and non-trivial. But this is contrary to observations! We routinely observe results of long-distance space translations in everyday life. If there were non-trivial effects due to the presence of the "potential linear momentum" U, we would have noticed them a long time ago.

Another evidence against the point form dynamics will be presented in section **??**, where we will see that point-form neutrino oscillation formula is in direct contradiction with available experiments.

Bibliography

[1] P. A. M. Dirac. Forms of relativistic dynamics. *Rev. Mod. Phys.*, **21**:392, 1949.

[2] S. D. Głazek and A. P. Traviński. Neutrino oscillations in the front form of Hamiltonian dynamics. *Phys. Rev. D*, **87**:025002, 2013.

[3] B. D. Keister and W. N. Polyzou. Relativistic quantum theories and neutrino oscillations. *Phys. Scr.*, **81**:055102, 2010.

[4] E. V. Stefanovich. Is Minkowski space-time compatible with quantum mechanics? *Found. Phys.*, **32**:673, 2002.

[5] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan. Relativistic invariance and Hamiltonian theories of interacting particles. *Rev. Mod. Phys.*, **35**:350, 1963.

[6] L. L. Foldy. Relativistic particle systems with interaction. *Phys. Rev.*, **122**:275, 1961.