A GENTLE INTRODUCTION TO NEUTRINO OSCILLATIONS

Lecture 6: Relativistic theory of oscillations

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Contents

Chapter 1

RELATIVISTIC THEORY OF OSCILLATIONS

In the previous chapter we arrived at a conclusion that certain preference should be given to instant form interactions. Here we will see how neutrino oscillations can be described in the instant and point forms of dynamics. Can we use experimental data to decide which form of dynamics was chosen by nature?

1.1 Mixing interaction in the instant form of dynamics

Let us now try to build neutrino instant form generators P, H, K explicitly. First, the total momentum remains the same as in the non-interacting case (**??**)

$$P = P_0 = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}. \tag{1.1}$$

We don't know much about the Hamiltonian H except that it must commute with P_0. The most general 2×2 Hermitian matrix having this property is

$$H = \begin{bmatrix} \Omega_\mu(p) & f(p) \\ f^*(p) & \Omega_\tau(p) \end{bmatrix}, \tag{1.2}$$

where $\Omega_\mu(p)$, $\Omega_\tau(p)$ are some arbitrary real functions and $f(p)$ is an arbitrary complex function. In fact, these matrix elements cannot be chosen arbitrarily, because our Hamiltonian must satisfy conditions of relativistic invariance. One such condition demands that after diagonalization matrix elements of H assume the standard momentum dependence. Keeping with our rule to use parentheses for vectors/matrices in the energy basis, we can write

$$H = \begin{pmatrix} E_2(p) & 0 \\ 0 & E_3(p) \end{pmatrix} = \begin{pmatrix} \sqrt{m_2^2c^4 + p^2c^2} & 0 \\ 0 & \sqrt{m_3^2c^4 + p^2c^2} \end{pmatrix}, \qquad (1.3)$$

where

$$E_2(p) = \sqrt{m_2^2c^4 + p^2c^2}, \qquad (1.4)$$

$$E_3(p) = \sqrt{m_3^2c^4 + p^2c^2} \qquad (1.5)$$

are momentum-dependent energy eigenvalues and m_2 and m_3 are eigenvalues of the mass operator

$$M = \begin{pmatrix} m_2 & 0 \\ 0 & m_3 \end{pmatrix}.$$

Just as in the simple 2-level system, we can use "rotation" matrices[1]

$$U = \begin{pmatrix} C & S \\ -S & C \end{pmatrix}$$

and

$$U^{-1} = \begin{bmatrix} C & -S \\ S & C \end{bmatrix}$$

[1]Here we assumed that coefficients C and S are constants that do not depend on momentum. In principle, we could take into account non-trivial momentum dependencies $C(p), S(p)$. We will not do that, because experimental data suggest that $C(p)$ and $S(p)$ are nearly constant.

to make transitions from the flavor basis to the energy basis and back. For, example, in the flavor basis the matrix of the momentum operator (1.1) is proportional to the unit matrix

$$P = p \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right],$$

Then in the energy basis the matrix remains the same, as expected

$$P = pU \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] U^{-1} = pUU^{-1} = \left(\begin{array}{cc} p & 0 \\ 0 & p \end{array} \right). \tag{1.6}$$

Now we have two interacting generators in explicit matrix forms: P in (1.6) and H in (1.3). In order to prove relativistic invariance we have to provide an expression for K, such that commutators (**??**) - (**??**) are obeyed. This operator is easy to construct in the energy basis by analogy with (**??**)

$$K = -i\hbar \left(\begin{array}{cc} \frac{E_2(p)}{c^2}\frac{d}{dp} + \frac{p}{2E_2(p)} & 0 \\ 0 & \frac{E_3(p)}{c^2}\frac{d}{dp} + \frac{p}{2E_3(p)} \end{array} \right). \tag{1.7}$$

Then commutators for P, H, K in the energy basis can be proven exactly by the same arguments as commutators for P_0, H_0, K_0 in the flavor basis. Compare, for example, with our proof of $[\hat{K}, \hat{P}]$ in (**??**).

1.2 Oscillations in the instant form of dynamics

Now we can proceed to calculation of oscillations of a moving neutrino. Suppose that we managed to prepare a μ-neutrino in a state $|\Psi\rangle$ whose momentum-space wave function is $\psi(p)$. This is an eigenstate of the non-interacting mass operator M_0 with eigenvalue m_μ. In the non-interacting (flavor) basis $|\mu, p\rangle, |\tau, p\rangle$, this state is represented by the normalized column vector

$$|\Psi\rangle = \left[\begin{array}{c} \psi(p) \\ 0 \end{array} \right], \tag{1.8}$$

$$\int |\psi(p)|^2 dp = 1. \tag{1.9}$$

Next we transform components of this vector to the energy basis

$$|\Psi\rangle = \left[\begin{array}{cc} C & -S \\ S & C \end{array} \right] \left[\begin{array}{c} \psi(p) \\ 0 \end{array} \right] = \left(\begin{array}{c} C\psi(p) \\ S\psi(p) \end{array} \right) \qquad (1.10)$$

and apply the time evolution operator, which is diagonal in this basis

$$|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi\rangle = \left(\begin{array}{cc} e^{-iE_2t/\hbar} & 0 \\ 0 & e^{-iE_3t/\hbar} \end{array} \right) \left(\begin{array}{c} C\psi(p) \\ S\psi(p) \end{array} \right) = \left(\begin{array}{c} C\psi(p)e^{-iE_2t/\hbar} \\ S\psi(p)e^{-iE_3t/\hbar} \end{array} \right). \qquad (1.11)$$

Then return to the flavor basis

$$\left[\begin{array}{c} \Psi_\mu(p,t) \\ \Psi_\tau(p,t) \end{array} \right] = \left(\begin{array}{cc} C & S \\ -S & C \end{array} \right) \left(\begin{array}{c} C\psi(p)e^{-iE_2t/\hbar} \\ S\psi(p)e^{-iE_3t/\hbar} \end{array} \right) = \psi(p) \left[\begin{array}{c} C^2e^{-iE_2t/\hbar} + S^2e^{-iE_3t/\hbar} \\ SC(e^{-iE_3t/\hbar} - e^{-iE_2t/\hbar}) \end{array} \right]. \qquad (1.12)$$

The upper component of this vector is the momentum space wave function of the μ-neutrino component at time t. $|\Psi_\mu(p,t)|^2$ is the corresponding probability distribution. In order to calculate the total probability of finding μ-neutrino we should integrate $|\Psi_\mu(p,t)|^2$ over all values of p

$$\rho_\mu(t) = \int |\Psi_\mu(p,t)|^2 dp = \int |\psi(p)|^2 \left|C^2e^{-iE_2(p)t/\hbar} + S^2e^{-iE_3(p)t/\hbar}\right|^2 dp.$$

Now, let us assume that the wave packet $|\psi(p)|^2$ is well localized in the momentum space around value p_0, so that within this localization domain functions $E_2(p)$ and $E_3(p)$ may be approximated by constants[2]

$$E_2(p) \approx E_2(p_0),$$
$$E_3(p) \approx E_3(p_0).$$

[2]Some interesting effects can be predicted if one goes beyond the constant approximation. For example, it appears that flavor oscillations are not permanent. They decay gradually [1]. Another example is velocity oscillations, which will be discussed in detail in chapter **??**.

Then the second factor in the integrand does not depend on p and can be moved out of the integral

$$\rho_\mu(t) \approx \left|C^2 e^{-iE_2(p_0)t/\hbar} + S^2 e^{-iE_3(p_0)t/\hbar}\right|^2 \int |\psi(p)|^2 dp. \tag{1.13}$$

The remaining integral is equal to 1, due to the normalization condition (1.9). The resulting expression is exactly the one we evaluated in the simple 2-level case (**??**), (**??**).

$$\begin{aligned} \rho_\mu(t) &\approx \left|C^2 e^{-iE_2(p_0)t/\hbar} + S^2 e^{-iE_3(p_0)t/\hbar}\right|^2 \\ &= C^4 + S^4 + C^2 S^2 \cos(\gamma(p)t/\hbar) \qquad (1.14) \\ &= 1 - \sin^2(2\theta_{23}) \sin^2(\gamma(p)t/2\hbar), \qquad (1.15) \end{aligned}$$

where the momentum-dependent energy gap is

$$\gamma(p) \equiv E_3(p_0) - E_2(p_0).$$

Note that this energy difference is taken between two states having equal values of momentum. So, in the instant form of dynamics we are working with equal-momentum representation of neutrino.

Recall that in section **??** neutrino states were represented by plane waves. These wave functions are characterized by constant probabilities of finding the particle anywhere in space. Therefore, there was some inconsistency in claiming that neutrino moves from the source- to the detector and covers this distance L in time

$$T \approx L/c. \tag{1.16}$$

Nevertheless, this assumption played an important role in our calculations. In the present derivation, neutrino is modeled by a localized moving[3] wave packet, so there is no contradiction in replacing time t in (1.15) with parameter (1.16). Taking into account formula (**??**) for the energy gap

[3]In chapter **??** we will discuss how the wave packet $\psi(p)$ moves in the position space.

$$\gamma(p) \approx \frac{(m_3^2 - m_2^2)c^3}{2p} \tag{1.17}$$

and ultra-relativistic relationship $p \approx E/c$, we obtain the final expression for the probability of finding μ-neutrino at the distance L from the source

$$\rho_\mu(L) \approx 1 - \sin^2(2\theta_{23}) \sin^2 \frac{(m_3^2 - m_2^2)c^3 L}{4\hbar E}. \tag{1.18}$$

This is exactly the same textbook oscillation formula (**??**) as was derived in section **??**. One characteristic property of this result is that oscillation period increases as

$$\mathcal{T} \propto E \propto \frac{1}{\sqrt{1 - v^2/c^2}}$$

when neutrino energy E and velocity v go up. This may be interpreted as the usual relativistic time dilation applied to the oscillation period.

We see that oscillations depend on two parameters: the mixing angle θ_{23} and the difference of squared masses $m_3^2 - m_2^2$. The first parameter controls the oscillations amplitude $\sin^2(2\theta_{23})$. The second parameter is related to the spatial period of oscillations. Values of these parameters can be extracted from observations [2]

$$\sin^2(2\theta_{23}) > 0.92,$$
$$m_3^2 - m_2^2 = 24.4 \times 10^{-4}\ eV^2/c^4.$$

Unfortunately, oscillation studies cannot provide neutrino mass eigenvalues m_2 and m_3.[4]

1.3 What if the initial state is not "equal momentum"?

From formula (1.10) it follows that wave functions of the two mass components are localized in the same neighborhood of the momentum space. This

[4]As we mentioned previously, experimental data indicate that these values do not exceed $\approx 1\ eV/c^2$.

looks like an arbitrary assumption. The exact form of the wave packet should depend on the method of preparation of the neutrino state. The most common source of neutrinos are weak decays, such as the two-body decay of a pion

$$\pi^+ \to \mu^+ + \nu_\mu. \tag{1.19}$$

Let us now try to determine momentum composition of the neutrino produced in this decay. From our discussion in section **??** we know that ν_μ momentum and energy are determined uniquely. Of course, this can be true only for fixed mass neutrino components of ν_μ. Let us then split reaction (1.19) into two sub-processes

$$\pi^+ \to \mu^+ + \nu_2, \tag{1.20}$$

$$\pi^+ \to \mu^+ + \nu_3 \tag{1.21}$$

and apply momentum-energy conservation law to each of them separately [3].

First consider reaction (1.20) and assume that the initial pion is at rest. The momentum conservation law requires that momenta of the two products μ^+ and ν_2 are opposite: p_2 and $-p_2$. Then the energy conservation law reads

$$m_\pi c^2 = \sqrt{m_\mu^2 c^4 + p_2^2 c^2} + \sqrt{m_2^2 c^4 + p_2^2 c^2}.$$

We can solve this equation with respect to p. Ignoring negligible contributions $\propto m_2^4$, we obtain

$$p_2 \approx p_0 \sqrt{1 - \frac{2m_2^2 c^2 (m_\pi^2 + m_\mu^2)}{4p_0^2 m_\pi^2}}, \tag{1.22}$$

where

$$p_0 \equiv \frac{(m_\pi^2 - m_\mu^2)c}{2m_\pi} \tag{1.23}$$

is the momentum of the products in the (hypothetical) case of massless neutrino. Neutrino ν_2 with momentum (1.22) has energy

$$E_2 = \sqrt{m_2^2 c^4 + p_2^2 c^2} \approx p_0 c \sqrt{1 + \frac{m_2^2 c}{p_0 m_\pi}}$$

Using the same approach, we can obtain momentum and energy for the ν_3 neutrino component

$$p_3 \approx p_0 \sqrt{1 - \frac{2m_3^2 c^2 (m_\pi^2 + m_\mu^2)}{4p_0^2 m_\pi^2}}, \tag{1.24}$$

$$E_3 \approx p_0 c \sqrt{1 + \frac{m_3^2 c}{p_0 m_\pi}}. \tag{1.25}$$

This means that the two mass components produced in π^+ decay have neither equal momenta nor equal energies. The momentum difference is

$$p_2 - p_3 \approx \frac{(m_3^2 - m_2^2)(m_\pi^2 + m_\mu^2)c^2}{4p_0 m_\pi^2}$$

and the energy difference is

$$E_3 - E_2 \approx \frac{(m_3^2 - m_2^2)c^2}{2m_\pi}$$

Substituting numerical values relevant to the pion decay

$$\begin{aligned} m_\pi &= 139.6\ MeV, \\ m_\mu &= 105.7\ MeV, \\ p_0 c &= 29.8\ MeV, \\ (m_3^2 - m_2^2) &= 24.4 \times 10^{-4}\ eV^2/c^4, \end{aligned}$$

we obtain

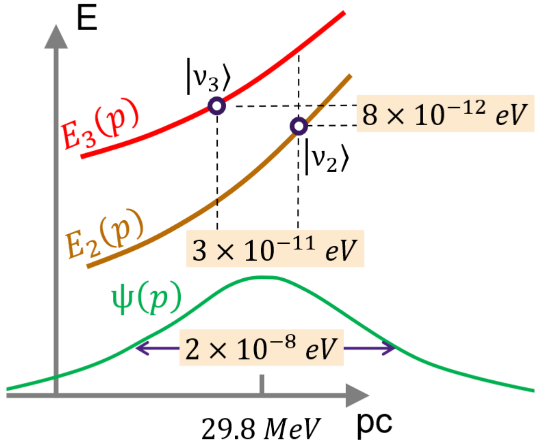

Figure 1.1: Momenta and energies of neutrino mass eigenstates $|\nu_2\rangle$ and $|\nu_3\rangle$ produced in pion decay. The width of the momentum-space wave packet $\psi(p)$ is much greater than the momentum difference $p_2 - p_3$.

$$p_2c - p_3c = 30 \times 10^{-12}\ eV,$$
$$E_3 - E_2 = 8 \times 10^{-12}\ eV.$$

See Fig. 1.1. Apparently, this result contradicts our instant-form assumption about the equal momentum composition of a μ-neutrino (1.10). Does this mean that our oscillation formula (1.15) is incorrect?

We have to remember that realistic particle states are described by wave packets. Let us try to estimate the width of the neutrino wave packet $\psi(p)$ created as a result of pion decay. We can do that by taking into account that pion mass is not well defined. Pion lifetime is $\tau = 2.6 \times 10^{-8}s$.[5] The corresponding mass uncertainty is

$$\Delta m_\pi = \frac{\hbar}{\tau c^2} = \frac{6.6 \times 10^{-16} eV \cdot s}{2.6 \times 10^{-8} s \cdot c^2} = 2.5 \times 10^{-8} eV/c^2.$$

Then the uncertainty of the neutrino momentum (i.e., the width of the momentum-space wave function) can be estimated from (1.23)

[5]Muon is also unstable, but its lifetime is much longer $\tau = 2.2 \times 10^{-6}$ s, so for our purposes it is permissible to treat muon as a stable particle.

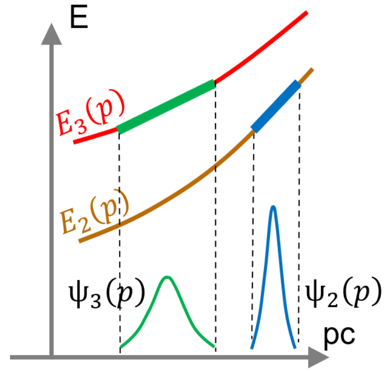

Figure 1.2: Two mass components of a neutrino state have wave functions $\psi_2(p)$ and $\psi_3(p)$ with zero overlap.

$$\Delta pc = \frac{dp_0}{dm_\pi}\Delta m_\pi c = \left(\frac{1}{2} + \frac{m_\mu^2}{2m_\pi^2}\right)\Delta m_\pi c = 0.79 \times 2.5 \times 10^{-8} eV = 2 \times 10^{-8} eV.$$

This uncertainty is much greater than the momentum difference $p_2c - p_3c$ (see Fig. 1.1). Therefore it is legitimate to treat neutrinos produced in pion decays as equal-momentum states and apply to them the instant-form approach developed above.

We have to admit that the argument about the wide momentum-space wave packet does not work in all cases. There are beta decays whose lifetimes are very long, the mass uncertainty of the parent particle is small and the width of the wave packet is comparable with the difference $p_2 - p_3$. What can we say about oscillations in these cases?

Let us see what happens if wave functions of the two mass components are localized in different non-overlapping momentum regions, as shown in Fig. 1.2.

$$|\Psi\rangle = \left(\begin{array}{c} C\psi_2(p) \\ S\psi_3(p) \end{array}\right), \tag{1.26}$$

$$\int |\psi_2(p)|^2 dp = \int |\psi_3(p)|^2 dp = 1,$$

$$\int \psi_2(p)\psi_3^*(p) dp = 0. \tag{1.27}$$

Recall that the plane-wave approach from section ?? predicted that formula (1.15) should remain essentially valid in this case as well. Are we going to get the same result in our wave packet approach?

After time evolution state (1.26) becomes

$$|\Psi(t)\rangle = \left(\begin{array}{c} C\psi_2(p)e^{-iE_2t/\hbar} \\ S\psi_3(p)e^{-iE_3t/\hbar} \end{array} \right).$$

Returning to the flavor basis we obtain

$$\begin{aligned} \left[\begin{array}{c} \Psi_\mu(p,t) \\ \Psi_\tau(p,t) \end{array} \right] &= \left(\begin{array}{cc} C & S \\ -S & C \end{array} \right) \left(\begin{array}{c} C\psi_2(p)e^{-iE_2t/\hbar} \\ S\psi_3(p)e^{-iE_3t/\hbar} \end{array} \right) \\ &= \left[\begin{array}{c} C^2\psi_2(p)e^{-iE_2t/\hbar} + S^2\psi_3(p)e^{-iE_3t/\hbar} \\ -SC\psi_2(p)e^{-iE_2t/\hbar} + SC\psi_3(p)e^{-iE_3t/\hbar} \end{array} \right] \end{aligned}$$

The probability of finding μ-neutrino is given by the integral

$$\begin{aligned} \rho_\mu(t) &= \int |\Psi_\mu(p,t)|^2 dp = \int \left| C^2\psi_2(p)e^{-iE_2(p)t/\hbar} + S^2\psi_3(p)e^{-iE_3(p)t/\hbar} \right|^2 dp \\ &= \int dp \left(C^2\psi_2(p)e^{-iE_2(p)t/\hbar} + S^2\psi_3(p)e^{-iE_3(p)t/\hbar} \right) \left(C^2\psi_2^*(p)e^{iE_2(p)t/\hbar} + S^2\psi_3^*(p)e^{iE_3(p)t/\hbar} \right) \\ &= \int dpC^4|\psi_2(p)|^2 + \int dpS^4|\psi_3(p)|^2 + \int dpC^2S^2\psi_2(p)\psi_3^*(p)e^{-i(E_2(p)-E_3(p)t/\hbar} \\ &+ \int dpC^2S^2\psi_2^*(p)\psi_3(p)e^{i(E_2(p)-E_3(p)t/\hbar}. \end{aligned}$$

Due to the zero overlap condition (1.27), the third and fourth terms on the right hand side vanish, so the μ-neutrino probability remains constant in time

$$\rho_\mu(t) = C^4 \int dp|\psi_2(p)|^2 + S^4 \int dp|\psi_3(p)|^2 = C^4 + S^4$$

There is no oscillation! **In the instant form of dynamics we are free to prepare neutrino state with different momentum-space wave functions of mass components. But we will get oscillations only if these wave functions overlap.**

1.4 Oscillations in the point form of dynamics

Return to the non-interacting neutrino system. In section **??** we introduced wavefunctions and operators in the basis $|\mu, p\rangle, |\tau, p\rangle$ of common eigenvectors of momentum P_0, energy H_0 and mass M_0. But in quantum mechanics, we are free to choose any convenient basis set. Let us introduce operator Q_0 which is the ratio of the non-interacting linear momentum and mass: $Q_0 \equiv P_0/M_0$. Eigenvalues of this operator will be denoted by q; their spectrum occupies the entire real line $q \in [-\infty, \infty]$ provided that the mass operator M_0 is strictly positive. Then we can rewrite non-interacting operators (**??**) - (**??**) as matrix functions defined on this spectrum, i.e., in the basis $|\mu, q\rangle, |\tau, q\rangle$

$$Q_0 = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} = q \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{1.28}$$

$$M_0 = \begin{bmatrix} m_\mu & 0 \\ 0 & m_\tau \end{bmatrix}, \tag{1.29}$$

$$P_0 = M_0 Q_0 = \begin{bmatrix} m_\mu & 0 \\ 0 & m_\tau \end{bmatrix} q, \tag{1.30}$$

$$H_0 = M_0 c^2 \sqrt{1 + Q_0^2/c^2} = \begin{bmatrix} m_\mu & 0 \\ 0 & m_\tau \end{bmatrix} c^2 \sqrt{1 + q^2/c^2}, \tag{1.31}$$

$$K_0 = -i\hbar \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\sqrt{1 + q^2/c^2} \frac{d}{dq} + \frac{q}{2c^2\sqrt{1 + q^2/c^2}} \right). \tag{1.32}$$

Next we have to construct interacting energy H, momentum P, and boost momentum K. This might be a difficult task in the flavor basis, but it becomes almost trivial in the energy basis. First note that Q_0 and K_0 are proportional to the unit matrix, so they retain their forms in the new basis

$$Q_0 = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}, \tag{1.33}$$

$$K = K_0 = -i\hbar \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(\sqrt{1 + q^2/c^2} \frac{d}{dq} + \frac{q}{2c^2\sqrt{1 + q^2/c^2}} \right) \tag{1.34}$$

Let us now postulate expressions for the interacting generators of space and time translations

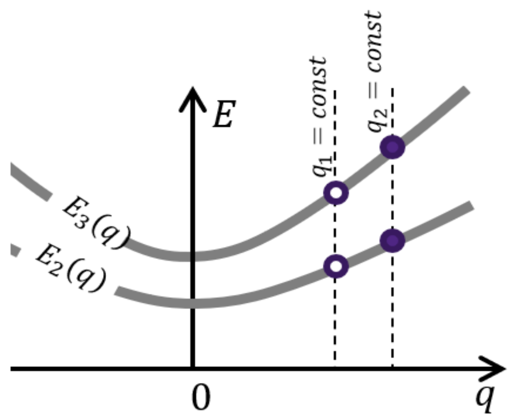

Figure 1.3: Neutrino energy eigenvalues $E_i(q) = m_i c^2 \sqrt{1 + q^2/c^2}$ as functions of the parameter q in Dirac's point form of dynamics. Note that the energy gap $E_3(q) - E_2(q)$ grows as a function of velocity (or q).

$$P = MQ_0 = \begin{pmatrix} m_2 & 0 \\ 0 & m_3 \end{pmatrix} q, \quad (1.35)$$

$$H = Mc^2\sqrt{1 + Q_0^2/c^2} = \begin{pmatrix} E_2(q) & 0 \\ 0 & E_3(q) \end{pmatrix}, \quad (1.36)$$

Here, instead of p-dependent instant form energy eigenvalues (1.4) - (1.5) we use q-dependent point form eigenvalues

$$E_2(q) = m_2 c^2 \sqrt{1 + q^2/c^2}, \quad (1.37)$$

$$E_3(q) = m_3 c^2 \sqrt{1 + q^2/c^2}. \quad (1.38)$$

See Fig 1.3.[6]

It is not difficult to verify that matrices P, H, K satisfy the required commutation relations (**??**) - (**??**). Therefore our theory is relativistically invariant. From (1.34) it follows that the boost momentum operator K remains non-interacting. This means that we managed to construct a point form representation of the Poincaré Lie algebra, as promised.

[6]Note that $E(q) = mc^2\sqrt{1 + q^2/c^2} = \sqrt{m^2c^4 + m^2q^2c^2} = \sqrt{m^2c^4 + p^2c^2} = E(p)$.

Now calculations of neutrino flavor oscillations can proceed in full analogy with the instant form. For our initial state, we choose the same state as in (1.8). The only difference is that now we express the wave function $\psi(p)$ through variable q

$$\begin{aligned} \phi(q) &\equiv \psi(m_\mu q), \\ |\Psi\rangle &= \left[\begin{array}{c} \phi(q) \\ 0 \end{array} \right]. \end{aligned}$$

In the energy basis

$$|\Psi\rangle = \left[\begin{array}{cc} C & -S \\ S & C \end{array} \right] \left[\begin{array}{c} \phi(q) \\ 0 \end{array} \right] = \left(\begin{array}{c} C\phi(q) \\ S\phi(q) \end{array} \right)$$

The time evolution of this vector is

$$|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi\rangle = \left(\begin{array}{cc} e^{-iE_2t/\hbar} & 0 \\ 0 & e^{-iE_3t/\hbar} \end{array} \right) \left(\begin{array}{c} C\phi(q) \\ S\phi(q) \end{array} \right) = \phi(q) \left(\begin{array}{c} Ce^{-iE_2t/\hbar} \\ Se^{-iE_3t/\hbar} \end{array} \right)$$

The t-dependent μ-neutrino state in the flavor basis is (compare with (1.11))

$$|\Psi(t)\rangle = \phi(q) \left[\begin{array}{c} C^2 e^{-iE_2(q)t/\hbar} + S^2 e^{-iE_3(q)t/\hbar} \\ CS\left(e^{-iE_3(q)t/\hbar} - e^{-iE_2(q)t/\hbar}\right) \end{array} \right].$$

The probability for finding μ-neutrino in this state is (compare with (1.13))

$$\begin{aligned} \rho_\mu(t) = & \int \left(C^2 e^{iE_2(q)t/\hbar} + S^2 e^{iE_3(q)t/\hbar}\right) \times \\ & \left(C^2 e^{-iE_2(q)t/\hbar} + S^2 e^{-iE_3(q)t/\hbar}\right) |\phi(q)|^2 dq \\ = & \, C^4 + S^4 + 2C^2S^2 \int |\phi(q)|^2 \cos\left(\frac{\delta(q)t}{\hbar}\right) dq, \end{aligned} \quad (1.39)$$

where $\delta(q)$ is the gap between energy eigenvalues

$$\begin{aligned} &\delta(q) \equiv E_3(q) - E_2(q) = \Delta m c^2 \sqrt{1 + q^2/c^2}, \\ &\Delta m \equiv m_3 - m_2. \end{aligned}$$

Assuming that neutrino wave function is well localized near value q_0, we can replace $\delta(q)$ in (1.39) with a constant

$$\delta(q) \approx \delta(q_0) = E_3(q_0) - E_2(q_0) = \Delta mc^2\sqrt{1 + q_0^2/c^2}, \tag{1.40}$$

take the q_0-independent oscillating factor $\cos(\delta(q_0)t/\hbar)$ out of the integral and finally obtain (compare with (1.15))

$$\begin{aligned}\rho_\mu(t) &\approx C^4 + S^4 + 2C^2S^2\cos\left(\frac{\delta(q_0)t}{\hbar}\right) \\ &= 1 - \sin^2 2\theta_{23}\sin^2\frac{\Delta mc^2\sqrt{1+q_0^2/c^2}t}{2\hbar}\end{aligned} \tag{1.41}$$

At high energies $q_0 \gg c$ and $t \approx L/c$, therefore, we can approximate

$$\rho_\mu(t) \approx 1 - \sin^2 2\theta_{23}\sin^2\frac{\Delta mq_0L}{2\hbar}.$$

Our result (1.40) means that in the point form of dynamics the two neutrino mass components have equal q values. To clarify the physical meaning of this result, notice that we can switch variables from q_0 to neutrino velocity

$$v \equiv \frac{pc^2}{h} = \frac{q_0}{\sqrt{1+q_0^2/c^2}}.$$

This means, in particular, that the two mass components of oscillating neutrino have equal velocities, as shown in Fig. 1.4.[7] In contrast to the instant form (**??**), the energy gap δ *increases* with velocity. This means that oscillations of a fast moving neutrino (1.41) become *faster* rather than slower. This prediction is completely contradicted by Einstein's special relativity and by existing experiments. However, it is important to realize that acceleration of time-dependent processes in moving systems is an inherent property of the point form dynamics. This property can be proved under very general

[7] Of course, it should be possible to prepare neutrino state which does not respect the equal velocity condition. Then the oscillation phenomenon may disappear in full analogy with our discussion in section 1.3.

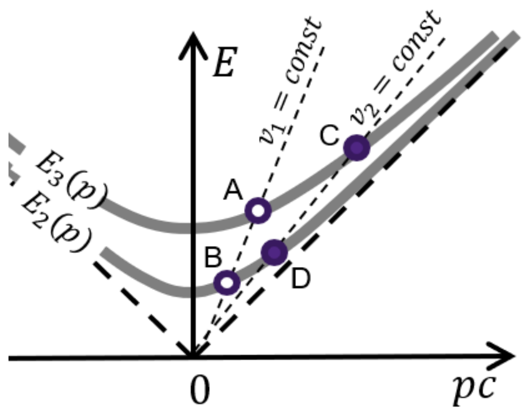

Figure 1.4: Neutrino mass-momentum-energy eigenvalues in $pc - E$ coordinates. In point-form dynamics, neutrino mass components have equal velocities, as in pairs $A - B$ and $C - D$. The energy gap $E_3 - E_2$ grows as a function of velocity, therefore oscillations become *faster* at higher energies. This contradicts observations.

conditions. An example of such a proof applied to decay laws of fast-moving particles can be found in section 4.4.4 of [4].

Our discussion in this chapter should lead us to the following conclusion: **Neutrino mixing interaction belongs to the instant form of dynamics.**

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