Basic formulation of Lagrangian and Hamiltonian classical and quantum field theory

Bill Celmaster

March 3, 2024

1 Introduction

Over the past few years, we've focused on relativistic quantum field theory as it applies to fundamental physics. Similar mathematical techniques can be used to study non-relativistic quantum many-body physics. Lancaster takes on this subject matter towards the end of the book, with the first topic being SuperFluids, in Chapter 42. To study this chapter, we need some **selected** material from earlier chapters. This set of notes covers the basic quantum field theoretic formulation for both relativistic fundamental particle theory and non-relativistic many-body theory. The following topics will be covered (the Chapter numbers refer to Lancaster):

- Harmonic oscillators, phonons (Chapter 2) and fields (Chapters 3 and 4)
- Digression on Kabbalah
- Complex scalar field (Chapter 7)
- Canonical quantization and non-relativistic limit for complex scalar fields (Chapter 12)
- Hamiltonians and Lagrangians for non-relativistic many-body fluid physics (Chapter 4)

Believe it or not, much of this will look familiar.

2 Harmonic Oscillators

"The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction."

Sidney Coleman

This section is important because it introduces ladder operators. The harmonic oscillator itself, isn't especially germane to field theory. However, what is important is the technique of using ladder operators for finding the spectrum of Hamiltonians. The technique has to do with solutions of certain kinds of differential equations, of which the harmonic oscillator is the simplest.

2.1 Mass on a spring

(Lancaster 2.2) Classically, the energy of a spring (linear harmonic oscillator $\equiv$ LHO) is given by

$$E = \frac{p^2}{2m} + \frac{Kq^2}{2}. \tag{1}$$

The rules for (canonical) quantization are:

- Replace q and p by (non-commuting) operators $\hat{q}$ and $\hat{p}$ on a Hilbert space.
- Require that those operators obey

$$[\hat{q}, \hat{p}] \equiv \hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar. \tag{2}$$

- Replace E with the operator $\hat{H}$ (the Hamiltonian).

The possible energy values for the system are obtained by solving the equation

$$\hat{H}|\psi\rangle = E|\psi\rangle \tag{3}$$

where $|\psi\rangle$ denotes a vector of the Hilbert space. In this equation, $|\psi\rangle$ is called an eigenvector of $\hat{H}$, and E is its eigenvalue.

We solve this eigenvalue equation by a set of tricks that we'll use over and over again in various contexts.

- Define the operator $\hat{a}$ by

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{q} + \frac{i}{m\omega} \hat{p} \right) \tag{4}$$

  where $\omega \equiv \sqrt{K/m}$. The adjoint (effectively the complex conjugate of an operator) of a is

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{q} - \frac{i}{m\omega} \hat{p} \right) \tag{5}$$

- Plug into Eq. (2) to obtain

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{6}$$

  and into Eq. (1) to obtain

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right). \tag{7}$$

  Eq. (6) is a critical commutation relationship. When operators satisfy this equation, they are known as lowering ($\hat{a}$) and raising ($\hat{a}^\dagger$) operators because they lower and raise the energy. We'll see shortly that $\hat{a}$ and $\hat{a}^\dagger$ can also be interpreted as particle annihilation and creation operators.

- **I'm about to change the conventional presentation because I think the conventional notation leads to an unintended sleight of hand.** Here's an example to illustrate what we can do with $\hat{a}$ and $\hat{a}^\dagger$. Suppose there is a vector $|\gamma_n\rangle$ with the property that

$$\hat{a}^\dagger \hat{a} |\gamma_n\rangle = n|\gamma_n\rangle. \tag{8}$$

  Then we can show that $\hat{a}^\dagger|\gamma_n\rangle \propto |\gamma_{n+1}\rangle$. Here's the proof. We want to show that

$$\hat{a}^\dagger \hat{a} \left(\hat{a}^\dagger |\gamma_n\rangle\right) = (n+1)\left(\hat{a}^\dagger|\gamma_n\rangle\right). \tag{9}$$

  Manipulate the LHS.

$$\begin{aligned} \hat{a}^\dagger \hat{a} \hat{a}^\dagger |\gamma_n\rangle &= \left(\hat{a}^\dagger \hat{a}^\dagger \hat{a} + \hat{a}^\dagger\right)|\gamma_n\rangle \\ &= \left(\hat{a}^\dagger (\hat{a}^\dagger \hat{a}) + \hat{a}^\dagger\right)|\gamma_n\rangle \\ &= \left(n\hat{a}^\dagger + \hat{a}^\dagger\right)|\gamma_n\rangle \\ &= (n+1)\hat{a}^\dagger|\gamma_n\rangle. \end{aligned} \tag{10}$$

  where the first line is derived from Eq. (6).

- From the above, we immediately find that the Hamiltonian (Eq. (48) has eigenvectors $|\gamma_n\rangle$ with eigenvalues (energies) $\hbar\omega(n+\frac{1}{2})$.

- The lowest-energy state has $n = 0$ and is called **the single oscillator vacuum state** $|\gamma_0\rangle$. Notice that it's energy is **not** 0 (it's $\hbar\omega/2$).

- See Lancaster 2.2 for the remaining parts of this analysis but notice that he uses the (conventional) state notation $|n\rangle$ rather than $|\gamma_n\rangle$.

- One final thing. The usual quantum LHO introduction starts with the Schrodinger equation. Instead of a Hilbert space and operators, we deal with wave functions and, for example, the differential operators q and $-i\hbar\frac{\partial}{\partial q}$. Then our eigenvalue equation becomes a differential equation. What we learn from the operator technique above, is that the essential facts about the theory are obtained from the canonical commutation relations.

2.2 Many independent LHO's

Lancaster 2.3.

- First, let's simplify notation.
  - Don't put carats on operators. So

$$\begin{aligned} \hat{a} &\rightarrow a \\ \hat{a}^\dagger &\rightarrow a^\dagger \\ \hat{q} &\rightarrow q \\ \hat{p} &\rightarrow p \\ \hat{H} &\rightarrow H \\ &... \end{aligned} \tag{11}$$

    There will occasionally be some notational ambiguity which hopefully can be resolved by context.
  - Pick dimensions where $\hbar = 1$.
- Next consider a whole bunch of non-interacting (independent) LHO's. The Hamiltonian is

$$H = \sum_{i=1}^{N} H_i \tag{12}$$

  where

$$H_i = \frac{p_i^2}{2m_i} + \frac{m_i \omega_i^2 q_i^2}{2}. \tag{13}$$

  The independence of the LHO's is expressed by the fact that there are no terms involving products of operators with different indices. This independence also implies that there are no connections between the various q_k and therefore we can't assert any kind of spacial relationship between the various springs (they may as well be laid out randomly in space).

  The q_i and p_i operators obey the CCR (canonical commutation relations)

$$\begin{aligned} [q_i, p_j] &= i\delta_{i,j} \\ [q_i, q_j] &= 0 \\ [p_i, p_j] &= 0. \end{aligned} \tag{14}$$

  Notice that we've set $\hbar = 1$. Also notice that since there are more operators (N pairs), we explicitly show that the commutators are 0 when the indices are different.

- We want to know the possible values of total energy for this system. As before, we use the lowering-raising operator trick. Define the i^{th} lowering operator by

$$a_i = \sqrt{\frac{m_i\omega_i}{2}}\left(q_i + \frac{i}{m_i\omega_i}p_i\right). \qquad (15)$$

Then the i^{th} raising operator is $a_i^\dagger$.

We generalize the observations of last section by introducing eigenvectors $|\gamma_{n_1}, \gamma_{n_2}, ..., \gamma_{n_k}, ..., \gamma_{n_N}\rangle$ with the property that

$$\hat{a_k}^\dagger \hat{a_k}|\gamma_{n_1}, \gamma_{n_2}, ..., \gamma_{n_k}, ..., \gamma_{n_N}\rangle = n_k|\gamma_{n_1}, \gamma_{n_2}, ..., \gamma_{n_k}, ..., \gamma_{n_N}\rangle. \qquad (16)$$

As before, we can show that $a_k^\dagger|\gamma_{n_1}, \gamma_{n_2}, ..., \gamma_{n_k}, ..., \gamma_{n_N}\rangle \propto |\gamma_{n_1}, \gamma_{n_2}, ..., \gamma_{n_{k+1}}, ..., \gamma_{n_N}\rangle$, and that

$$H|\gamma_{n_1}, \gamma_{n_2}, ..., \gamma_{n_k}, ..., \gamma_{n_N}\rangle = \left(\sum_{k=1}^{N}\omega_k(n_k + \frac{1}{2})\right)|\gamma_{n_1}, \gamma_{n_2}, ..., \gamma_{n_k}, ..., \gamma_{n_N}\rangle. \qquad (17)$$

The vacuum state is $|\gamma_0, \gamma_0, ..., \gamma_0, ..., \gamma_0\rangle$.

2.2.1 Digression on Fourier transforms

When dealing with independent oscillators, there is no special reason why we should consider the Fourier transforms of the position and momentum operators. However, we may choose to do so. I'll do so in order to illustrate a different set of raising and lowering operators.

For now, turn to a 1D example with an even number N of particles numbered from $-N/2$ through $N/2-1$. We'll relabel the coordinates from q_i to x_i to emphasize the one-dimensionaity. Let

$$\begin{aligned}\tilde{x}_k &= \frac{1}{\sqrt{N}}\sum_{j=-\frac{N}{2}}^{\frac{N}{2}-1} x_j e^{-ijka} \\ \tilde{p}_k &= \frac{1}{\sqrt{N}}\sum_{j=-\frac{N}{2}}^{\frac{N}{2}-1} p_j e^{-ijka}\end{aligned} \qquad (18)$$

where $a = \frac{2\pi}{N}$. Then, assuming the previous CCR, we can derive that

$$[\tilde{x}_k, \tilde{p}_{k'}] = i\delta_{k,-k'}. \tag{19}$$

Lancaster goes on to define new lowering and raising operators[1]

$$\begin{aligned} \tilde{a}_k &= \sqrt{\frac{m\omega_k}{2}}\left(\tilde{x}_k + \frac{i}{m\omega_k}\tilde{p}_k\right) \\ \tilde{a}_k^\dagger &= \sqrt{\frac{m\omega_k}{2}}\left(\tilde{x}_{-k} + \frac{i}{m\omega_k}\tilde{p}_{-k}\right) \end{aligned} \tag{20}$$

where

$$\omega_k = \sqrt{4K/m}\sin\left(\frac{ka}{2}\right). \tag{21}$$

and from which (see Lancaster exercise 2.3) we can derive

$$x_j = \frac{1}{\sqrt{mN}}\sum_k \frac{1}{\sqrt{2\omega_k}}\left[\tilde{a}_k e^{ikja} + \tilde{a}_k^\dagger e^{-ikja}\right]. \tag{22}$$

From the definitions in Eq. (20), we can show that $\tilde{a}_k$ and $\tilde{a}_k^\dagger$ obey the usual commutation relations for lowering and raising operators, such as

$$[\tilde{a}_k, \tilde{a}_k^\dagger] = \delta_{k,k'}. \tag{23}$$

Now, it's important to observe that these lowering-raising operators **do not** lower and raise the particle-states discussed above. In particular,

$$\tilde{a}_k^\dagger|\gamma_{n_1}, \gamma_{n_2}, ..., \gamma_{n_k}, ..., \gamma_{n_N}\rangle \not\propto |\gamma_{n_1}, \gamma_{n_2}, ..., \gamma_{n_{k+1}}, ..., \gamma_{n_N}\rangle. \tag{24}$$

This should not be a surprise, since we've changed variables to be linear combinations of the particle-variables. We can understand this in terms of Fock space states (multiparticle Hilbert space). The basis states we were using with the previous lowering-raising operators, were the "canonical Fock space basis states" [2] $|\gamma_{n_1}, \gamma_{n_2}, ..., \gamma_{n_k}, ..., \gamma_{n_N}\rangle = |\gamma_{n_1}\rangle|\gamma_{n_1}\rangle...|\gamma_{n_k}\rangle|\gamma_{n_N}\rangle$. The basis states used with the operators a_k are linear combinations of the canonical Fock basis states. We will see later that $\tilde{a}_k$ and $\tilde{a}_k^\dagger$ are best interpreted as "annihilation and creation" operators – to be defined shortly.

[1]Beware of a few typos in Lancaster.

[2]For now I'm ignoring all matters having to do with symmetrization – see Lancaster.

2.3 Phonons

Lancaster 2.4

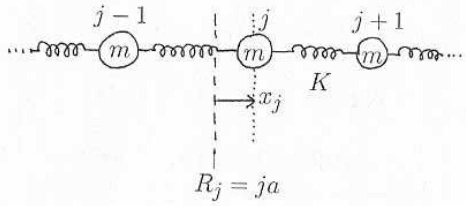


Fig. 2.5 A linear chain of atoms.

Now we take a string of harmonic oscillators, each of which interacts with its neighbors. As in the previous subsection, we'll use the position operator x rather than q, to indicate that the LHO's lie along the x direction and are ordered so that consecutive indices are neighbors.

$$H = \sum_{j=-\frac{N}{2}}^{\frac{N}{2}-1} \frac{p_j^2}{2m} + \frac{1}{2}K(x_{j+1} - x_j)^2. \tag{25}$$

In order to simplify the math, we will assume periodic boundary conditions.[3]

It turns out that the system can be most easily mathematically described by transforming to a new basis obtained by the Fourier transformations described in subsection 2.2.1. We can then show (using notation of subsection 2.2.1)

$$H = \sum_k \omega_k \left(\tilde{a}_k^\dagger \tilde{a}_k + \frac{1}{2} \right). \tag{26}$$

To summarize, we start with equal-mass particles connected by coupled springs, each of which has the same spring constant. We derive the quantum Hamiltonian for this system, as a sum over collective modes labeled

[3]The rationale for various boundary conditions, is that physics "in the bulk" (i.e., in the interior of the system) is insensitive to what happens at the boundary. That isn't always the case, but physicists tend to check before making the insensitivity-assumption. In 1D systems, one can actually construct periodic boundary conditions by closing the string on itself.

by the index k. Because of the commutation relations for a_k and $a_k^\dagger$, each mode can have energy in integer-multiples of the mode-frequency $\omega_k = \sqrt{4K/m}\sin\left(\frac{ka}{2}\right)$. We might think of these modes as harmonic oscillators with energy in quanta of ω_k. Alternatively we can regard these modes as consisting of states of n particles called ***phonons*** each of which have energy in quanta of ω_k – where $\tilde{a}_k^\dagger \tilde{a}_k$ has eigenvalue n, i.e. $\tilde{a}_k^\dagger \tilde{a}_k$ is the number operator for the k^{th} mode. Either interpretation would be valid since the harmonic oscillator has energy levels in integer multiples of ω_k. However, we'll see later when we consider independent particles under the influence of a non-LHO potential, and therefore with energies not proportional to an integer, that the number-operator interpretation is more appropriate.

2.3.1 Annihilation and creation operators

Up to now, I've tried to refer to a_j and $\hat{a}_j$ as lowering and raising operators rather than as annihilation and creation operators. That's because these operators change the states of particles but they don't change numbers of particles. As we saw in the previous paragraphs, the operators a_k and $a_k^\dagger$ may deserve a different interpretation. We'll explore this now.

In this section, I'll define **DIFFERENT** operators that we can use instead, which change numbers of particles, and which lead to the same physics. We'll refer to these operators as annihilation and creation operators. They happen to have the same commutation relations as the lowering and raising operators. **Lancaster goes through this analysis in chapters 3 and 4, but in my opinion he obscures some of the points by his choice of terminology and notation. A clearer treatment – in my opinion – is given in section 64 of Volume 3 (on QM) of the Landau and Lifshitz series.**

Before we start, here's why all this important.

- Despite mathematical equivalence, lowering-raising operators are **not** interpreted the same as annihilation-creation operators. They have similar properties. Operators with those kind of commutation relations are sometimes called **ladder operators**.
- In general, ladder operators can have multiple interpretations depending on context. Even when they are interpreted as particle annihilation and creation operators, they can be transformed into different sets of

annihilation and creation operators for entities that can best be regarded as linear combinations of the original particles. The physicist Bogoliubov, whose contributions are discussed in Lancaster Chapter 42, is responsible for many insights having to do with the fungibility of particles.

- The lowering-raising formalism depends on a generalized LHO many-body Hamiltonian (the phonon or independent-oscillator Hamiltonian) where single-particle energy levels are evenly spaced. The annihilation-creation formalism is more flexible.

- Many-body physicists use the annihilation-creation formalism rather than the lowering-raising formalism.

- The connection to quantum field theory is clearer with the annihilation-creation formalism. In QFT, particles can be actually annihilated and created and therefore the strict lowering-raising interpretation isn't sufficient. Besides, in QFT the fields at a point $(t, \mathbf{x})$ don't directly represent particles whose state-energies are of interest.

Consider a system with an infinite number of distinct (indistinguishable) particles. We can regard such a system as the $N \to \infty$ limit of an N-particle system, and we write the canonical Fock state basis[4]in the form of vectors

$$|\gamma_{n_1}, \gamma_{n_2}, ..., \gamma_{n_k}, ...,\rangle = |\gamma_{n_1}\rangle|\gamma_{n_2}\rangle...|\gamma_{n_k}\rangle... \tag{27}$$

These denote states where the k^{th} particle is in state $|\gamma_{n_k}\rangle$. An alternate notation for such a vector is the "occupation number" representation (see Lancaster Chapter 3)

$$|N_{\gamma_0}, N_{\gamma_1}, ..., N_{\gamma_k}, ...\rangle. \tag{28}$$

This is interpreted as meaning "N_{γ_0} particles are in state $|\gamma_0\rangle$, N_{γ_1} particles are in state $|\gamma_1\rangle$... N_{γ_k} particles are in state $|\gamma_k\rangle$, etc." We can see that this notation is essentially a different way of writing the Fock basis states **except that we don't count as separate states $|\gamma_{n_1}\rangle|\gamma_{n_2}\rangle...|\gamma_{n_k}\rangle$ and $|\gamma_{n_2}\rangle|\gamma_{n_1}\rangle...|\gamma_{n_k}\rangle$.**[4]

[4]I have swept under the rug, the entire discussion of symmetrization of wave states. In point of fact, the canonical Fock states aren't states of the physical Hilbert space. Instead, linear combinations are required such that the state is symmetric (or for fermions, antisymmetric) under exchange of particles. This is discussed at length in Lancaster and comes out naturally in the occupation number representation. I especially like a treatment given by Neumaier in `https://www.physicsforums.com/insights/clarifying-common-issues-with-indistinguishable-particles/`.

Now we will use this notation to define the ladder operators a_{γ_i} and $a^\dagger_{\gamma_i}$.

$$\begin{aligned} a_{\gamma_k}|N_{\gamma_0}, N_{\gamma_1}, ..., N_{\gamma_k}, ...\rangle &= \sqrt{N_{\gamma_k}}|N_{\gamma_0}, N_{\gamma_1}, ..., N_{\gamma_k} - 1, ...\rangle, \\ a^\dagger_{\gamma_k}|N_{\gamma_0}, N_{\gamma_1}, ..., N_{\gamma_k}, ...\rangle &= \sqrt{N_{\gamma_k} + 1}|N_{\gamma_0}, N_{\gamma_1}, ..., N_{\gamma_k} + 1, ...\rangle. \end{aligned} \tag{29}$$

We can show that these operators obey the usual ladder-operator commutation rules. As we see, these ladder operators respectively annihilate and create particles and we therefore call them annihilation and creation operators. From now on, when we encounter ladder operators in many-body theory or QFT, they will be interpreted as annihilation/creation operators. We will simplify the above notation to

$$\begin{aligned} a_k|N_0, N_1, ..., N_k, ...\rangle &= \sqrt{N_k}|N_0, N_1, ..., N_k - 1, ...\rangle, \\ a^\dagger_k|N_0, N_1, ..., N_k, ...\rangle &= \sqrt{N_k + 1}|N_0, N_1, ..., N_k + 1, ...\rangle. \end{aligned} \tag{30}$$

2.3.2 A Lagrangian digression on phonons

This next blurb is an after-thought to the material in Lancaster. So far in these notes, I haven't introduced Lagrangians or the rules governing the use of Lagrangians. That will come later. For now, I'll assume we all have encountered Lagrangians in the past, and I'll briefly summarize the situation for the preceding phonon example. Let the phonon Lagrangian be

$$\mathcal{L}_{\text{phonon}} = \sum_j \frac{m}{2}\partial_0 x_j \partial_0 x_j - \frac{\delta}{2} K \hat{\partial} x_j \hat{\partial} x_j \tag{31}$$

where we define $\hat{\partial} x_j \equiv \frac{x_j - x_{j-1}}{\delta}$. Although I won't go through the derivation here, it can be shown that (for an infinite number of particles) this Lagrangian has the phonon Hamiltonian H of Eq. (25)).

This notation is suggestive of fields, if we make the following interpretation: Let x_j denote the displacement – for example in the x direction of an atom in the j^{th} position along a lattice in the z direction. Rewrite x_j as $x(t, y)$ and the above expression ends up resembling the kinetic part of a relativistic scalar field theory.

2.4 Fields

Lancaster 11 and also see Lancaster Chapters 3 and 4

As we've seen in the previous sections[5] , many-body theories can be expressed in terms of either x_j which describe the original particle positions, or in terms of "wave numbers", which describe matter waves.

One critical difference between quantum field theory and many-body theory is that in many-body theory, the number of particles is conserved. Therefore all expressions for observables, must involve products with equal numbers of annihilation and creation operators.

For phonons, we saw that particle-position description can be related to the wave number description (using annihilation and creation operators) by

$$x_j = \frac{1}{\sqrt{mN}} \sum_k \frac{1}{\sqrt{2\omega_k}} \left[\tilde{a}_k e^{ikja} + \tilde{a}_k^\dagger e^{-ikja} \right]. \tag{32}$$

We can write this equation in suggestive notation as

$$\phi(j) = \frac{1}{\sqrt{mN}} \sum_k \frac{1}{\sqrt{2\omega_k}} \left[\tilde{a}_k e^{ikja} + \tilde{a}_k^\dagger e^{-ikja} \right], \tag{33}$$

where we've replaced the position operator x_j by a (discrete) field operator $\phi(j)$.

As the number, N, of particles becomes very large, as we go from 1 to 3 dimensions, and as we restrict motion to a box of volume $\mathcal{V}$ with periodic boundary conditions, we end up with a continuum description

$$\phi(\mathbf{x}) \propto \sqrt{\frac{1}{\mathcal{V}}} \sum_{\mathbf{p}} \frac{1}{\sqrt{\omega(\mathbf{p})}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \tag{34}$$

where ω is model-dependent.

Lancaster points out that this expansion can be extended to the infinite-volume limit to an expression such as

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}} (2E_{\mathbf{p}})^{\frac{1}{2}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right). \tag{35}$$

This is a time-independent version of a scalar quantum field, so we see a resemblance between QFT and a phonon-like description of many-body theory.

[5] In Subsection 2.2 we didn't change to a Fourier transformed basis but we could have done so following precisely the same technique as what we did with phonons. Results would have looked the same, except for the values of ω_k.

Lancaster then organizes these observations in Chapters 3 and 4.

- Begin by adopting a particle-centric notation (instead of an LHO-centric approach) where the fundamental quantum operators are the annihilation and creation operators corresponding to various wave numbers (interpreted as momenta), and the fundamental quantum states are described by specifying how many quanta (levels of energy excitations) there are for each wave number (the "occupation-number" representation).

- Introduce a position-centric object which we'll call a "field" (sometimes, depending on context we'll call it a creation or annihilation field)

$$\psi^\dagger(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} a^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}. \tag{36}$$

  (Later, for notational consistency, we'll substitute ϕ for ψ.)

- The connection between particles, fields and harmonic oscillators, will ultimately arise because of the equations of motion governing behaviors of fields and/or particles in a many-body theory.

- In a full treatment of the many-body system at finite temperatures, pure-state expectation values are replaced by thermodynamic expectation values which involve something called *the density matrix.* Lancaster manages to stay clear of that subject.

- In chapters 3 and 4, Lancaster covers how to obtain the continuum limit for large systems, and also how to incorporate relativity by putting time on an equal footing with space.

3 Many-body physics, Quantum field theory and Kabbalah

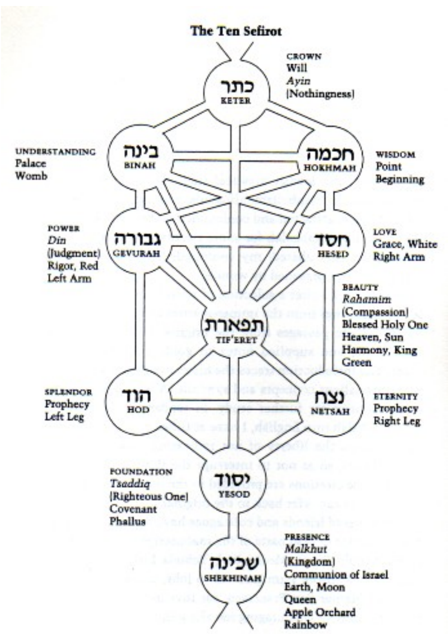
Many-body physics is about a **large finite number** of discrete particles, interacting either with classical or quantum forces.

Quantum Field Theory is about a continuum of entities (the field, at a position $\mathbf{x}$).

Both kinds of theory involve the "continuum" of time, but because of relativity, QFT puts space on an equivalent footing to time. In many-body physics, we often take continuum spacial limits in order to simplify the general mathematics. In QFT, we often discretize the theory in order to solve specific models.

The conceptual difference is profound. In many-body theory, the continuum is a mathematical convenience. In QFT, the continuum is *The Infinite*. Ein Sof.

אין סוף



History of infinity: See Wikipedia. An early example is Zeno's paradox: Achilles is trying to catch up to a tortoise. Achilles advances to where the tortoise was at time t_0. He arrives at time t_1. In the meantime, the tortoise advances to a new position. Achilles then advances to where the tortoise was at t_1. But in the meantime the tortoise advances to a new position. The paradox is "this process continues ad infinitum so Achilles doesn't catch up".

The notion of infinity shows up in natural science in the time of Newton. For example, Newton and Leibniz invented calculus, which relies on the idea of taking the limit of the infinitesimally small as in

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}. \tag{37}$$

The philosophical use of ∞ is that it represents a (Platonic) idea, whose mathematical definition requires the precise description of a limiting process.

The limiting process of calculus is quite straightforward. Quantum Field Theory introduces a new kind of ∞ with a limiting process described by the techniques of renormalization, associated with the idea of infinitesimally-small amounts of infinite intermediate energies.

Yet another kind of ∞ shows up in modern cosmology. The boundaries of our universe are infinitely far away, but again, this concept requires description of the process of approaching those infinite boundaries. The resultant mathematics leads to the study of celestial holography.

As a completely different manifestation of ∞ is the theory of probability. Mathematically, this doesn't appear to require infinity. However, when using probability to describe physical events (tossing a coin, predicting the weather, calculating specific heat of a solid), the mathematical principles of probability only hold when the event – or their circumstances – are repeated an infinite number of times and therefore the applicability of probability again requires a description of the limiting process (which is built into the theory).

The miracle (in my opinion) of nature, is that these various kinds of infinities (which include the limiting behavior) are relevant. Borrowing from Wittgenstein, imagine the following world: All coins have heads and tails, and absolutely nothing distinguishes the weight or physical attributes (other than the etching) of the two sides of the coin. Yet, coin-flips don't behave according to the usual laws of probability. For example, I flip a coin and exactly 3 times it ends up heads and exactly 3 times it ends up tails, again followed by 3 heads etc. But then you flip the coin and you get HTTHT-THTT... And this is true for all coins. Also something similar for dice, and for all the usual things we think about with games of chance. Now, I could still construct a mathematical theory of probability. But it wouldn't correspond to any of my life experiences and would be of no interest. Worse than that, life would probably be chaotic and perhaps impossible to understand. Indeed, maybe there'd be no life since life appears to require some organizing principles.

4 Complex scalar fields

See Chapter 7 of Lancaster. Some of this theory is used in Chapter 42.

Recall that classical field theory can be described by the following rules:

- Identify the fields of interest.
- Construct a Lagrangian as a function of the fields. Its integral is called the Action.
- As the fields are varied subject to boundary-value constraints, the Action changes. Find the field configurations for which the variations are extrema of the Action (e.g. a minimum of the Action). These configurations represent the physics (i.e., what happens in nature).
- These extrema correspond to solutions of partial differential equations known as the Euler-Lagrange equations. So if you solve those equations, you predict the future.

Many-body theory follows a slightly different set of rules:

- Identify the single-particle basis states. The Fock space (multiparticle) basis states will be constructed by providing occupation numbers for these states.
- Construct a multi-particle Hamiltonian. This is an operator on Fock space. All such operators can be built as linear combinations of products of annihilation and creation operators.

- Construct fields out of linear combinations of annihilation and creation operators and rewrite the Hamiltonian in terms of fields.
- If desired, perform a Legendre transform to bring the theory into Lagrangian form (I believe this can always be done), with equations of motion obtained from the action principal.

4.1 Example Lagrangian

Define a complex field ψ as

$$\psi(x) = \frac{1}{\sqrt{2}}[\phi_1(x) + i\phi_2(x)], \tag{38}$$

where ϕ_1 and ϕ_2 are real. Then

$$\psi^\dagger(x) = \frac{1}{\sqrt{2}}[\phi_1(x) - i\phi_2(x)], \tag{39}$$

where the dagger indicates conjugation (or later, when we deal with operators, it indicates the adjoint).

The Lagrangian we'll construct is

$$\begin{aligned}\mathcal{L}(\psi) &= \partial^\mu\psi^\dagger\partial_\mu\psi - m^2\psi^\dagger\psi - g(\psi^\dagger\psi)^2 \\ &= \frac{1}{2}\left(\partial^\mu\phi_1\partial_\mu\phi_1 + \partial^\mu\phi_2\partial_\mu\phi_2 - m^2(\phi_1^2+\phi_2^2) - \frac{1}{2}g\left(\phi_1^2+\phi_2^2\right)^2\right).\end{aligned} \tag{40}$$

In the second line I've expanded ψ into its real components ϕ_1 and ϕ_2, to illustrate that the Lagrangian could have been directly written in terms of the real fields ϕ_1 and ϕ_2. The use of complex fields is just a notational convenience. Also, note the resemblance of the first term to the phonon Lagrangian described in subsection 2.3.2.

4.1.1 Euler-Lagrange equations

The general Euler-Lagrange equations (whose solutions are extrema of the action) for a complex scalar theory consisting of fields ψ and $\psi^\dagger$ are:

$$\begin{aligned}\partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)} &= \frac{\partial\mathcal{L}}{\partial\psi}, \\ \partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi^\dagger\right)} &= \frac{\partial\mathcal{L}}{\partial\psi^\dagger}.\end{aligned} \tag{41}$$

Noting that $\mathcal{L}(\psi) = \partial^\mu \psi^\dagger \partial_\mu \psi - m^2 \psi^\dagger \psi - g(\psi^\dagger \psi)^2$, we find

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi\right)} &= \partial^\mu \psi^\dagger, \\
\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi^\dagger\right)} &= \partial^\mu \psi, \\
\frac{\partial \mathcal{L}}{\partial \psi} &= \left(m^2 + 2g\psi^\dagger \psi\right) \psi^\dagger \\
\frac{\partial \mathcal{L}}{\partial \psi^\dagger} &= \left(m^2 + 2g\psi^\dagger \psi\right) \psi
\end{aligned} \tag{42}$$

so that the Euler-Lagrange equations become

$$\begin{aligned}
\left(\partial_\mu \partial^\mu + m^2 + 2g\psi^\dagger \psi\right) \psi = 0 \\
\left(\partial_\mu \partial^\mu + m^2 + 2g\psi^\dagger \psi\right) \psi^\dagger = 0.
\end{aligned} \tag{43}$$

4.2 Symmetry

Let

$$\psi \to \psi e^{i\alpha}, \qquad \psi^\dagger \to \psi^\dagger e^{-i\alpha}. \tag{44}$$

The Lagrangian doesn't change. This is a continuous symmetry.

5 Canonical quantization of complex scalar field theory

Lancaster Chapter 12

Recall from the discussion of the harmonic oscillator, Eq. (14), that

$$\begin{aligned}
[q_k, p_j] &= i\delta_{k,j} \\
[q_j, q_k] &= 0 \\
[p_j, p_k] &= 0.
\end{aligned} \tag{45}$$

As Coleman pointed out, the rest of physics seems to be some kind of generalization of the harmonic oscillator.

- In the classical theory, find the analogue of q_i and the analogue of p_j (*canonical coordinates* and *canonical momenta*), then promote these to operators and construct the Hilbert space by requiring that they obey the CCR (Eq. (2)) and the Euler-Lagrange equations. (In practice, we only construct an approximate Hilbert space since our methods can only be used when we assume that it is OK to treat the interaction term as 0.) Furthermore, use the canonical coordinates and momenta, together with the Lagrangian, to construct the Hamiltonian.

- For the complex scalar field theory,

$$\begin{aligned} q_i &\to \mathbf{\Psi}(x) = \left(\psi(x), \psi^\dagger(x)\right) \\ p_j &\to \mathbf{\Pi}^0(x') \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \mathbf{\Psi}\right)}(x') = \left(\partial^0\psi^\dagger(x'), \partial^0\psi(x')\right) \end{aligned} \tag{46}$$

- Promote the fields to operators (I won't put hats on them.) Then the CCR are

$$\begin{gathered} [\psi(t,\mathbf{x}), \partial^0\psi^\dagger(t,\mathbf{x}')] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}') \\ [\psi(t,\mathbf{x}), \psi^\dagger(t,\mathbf{x}')] = [\psi(t,\mathbf{x}), \psi(t,\mathbf{x}')] = [\partial^0\psi(t,\mathbf{x}), \partial^0\psi^\dagger(t,\mathbf{x}')] = [\partial^0\psi(t,\mathbf{x}), \partial^0\psi(t,\mathbf{x}')] = 0 \end{gathered} \tag{47}$$

- The standard classical construction of the Hamiltonian is, for the full interacting scalar Lagrangian of Eq. (40)

$$\mathcal{H} = \partial_0\psi^\dagger(x)\partial_0\psi(x) + \boldsymbol{\nabla}\psi^\dagger(x)\boldsymbol{\nabla}\psi(x) + m^2\psi^\dagger(x)\psi(x) + g\left(\psi^\dagger(x)\psi(x)\right)^2 \tag{48}$$

- The Euler-Lagrange equations are, from Eq. (43):

$$\begin{aligned} \left(\partial_\mu\partial^\mu + m^2 + 2g\psi^\dagger\psi\right)\psi &= 0 \\ \left(\partial_\mu\partial^\mu + m^2 + 2g\psi^\dagger\psi\right)\psi^\dagger &= 0. \end{aligned} \tag{49}$$

- We can use the CCR and free-particle ($g = 0$) Euler-Lagrange equations to construct the Hilbert space, the operators of interest and therefore the (perturbative approximation to) quantum theory. This is covered in Lancaster 12.1. The steps are as follows:

  - If we set the interaction coefficient g to 0, the Euler-Lagrange equations are

  $$\left(\partial_\mu\partial^\mu + m^2\right)\psi(x) = 0. \tag{50}$$

  The solution is a linear combination of $e^{ip_0t}e^{-i\mathbf{p}\cdot\mathbf{x}} = e^{ip\cdot x}$ and $e^{-ip_0t}e^{i\mathbf{p}\cdot\mathbf{x}} = e^{-ip\cdot x}$ where $p^2 = p_0^2 - \mathbf{p}^2 = m^2$.

– By convention we write, where $a_{\mathbf{p}}(t)$ and $b_{\mathbf{p}}(t)$ are coefficients and $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$,

$$\begin{aligned}\psi(t, \mathbf{x}) &= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}(2E_{\mathbf{p}})^{\frac{1}{2}}} \left(a_{\mathbf{p}}(t)e^{i\mathbf{p}\cdot\mathbf{x}} + b_{\mathbf{p}}^{\dagger}(t)e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \\ \psi^{\dagger}(t, \mathbf{x}) &= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}(2E_{\mathbf{p}})^{\frac{1}{2}}} \left(a_{\mathbf{p}}^{\dagger}(t)e^{-i\mathbf{p}\cdot\mathbf{x}} + b_{\mathbf{p}}(t)e^{i\mathbf{p}\cdot\mathbf{x}}\right)\end{aligned} \tag{51}$$

The second line is obtained from the first by complex conjugation. These equations can be interpreted as classical equations with complex fields, or quantum equations with operator-valued fields (the dagger would then mean "the adjoint" of the operator). Since these fields must be solutions of the E-L equations, the t-dependence is given as $a_{\mathbf{p}}(t) = b_{\mathbf{p}}(t) = e^{-iE_{\mathbf{p}}t}$.

– If we substitute the field-expansions into the canonical commutation relations, that will lead to equal-time commutation relations for the coefficients $a_{\mathbf{p}}(t)$ and $b_{\mathbf{p}}(t)$.

$$[a_{\mathbf{p}}(t), a_{\mathbf{q}}^{\dagger}(t)] = [b_{\mathbf{p}}(t), b_{\mathbf{q}}^{\dagger}(t)] = \delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{52}$$

with all other commutator combinations vanishing.

– The $a_{\mathbf{p}}$ are known as the particle annihilation operators for momentum $\mathbf{p}$ and the $b_{\mathbf{p}}$ are known as the antiparticle annihilation operators for momentum $\mathbf{p}$. There are reasons for referring to particles and antiparticles but so far, we haven't motivated those names. Starting with the lowest-energy eigenstate of the free Hamiltonian we create all other states of the Hilbert space by successive applications of $a_{\mathbf{p}}^{\dagger}$ and $b_{\mathbf{p}}^{\dagger}$.

– All operators of interest can be constructed perturbatively as products of fields or their derivatives, and therefore can be constructed out of the annihilation and creation operators. Since these operators have known behavior on the constructed Hilbert space, we then know how all operators of interest act on the Hilbert space.

– There is at least one kind of ambiguity in promoting classical observables to quantum operators. For example, consider the charge operator from Lancaster Eq. (12.12).

$$Q_N = i \int d^3x \left(\psi(x)\partial^0\psi^{\dagger}(x) - i\psi^{\dagger}(x)\partial^0\psi(x)\right). \tag{53}$$

Terms classically involving the product of $\psi(x)$ and $\partial^0\psi^\dagger$ can be written in operator language either as $\psi(x)\partial^0\psi^\dagger(x)$ or $\partial^0\psi^\dagger(x)\psi(x)$ or as some linear combination of these. But $\psi(x)$ and $\partial^0\psi^\dagger(x)$ don't commute with each other, so these various operator versions aren't equal to one another. Physicists resolve this ambiguity by picking a particular ordering known as *normal ordering.* In general, physical results are independent of this ordering, but sometimes additional criteria must be invoked in order to obtain an acceptable ordering.

5.1 Some free-theory operators

In scattering problems, a variety of questions can be addressed concerning physics far from the interaction point. These questions are generally answered by setting the interaction term to 0 and then examining the free theory. Here we will look at two interesting observables. First we define the number operators.

$$\begin{aligned}\hat{n}^a_{\mathbf{p}} &= a^\dagger_{\mathbf{p}} a_{\mathbf{p}},\\ \hat{n}^b_{\mathbf{p}} &= b^\dagger_{\mathbf{p}} b_{\mathbf{p}}.\end{aligned} \tag{54}$$

(I have suppressed the time argument of the annihilation and creation operators since they are not important here.) These number operators count the number of particles and antiparticles having momentum $\mathbf{p}$. For example, if we have a state with only particles (no antiparticles) and if the state is labelled by the number, $n_{\mathbf{p}}$, of particles of each possible momentum $\mathbf{p}$, then

$$\hat{n}^a_{\mathbf{p}} |n_{\mathbf{p}_1}, n_{\mathbf{p}_2}, ..., n_{\mathbf{p}}, ...\rangle = n_{\mathbf{p}} |n_{\mathbf{p}_1}, n_{\mathbf{p}_2}, ..., n_{\mathbf{p}}, ...\rangle. \tag{55}$$

This can be proven by successive applications of the annihilation-creation commutation relations.

5.1.1 The Hamiltonian

The free Hamiltonian density is given by Eq. (48) with the interaction coupling g set to 0.

$$\mathcal{H} = \partial_0\psi^\dagger(x)\partial_0\psi(x) + \boldsymbol{\nabla}\psi^\dagger(x)\boldsymbol{\nabla}\psi(x) + m^2\psi^\dagger(x)\psi(x). \tag{56}$$

Then the Hamiltonian is $H = \int d^3x\mathcal{H}$. If we plug in the Fourier expansion (aka mode expansion) of Eq. (51), we find

$$H = \int d^3p E_{\mathbf{p}} \left(\hat{n}^a_{\mathbf{p}} + \hat{n}^b_{\mathbf{p}}\right) + \text{constant} \tag{57}$$

As it happens, the *constant* is infinite. However, by re-ordering some of the classical operators in the mode-expansion of the Hamiltonian, we can eliminate the constant. This is called *normal-ordering* and Lancaster writes $N[H]$.

The Hamiltonian expression is interpreted as the sum (integral) over all momenta, of the total energy of particles plus antiparticles having that momentum. The total energy is simply the product of the single-particle energy $E_{\mathbf{p}}$ times the number of particles and antiparticles with momentum $\mathbf{p}$.

5.1.2 The charge current

We can mode-expand the fields appearing in the expression for the charge current from Lancaster Eq. (12.12) (also see my section on Noether's theorem, in my notes entitled "Topics prerequisite to Chapter 42").

$$Q_N = i \int d^3x \left(\psi(x)\partial^0\psi^\dagger(x) - i\psi^\dagger(x)\partial^0\psi(x)\right). \tag{58}$$

Similarly to the computation for the Hamiltonian, we obtain

$$Q_N = \int d^3p \left(\hat{n}^b_{\mathbf{p}} - \hat{n}^a_{\mathbf{p}}\right) + \text{constant}. \tag{59}$$

Again, we can eliminate the constant by normal-ordering. We interpret this by saying that the total charge is obtained for each momentum as the difference between the number of antiparticles and particles. This should be reminiscent of the usual idea of charge for electrons and positrons. A positron has a positive charge and an electron has a negative charge, so the total charge is the difference. **This fact ultimately is why we refer to the a and b as *particle* and *antiparticle* annihilation operators.**

5.2 Non-relativistic limit

Lancaster 12.3

Most of our many-body examples involve non-relativistic (small velocity) motion and therefore should be described by the non-relativistic limit of QFT. We examine several ways of describing non-relativistic physics using QFT.

5.2.1 The Schrodinger equation

Start with the Euler-Lagrange equation Eq. (43). However, change the interaction term to reflect the presence of an external potential.

$$\left(\partial_\mu \partial^\mu + m^2\right)\psi + 2mV(x)\psi = 0, \tag{60}$$

where $V(x)$ is a real-valued function and we multiply by $2m$ so that later, we can make the connection with the Schrodinger equation. This is not some limiting case of the original EL equation, but instead is the limiting case of a Lagrangian that involves fields other than the charged scalar ψ.

Define a new field (we'll see why, shortly)

$$\Psi(\mathbf{x}, t) = \sqrt{2mc^2}\psi(x)e^{imc^2} \tag{61}$$

so that

$$\psi(\mathbf{x}, t) = \frac{1}{\sqrt{2mc^2}}\Psi(x)e^{-imc^2}, \tag{62}$$

where we've re-introduced the constant c so that we can take the non-relativistic limit by letting $c \to \infty$. In Eq. (60 substitute ψ in terms of Ψ to obtain

$$\frac{\partial^2 \Psi}{\partial t^2} - 2imc^2\frac{\partial \Psi}{\partial t} - c^2\boldsymbol{\nabla}^2\Psi + 2mc^2V(x)\Psi = 0. \tag{63}$$

In the large-c limit, the first term can be dropped, leaving

$$i\frac{\partial}{\partial t}\Psi = -\frac{1}{2m}\boldsymbol{\nabla}^2\Psi + V\Psi. \tag{64}$$

This resembles the Schrodinger equation, but **in the above expression, Ψ is an operator, whereas in the usual Schrodinger equation, Ψ is a state**.

However, if we apply the operators to a state $|s\rangle$, then we obtain a Schrodinger equation for a state $\xi = \Psi|s\rangle$.

$$i\frac{\partial}{\partial t}\xi = -\frac{1}{2m}\nabla^2\xi + V\xi. \tag{65}$$

What we are especially interested in, is the evolution of a state that represents a single particle. For the time being, I don't know how to construct an arbitrary single-particle state by applying $\Psi(x)$ to some state $|s\rangle$.

5.2.2 Nonrelativistic canonical quantization – the free theory

Above, we began with the Euler-Lagrange equation and then took the relativistic limit to construct a Schrodinger equation. Here, we will be more systematic by starting with the Lagrangian, then taking the non-relativistic limit. The classical theory was covered in subsection 4.1. We'll start with the non-interacting theory ($g = 0$ or $V(x) = 0$). The resulting Euler-Lagrange equation will be the Schrodinger equation for the free theory. In addition, we can apply the CCR to obtain the mode-expansion.

Recall the Lagrangian we had from Eq. (40), but re-enstating the unit c and removing the interaction term.

$$\mathcal{L}(\psi) = \partial_0\psi^\dagger\partial_0\psi - c^2\nabla\psi^\dagger\cdot\nabla\psi - m^2c^4\psi^\dagger\psi. \tag{66}$$

As in Eq. (62) redefine the field as

$$\psi(\mathbf{x}, t) = \frac{1}{\sqrt{2mc^2}}\Psi(x)e^{-imc^2}. \tag{67}$$

Then the transformed Lagrangian becomes[6]

$$\mathcal{L}(\Psi) = \frac{1}{2mc^2}\partial_0\Psi^\dagger\partial_0\Psi + i\Psi^\dagger\partial_0\Psi - \frac{1}{2m}\nabla\Psi^\dagger\cdot\nabla\Psi. \tag{68}$$

As before, in the non-relativistic limit we can drop the first term because it is suppressed by a factor of c^2, thus leaving the non-relativistic Lagrangian as

$$\mathcal{L}_{NR}(\Psi) = i\Psi^\dagger\partial_0\Psi - \frac{1}{2m}\nabla\Psi^\dagger\cdot\nabla\Psi. \tag{69}$$

[6]See Lancaster Equation (12.22). Note that the action-term $\int d^3x\frac{-i}{2}\partial_0\Psi^\dagger\Psi$ can be integrated by parts (assuming the fields drop off quickly to 0 at infinity) to give $\int d^3x\frac{i}{2}\Psi^\dagger\partial_0\Psi$. As a result, the term in the Lagrangian proportional to i becomes $i\Psi^\dagger\partial_0\Psi$. Lancaster, in Example 12.3, has a different explanation for this, but I can't follow his explanation.

The Euler-Lagrange equation, becomes the (operator) Schrodinger equation Eq. (64). A solution Ψ_{free} to the (free) Schrodinger equation is

$$\begin{aligned}\Psi_{\mathbf{p}}(x) &= a_{\mathbf{p}} e^{-i(Et-\mathbf{p}\cdot\mathbf{x})} \\ &= a_{\mathbf{p}} e^{-ip\cdot x}\end{aligned} \tag{70}$$

where $E = p_0 = \mathbf{p}^2/2m$, and $a_{\mathbf{p}}$ is time-independent. The most general solution is a linear combination of the $\Psi_{\mathbf{p}}(x)$ and is written

$$\Psi_{\text{free}}(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} a_{\mathbf{p}} e^{-ip\cdot x}, \tag{71}$$

which we call the *mode expansion*.

We now proceed to canonical quantization following the same procedure as for Eqs. (45 - 52) at the beginning of this section.

$$\begin{aligned} q_i &\to \mathbf{\Psi}(x) = \left(\Psi(x), \Psi^\dagger(x)\right) \\ p_j &\to \mathbf{\Pi}^0(x') \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \mathbf{\Psi}\right)}(x') = \left(i\Psi^\dagger(x'), 0\right). \end{aligned} \tag{72}$$

Then the CCR becomes

$$[\Psi(t,\mathbf{x}), i\Psi^\dagger(t,\mathbf{x}')] = i\delta^3(\mathbf{x}-\mathbf{x}'). \tag{73}$$

EXERCISE: Confirm the above CCR (Lancaster exercise (12.2)b)

If we promote the coefficients $a_{\mathbf{p}}$ to operators and substitute the mode expansion into

$$[\Psi_{\text{free}}(t,\mathbf{x}), i\Psi^\dagger_{\text{free}}(t,\mathbf{x}')] = i\delta^3(\mathbf{x}-\mathbf{x}'), \tag{74}$$

we obtain

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] = \delta^{(3)}(\mathbf{p}-\mathbf{p}'). \tag{75}$$

EXERCISE: Confirm this equation.

If we follow the usual procedure for obtaining the Hamiltonian from the Lagrangian, we find that in the non-relativistic limit

$$\begin{aligned} H_{\text{NR}} &= \int d^3x \frac{1}{2m} \mathbf{\nabla}\Psi^\dagger \cdot \mathbf{\nabla}\Psi \\ &= \int d^3x \frac{d^3p d^3p'}{(2\pi)^3} \frac{\mathbf{p}\cdot\mathbf{p}'}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}'} e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \\ &= \int d^3p \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}. \end{aligned} \tag{76}$$

thus demonstrating that $a_{\mathbf{p}}$ are annihilation operators for particles of kinetic energy $\frac{\mathbf{p}^2}{2m}$.

5.2.3 Nonrelativistic canonical quantization – the interacting theory

What happens when $V(x) \neq 0$? In this section, I'll deviate slightly from Lancaster's presentation but I'll end with something close to Lancaster's result. I start by obtaining the nonrelativistic Euler-Lagrange equations, and solving these by a transformation to a non-interacting many-body theory whose individual particles have the eigen-energies of Schrodinger Hamiltonian with potential $V(\mathbf{x})$. I'll then show an alternative approach more like Lancaster's, and explain the relationship between these two approaches.

Taking the Lagrangian we had from Eq. (40), but re-instating the unit c and substituting the interaction term,

$$\mathcal{L}(\psi) = \partial_0\psi^\dagger\partial_0\psi - c^2\boldsymbol{\nabla}\Psi^\dagger \cdot \boldsymbol{\nabla}\Psi - m^2c^4\psi^\dagger\psi - 2mc^2V(x)\psi^\dagger\psi. \tag{77}$$

As in Eq. (62) redefine the field as

$$\psi(\mathbf{x}, t) = \frac{1}{\sqrt{2mc^2}}\Psi(x)e^{-imc^2}. \tag{78}$$

Then the transformed Lagrangian becomes

$$\mathcal{L}(\Psi) = \frac{1}{2mc^2}\partial_0\Psi^\dagger\partial_0\Psi + i\Psi^\dagger\partial_0\Psi - \frac{1}{2m}\boldsymbol{\nabla}\Psi^\dagger \cdot \boldsymbol{\nabla}\Psi - V(x)\Psi^\dagger\Psi. \tag{79}$$

As before, in the non-relativistic limit we can drop the first term because it is suppressed by a factor of c^2, thus leaving the non-relativistic Lagrangian as

$$\mathcal{L}_{NR}(\Psi) = i\Psi^\dagger\partial_0\Psi - \frac{1}{2m}\boldsymbol{\nabla}\Psi^\dagger \cdot \boldsymbol{\nabla}\Psi - V(x)\Psi^\dagger\Psi. \tag{80}$$

Notice this is the same as Lancaster Eq. (12.24). We now make an additional assumption that V is time-independent. I will indicate that by $\hat{V}(\mathbf{x}) \equiv V(0, \mathbf{x})$. **Going forward, I will drop the *hat* and write $V(\mathbf{x})$.**

The Euler-Lagrange equation, following the same procedure as before, gives us the Schrodinger equation

$$i\frac{\partial}{\partial t}\Psi = -\frac{1}{2m}\boldsymbol{\nabla}^2\Psi + V\Psi. \tag{81}$$

When the potential is 0 (the free theory), the Schrodinger equation is solved as a superposition of plane waves. Then our quantization procedure leads to a set of creation and annihilation operators for plane wave momentum states. For a more general potential, we have different solutions leading to different annihilation and creation operators. I'll treat the general situation by taking a brief detour to standard quantum mechanics.

We start by finding the eigenstates of the operator (acting on the Hilbert space of time-independent complex functions of $\mathbf{x}$).

$$\hat{E} = -\frac{1}{2m}\boldsymbol{\nabla}^2 + V(\mathbf{x}). \tag{82}$$

Let's label the eigenfunction as $\psi_n(\mathbf{x})$, noting that here, ψ_n is an ordinary complex function and not an operator. The index n labels the eigenfunction and, in general for 3D systems, is often described as a triplet $n = (n_1, n_2, n_3)$. We are solving

$$\hat{E}\psi_n(\mathbf{x}) = E_n\psi_n(\mathbf{x}), \tag{83}$$

thus

$$\left(-\frac{1}{2m}\boldsymbol{\nabla}^2 + V(\mathbf{x})\right)\psi_n(\mathbf{x}) = E_n\psi_n(\mathbf{x}). \tag{84}$$

As an example, if V is a 3D linear harmonic oscillator potential, then the eigenfunctions are products of Hermite polynomials and exponentials.

With appropriate normalization[7], it can be shown that the eigensolutions obey the orthonormality relations

$$\int d^3x\psi_n(\mathbf{x})\psi_m^*(\mathbf{x}) = \delta_{mn} \tag{85}$$

Next, with an abuse of notation, we define $\psi_n(t, \mathbf{x}) = e^{-iE_nt}\psi_n(\mathbf{x})$. It's easy to check that

$$i\frac{\partial}{\partial t}\psi_n(t, \mathbf{x}) = \left(-\frac{1}{2m}\boldsymbol{\nabla}^2 + V\right)\psi_n(t, \mathbf{x}), \tag{86}$$

and therefore that $\psi_n(t, \mathbf{x})$ is a solution to the Schrodinger equation. We can show that the most general solution to the Schrodinger equation is of the form $\psi(t, \mathbf{x}) = \sum_n a_n\psi_n(t, \mathbf{x})$, where the a_n are complex coefficients.

[7] To be more precise, we should focus for now on situations where there aren't two independent solutions with the same eigenvalue. That situation is known as degeneracy and can be resolved so that all our key results come out the same.

Now let's return to our operator Schrodinger equation, Eq. (81). We can solve this by taking the above general solution and promoting the coefficients a_n to operators $\hat{a}_n$. So

$$\begin{aligned}\Psi(x) &= \sum_n \hat{a}_n \psi_n(t, \mathbf{x}) \\ &= \sum_n \hat{a}_n \psi_n(\mathbf{x}) e^{-iE_n t}\end{aligned} \tag{87}$$

From the orthonormality relations, Eq. (85), we then have

$$\hat{a}_n = \int d^3x \Psi(t, \mathbf{x}) \psi_n^*(\mathbf{x}) e^{iE_n t}. \tag{88}$$

Then

$$\begin{aligned}[\hat{a}_n, \hat{a}_m^\dagger] &= \int d^3x d^3x' \left[\Psi(t, \mathbf{x}), \Psi^\dagger(t, \mathbf{x}')\right] \psi_n^*(\mathbf{x}) \psi_m(\mathbf{x}') e^{-i(E_m - E_n)t} \\ &= \int d^3x d^3x' \delta^3(\mathbf{x} - \mathbf{x}') e^{-i(E_m - E_n)t} \psi_n^*(\mathbf{x}) \psi_m(\mathbf{x}') \\ &= \int d^3x e^{-i(E_m - E_n)t} \psi_n^*(\mathbf{x}) \psi_m(\mathbf{x}) \\ &= \delta_{nm}\end{aligned} \tag{89}$$

where we used in the second line, the CCR from Eq. (73)[8], $[\Psi(t, \mathbf{x}), i\Psi^\dagger(t, \mathbf{x}')] = i\delta^3(\mathbf{x} - \mathbf{x}')$.

We can also derive the Hamiltonian density operator starting from Lancaster Eq. 12.26.

$$\mathcal{H} = \frac{1}{2m} \boldsymbol{\nabla} \Psi^\dagger \cdot \boldsymbol{\nabla} \Psi + V(x) \Psi^\dagger \Psi. \tag{90}$$

It is also worth pointing out that when integrated over $\mathbf{x}$ this expression for the Hamiltonian is also the expression for the "conserved Noether charge" associated with time-translation symmetry. In particular, the time-derivative

[8]The derivation of those CCR turns out to be exactly the same for our interacting theory as for the free theory.

of the Hamiltonian is 0. If we substitute Eq. (87), we get

$$
\begin{aligned}
H &= \int d^3x \mathcal{H}(\mathbf{x}) \\
&= \int d^3x \left(\frac{1}{2m} \boldsymbol{\nabla} \Psi^\dagger(\mathbf{x}) \cdot \boldsymbol{\nabla} \Psi(\mathbf{x}) + V(x) \Psi^\dagger(\mathbf{x}) \Psi(\mathbf{x}) \right) \\
&= \int d^3x \Psi^\dagger(\mathbf{x}) \left(-\frac{1}{2m} \boldsymbol{\nabla}^2 + V(x) \right) \Psi(\mathbf{x}) \\
&= \int d^3x \sum_n \sum_m \hat{a}_n^\dagger \hat{a}_m \psi_n^*(t, \mathbf{x}) \left(-\frac{1}{2m} \boldsymbol{\nabla}^2 + V(x) \right) \psi_m(t, \mathbf{x}) \\
&= \int d^3x \sum_n \sum_m \hat{a}_n^\dagger \hat{a}_m \psi_n^*(t, \mathbf{x}) E_m \psi_m(t, \mathbf{x}) \\
&= \sum_n E_n \hat{a}_n^\dagger \hat{a}_n
\end{aligned}
\tag{91}
$$

where the third line is obtained through integration-by-parts, the fourth line is obtained by substituting Eq. (87), the fifth line is obtained from the eigenvalue equation for $\hat{E}$ and the last line is obtained from the orthonormality relations.

It's important to note that in the final expression for the Hamiltonian operator, the role of $\hat{a}_n^\dagger \hat{a}_n$ is that of a counting operator, rather than as an operator which identifies the energy-state of a particle. We see this, because E_n is not proportional to an integer whereas the eigenvalues of $\hat{a}_n^\dagger \hat{a}_n$ are integers. We have therefore shown that the operators $\hat{a}_n$ and $\hat{a}^\dagger$ are annihilation and creation operators for particles of energy E_n. **Note that this is nothing like Lancaster's result.**

Now let's turn to an approach closer to Lancaster's. The field $\Psi(t, \mathbf{x})$ can be Fourier-expanded as

$$
\Psi(t, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{p}\cdot\mathbf{x}} a_{\mathbf{p}}(t). \tag{92}
$$

For notational consistency with other texts, define $a_{\mathbf{p}} \equiv a_{\mathbf{p}}(0)$.

For the free-field theory, the time-dependence of $a_{\mathbf{p}}(t)$ is required to be $a_{\mathbf{p}}(t) = a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t}$, in order for Ψ to be a solution of the field Schrodinger equation. $E_{\mathbf{p}}$ is the kinetic energy. However, for the interacting theory, that time-dependence would be wrong since it doesn't account for the potential V. Let's therefore proceed with an arbitrary time-dependence of the coefficient.

The reverse Fourier-transform becomes

$$a_{\mathbf{p}}(t) = \int \frac{d^3x}{(2\pi)^{\frac{3}{2}}} e^{-i\mathbf{p}\cdot\mathbf{x}} \Psi(t, \mathbf{x}). \tag{93}$$

Now we can promote the coefficients to operators and take the equal-time commutators.

$$\begin{aligned}
[a_{\mathbf{p}}(t), a^\dagger_{\mathbf{p}'}(t)] &= \int \frac{d^3x d^3x'}{(2\pi)^3} e^{-i(\mathbf{p}\cdot\mathbf{x} - \mathbf{p}'\cdot\mathbf{x}')} [\Psi(t, \mathbf{x}), \Psi^\dagger(t, \mathbf{x}')] \\
&= \int \frac{d^3x d^3x'}{(2\pi)^3} e^{-i(\mathbf{p}\cdot\mathbf{x} - \mathbf{p}'\cdot\mathbf{x}')} \delta^3(\mathbf{x} - \mathbf{x}') \\
&= \int \frac{d^3x}{(2\pi)^3} e^{-i(\mathbf{p} - \mathbf{p}')\cdot\mathbf{x}} \\
&= \delta^3(\mathbf{p} - \mathbf{p}').
\end{aligned} \tag{94}$$

We have therefore shown that $a_{\mathbf{p}}(t)$ is an annihilation operator and its adjoint is a creation operator. This is similar to what Lancaster has, except for the fact that the annihilation and creation operators are now time-dependent.

The full interacting Hamiltonian is given in Eq. (90) as

$$\mathcal{H} = \frac{1}{2m} \boldsymbol{\nabla}\Psi^\dagger \cdot \boldsymbol{\nabla}\Psi + V(x)\Psi^\dagger\Psi \tag{95}$$

and can be expanded similarly to the lines of Eq. (91) but by substituting Eq. (92).

$$\begin{aligned}
H &= \int d^3x \mathcal{H}(t, \mathbf{x}) \\
&= \int d^3x \left(\frac{1}{2m} \boldsymbol{\nabla}\Psi^\dagger(t, \mathbf{x}) \cdot \boldsymbol{\nabla}\Psi(t, \mathbf{x}) + V(\mathbf{x})\Psi^\dagger(t, \mathbf{x})\Psi(t, \mathbf{x}) \right) \\
&= \int d^3x \Psi^\dagger(t, \mathbf{x}) \left(-\frac{1}{2m} \boldsymbol{\nabla}^2 + V(\mathbf{x}) \right) \Psi(t, \mathbf{x}) \\
&= \int d^3x \frac{d^3p d^3q}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{x}} a^\dagger_{\mathbf{p}}(t) a_{\mathbf{q}}(t) \left(-\frac{1}{2m} \boldsymbol{\nabla}^2 + V(\mathbf{x}) \right) e^{i\mathbf{q}\cdot\mathbf{x}} \\
&= \int d^3p \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}}(t) a_{\mathbf{p}}(t) + \int d^3x \frac{d^3p d^3q}{(2\pi)^3} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} a^\dagger_{\mathbf{p}}(t) a_{\mathbf{q}}(t) V(\mathbf{x}) \\
&= \int d^3p \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}}(t) a_{\mathbf{p}}(t) + \int d^3p d^3q a^\dagger_{\mathbf{p}}(t) a_{\mathbf{q}}(t) \tilde{V}(\mathbf{p} - \mathbf{q}),
\end{aligned} \tag{96}$$

where, for an arbitrary function of $\mathbf{x}$,

$$\tilde{f}(\mathbf{p} - \mathbf{q}) \equiv \int \frac{d^3x}{(2\pi)^3} f(\mathbf{x}) e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}. \tag{97}$$

Above, the third line is obtained through integration-by-parts and the fourth line is obtained by substituting Eq. (92).

As observed earlier, the Hamiltonian is time-independent. Therefore we can evaluate it at time $t = 0$. We can then substitute our definition $a_{\mathbf{p}} \equiv a_{\mathbf{0}}(t)$ to obtain

$$H = \int d^3p \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \int d^3p d^3q a^\dagger_{\mathbf{p}} a_{\mathbf{q}} \tilde{V}(\mathbf{p} - \mathbf{q}) \tag{98}$$

Finally, we can find a relationship between these p-mode operators and those obtained earlier for a mode decomposition in terms of eigensolutions to the Schrodinger equation. I don't have any examples showing how this relationship would be used, but derive it here for completeness.

Recall Eq. (87)

$$\Psi(x) = \sum_n \hat{a}_n \psi_n(\mathbf{x}) e^{-iE_n t}, \tag{99}$$

and apply this to Eq. (93).

$$a_{\mathbf{p}}(t) = \sum_n \int \frac{d^3x}{(2\pi)^{\frac{3}{2}}} e^{-i\mathbf{p}\cdot\mathbf{x}} \hat{a}_n \psi_n(\mathbf{x}) e^{-iE_n t}. \tag{100}$$

Lancaster, at the end of Example 12.4, makes some comments about a time-dependent aspect of his equation for the Hamiltonian. I don't follow his remarks, but suspect that their origin might be similar to those having to do with the time-dependence of my annihilation operators.

6 Non-relativistic many-body theory of fluids

The idea is to cast NRMB (*non-relativistic many-body* theory) into the same formalism as quantum field theory. Here is a (hopefully) simple argument for why this can be done:

- We saw in the QFT discussion, that the Lagrangian formulation of quantum field theory leads directly to a Fock space with a momentum-state occupation-number representation that can be implemented with ladder operators.

- Conversely, if we start with such a representation and implementation, we can construct a Lagrangian quantum field theory for it.
- The resulting theory has the same free part as QFT. The interactive part is obtained from the interaction-Hamiltonian, noting that $\mathcal{L}_I = -\mathcal{H}_I$.[9]
- As for the question of non-relativistic versus relativistic, we can approach this in several ways but in principle it's simply a matter of taking the non-relativistic limit of the QFT theory.

6.1 Example with single-particle operators

I'll illustrate this with an example which begins by focusing on a particularly simple interaction Hamiltonian based on "single-particle operators". An excellent treatment of this is given in Lancaster 4.2 and 4.4 (note that 4.3 is a digression – a fact which originally confused me quite a bit). A similar excellent treatment can be found in Landau and Lifshitz Vol. 3 section 64 (in fact, I found this treatment a bit clearer but that's probably because I initially underestimated the power and importance of Lancaster).

The single-particle-operator example is meant to be an approximation describing some kinds of fluids.

$$H_I = \sum_{a=1}^{N} h_a \tag{101}$$

where the sum is over particles, and h_a is a single-particle Hamiltonian. We take N to be very large. Furthermore, h_a acts on the Hilbert space of a single particle, with momentum basis $|\mathbf{k}_i\rangle$, so that $\langle \mathbf{k}_i | h_a | \mathbf{k}_j \rangle$ is independent of the index a. This form of the interaction Hamiltonian assures that the Hamiltonian maintains the indistinguishability (e.g. symmetry) of particles.

We can show in general that H_I can be written as

$$H_I = \sum_{\alpha\beta} (H_I)_{\alpha\beta}\, a^\dagger_\alpha a_\beta \tag{102}$$

[9]Strictly speaking, this equation is true only when the interaction Hamiltonian doesn't involve time-derivatives.

where the annihilation and creation operators are with respect to single-particle basis functions. We'll illustrate this with a particular case using momentum basis states in a coordinate-representation.

Consider for example, single-particle momentum basis functions $\psi_{\mathbf{k}_1}(\mathbf{x}), \psi_{\mathbf{k}_2}(\mathbf{x}), ...$, which we can also write as $\langle \mathbf{x}|\mathbf{k}_1\rangle, \langle \mathbf{x}|\mathbf{k}_2\rangle, ...$ The multi-particle system has wavefunction $\psi(\mathbf{x}_1, \mathbf{x}_2, ...) = \psi_{\mathbf{k}_1}(\mathbf{x}_1)\psi_{\mathbf{k}_4}(\mathbf{x}_2)... + \psi_{\mathbf{k}_4}(\mathbf{x}_1)\psi_{\mathbf{k}_1}(\mathbf{x}_2)... + ...$ The sum is over all permutations of the single-particle states comprising the multi-particle wavefunction. Then let h_a act on a single-particle state as $V(\mathbf{x})\psi(\mathbf{x})$.

$$\begin{aligned} h_1\psi &= \left(V(\mathbf{x}_1)\psi_{\mathbf{k}_1}(\mathbf{x}_1)\right)\psi_{\mathbf{k}_4}(\mathbf{x}_2)... + \left(V(\mathbf{x}_1)\psi_{\mathbf{k}_4}(\mathbf{x}_1)\right)\psi_{\mathbf{k}_1}(\mathbf{x}_2)... \\ h_2\psi &= \psi_{\mathbf{k}_1}(\mathbf{x}_1)\left(V(\mathbf{x}_2)\psi_{\mathbf{k}_4}(\mathbf{x}_2)\right)... + \psi_{\mathbf{k}_5}(\mathbf{x}_1)\left(V(\mathbf{x}_2)\psi_{\mathbf{k}_1}(\mathbf{x}_2)\right)... \end{aligned} \tag{103}$$

etc. From this we immediately see that $H_I|\psi\rangle = \left(\sum_a V(\mathbf{x}_a)\right)|\psi\rangle$.

Now we need to write H_I in terms of the fundamental field ψ of our theory. We begin by writing H_I using the momentum-state annihilation and creation operators which we define as the operators that create momentum states from the vacuum, namely $a_{\mathbf{k}}|0\rangle = |\mathbf{k}\rangle$. For this, we invoke Lancaster Eqs. (4.17) and (4.18) (Lancaster provides the nontrivial proof of Eq. (4.18) in example 4.3.) First we take the continuum limit. Then we write

$$V = \int d^3k_1 d^3k_2 V_{\mathbf{k}_1\mathbf{k}_2}|\mathbf{k}_1\rangle\langle\mathbf{k}_2|. \tag{104}$$

We observe that the result of V operating on a momentum basis state[10] is

$$\langle\mathbf{x}|V|\mathbf{k}\rangle = V(\mathbf{x})\langle\mathbf{x}|\mathbf{k}\rangle = V(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{105}$$

so that

$$\int d^3x e^{-i\mathbf{k}'\cdot\mathbf{x}}\langle\mathbf{x}|V|\mathbf{k}\rangle = \int d^3x V(\mathbf{x})e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}. \tag{106}$$

We can also evaluate the LHS of the above equation by using Eq. (104).

$$\begin{aligned} \int d^3x e^{-i\mathbf{k}'\cdot\mathbf{x}}\langle\mathbf{x}|V|\mathbf{k}\rangle &= \int d^3x d^3k_1 d^3k_2 e^{-i\mathbf{k}'\cdot\mathbf{x}} V_{\mathbf{k}_1\mathbf{k}_2}\langle\mathbf{x}|\mathbf{k}_1\rangle\langle\mathbf{k}_2|\mathbf{k}\rangle \\ &= \int d^3x d^3k_1 d^3k_2 e^{-i(\mathbf{k}'-\mathbf{k}_1)\cdot\mathbf{x}} V_{\mathbf{k}_1\mathbf{k}_2}\delta^3(\mathbf{k}_2-\mathbf{k}) \\ &= (2\pi)^3 V_{\mathbf{k}'\mathbf{k}}. \end{aligned} \tag{107}$$

[10] For ease of notation I haven't included normalization factors in the momentum-state wavefunctions.

Finally, setting the LHS above to the RHS of Eq. (106) we get

$$V_{\mathbf{k}'\mathbf{k}} = \frac{1}{(2\pi)^3}\int d^3x V(\mathbf{x})e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}. \tag{108}$$

Compare this to Lancaster Eq. (4.36). Having obtained $V_{\mathbf{k}'\mathbf{k}}$ we now use Lancaster Eq. (4.18) to show, in the continuum limit, that [11]

$$H_I = \int d^3k_1 d^3k_2 V_{\mathbf{k}_1\mathbf{k}_2} a^\dagger_{\mathbf{k}_1} a_{\mathbf{k}_2}. \tag{109}$$

We also have, as usual, the free part of the Hamiltonian in the non-relativistic continuum limit, as

$$H_0 = \int d^3k \frac{\mathbf{k}^2}{2m} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}. \tag{110}$$

Setting $V_{\mathbf{k}_1\mathbf{k}_2} \to \tilde{V}(\mathbf{k}_1 - \mathbf{k}_2)$, and then changing the integration variables, the full Hamiltonian becomes

$$H = \int d^3p \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \int d^3p d^3q a^\dagger_{\mathbf{p}} a_{\mathbf{q}} \tilde{V}(\mathbf{p}-\mathbf{q}) \tag{111}$$

with

$$\tilde{V}(\mathbf{p}-\mathbf{q}) = \frac{1}{(2\pi)^3}\int d^3x V(\mathbf{x})e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}. \tag{112}$$

We see that this is precisely the same expression as we had in Eqs. (98) and (97) which we obtained from the Lagrangian theory given in Eq. (80).[12] EXERCISE: Confirm the above two equations, and then show the steps that would take us from Eq. (80) to the above equations.

6.2 Some observations

- As mentioned much earlier in this presentation, the continuum limit is a mathematical approximation of the many-body theory. It is meant to be valid in the limit where momenta spacings are small, momenta ranges (largest values) are large and number of particles is infinite. The

[11] Lancaster doesn't appear to explain, in chapter 4.2, why his expressions are written in the form $a^\dagger_\alpha a_\beta$ rather than $a_\beta a^\dagger_\alpha$. The order matters when $\alpha = \beta$ and ultimately the choice differs by a constant (unfortunately the constant is infinite). By convention, we place the creation operators to the left of the annihilation operators. That is called *normal ordering* and is discussed by Lancaster in chapter 4.4.

[12] Recall that this non-relativistic Lagrangian was obtained as the low-energy limit of a fully relativistic Lagrangian.

limiting procedure can be controlled by careful choices of boundary conditions are other regularization methods. When these limits are taken, field theory methods can be used to derive general behaviors of the many-body theory under consideration.

- The single-operator Hamiltonian can be interpreted as follows: The lowest-order (i.e. most important) interactions can be interpreted as transitions where a particle changes from one momentum state to another. The annihilation operator describes the momentum state that the particle was in before the transition, and the creation operator describes the momentum state that the paricle is in after the transition. By symmetry, the interaction-Hamiltonian is a sum over all such transitions, weighted by a potential function representing a common force acting on single particles in the fluid (for example, if we think of the particles as electrons, then the force might be an average of positive nuclear forces).

- So far, we've been vague about the meaning of particles and fields in many-body theory. For example, we've pointed out that coupled particles have collective behaviors that can be interpreted as other particles called phonons. This allows us to re-think our system as a collection of phonons with an interaction Hamiltonian pertaining to forces on the phonons. As indicated by our single-particle-operator example, we could then derive a field theory that embodies both the phonons and their Hamiltonian. In this way of looking at things, the field is purely a mathematical convenience. However, like all mathematical conveniences it has a life of its own. The field evolves partly based on its coupling to other parts of the system, and the field has correlation properties that can affect various observables, some of which are collective behaviors of the underlying phonons of the theory (and therefore of the original coupled particles). The field is then called an ***order parameter***. One might argue that all the same kinds of things are true about quantum field theory, so sometimes those fields are also known as order parameters. The main difference in approach is that for QFT, the fundamental interactions are hypothesized to occur between the fields, whereas in many-body theory, the fundamental interactions are hypothesized to occur between particles.

6.3 Two-body interactions

This section follows Lancaster chapter 4.4 and is also covered by Landau and Lifshitz Volume 3 section 64. The one-body example is unable to capture the effects of two-body interactions such as Coulomb interactions. Let the general two-body interaction be

$$H_I^{(2)} = \sum_{a<b}^{N} h_{ab}. \tag{113}$$

Following Lancaster, or Landau and Lifshitz, and with similar arguments for the case of the one-particle operators, we can show that the general form of two-body interaction Hamiltonian is

$$H_I^{(2)} = \sum_{\alpha\beta\gamma\delta} \left(H_I^{(2)}\right)_{\alpha\beta\gamma\delta} a_\alpha^\dagger a_\beta^\dagger a_\gamma a_\delta. \tag{114}$$

An example two-particle operator acting on a two-particle state, would be $V(\mathbf{x}-\mathbf{y})\left(\psi_1(\mathbf{x})\psi_2(\mathbf{y}) + \psi_1(\mathbf{y})\psi_2(\mathbf{x})\right)$. Lancaster shows for this case (in the continuum limit), that

$$H_V^{(2)} = \frac{1}{2}\int d^3p_1 d^3p_2 d^3q \tilde{V}(\mathbf{q}) a^\dagger_{\mathbf{p}_1+\mathbf{q}} a^\dagger_{\mathbf{p}_2-\mathbf{q}} a_{\mathbf{p}_2} a_{\mathbf{p}_1}, \tag{115}$$

where $\tilde{V}$ is defined by Eq. (112). Note that this can be described by the schematic

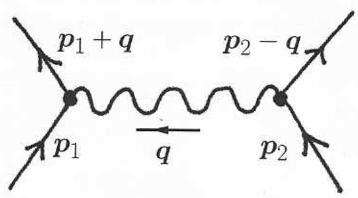


Fig. 4.8 A Feynman diagram for the process described in the text. Interpreting this diagram, one can think of time increasing in the vertical direction.

Through an argument similar to what we obtained starting with Eq. (80)

the above theory can be obtained from a (non-relativistic) action

$$S(\Psi) = \int d^3x \left[i\Psi^\dagger \partial_0 \Psi - \frac{1}{2m} \boldsymbol{\nabla}\Psi^\dagger \cdot \boldsymbol{\nabla}\Psi \right] - \frac{1}{2} \int d^3x d^3y \Psi^\dagger(\mathbf{x})\Psi^\dagger(\mathbf{y}) V(\mathbf{x}-\mathbf{y}) \Psi(\mathbf{y})\Psi(\mathbf{x})). \tag{116}$$

Notice that operators have been placed in normal order (creation operators to the left of annihilation operators). Lancaster explains this in chapter 4.4.