Summary of Field Theory Formulation

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1 Principles of a quantum many-body theory

Goal: Set up a mathematical theory of quantum many-body physics which satisfies the following principles:

- It can be described by an Action Principle involving the integral (known as *The Action*) of a real Lagrangian function of complex fields (a form of generalized coordinates) and their time-derivatives. The Action Principle is satisfied by solutions to the Euler-Lagrange equations.
- The Action is invariant under known physical symmetry transformations of the theory. Examples include Poincare invariance for relativistic theories or Galilean invariance for non-relativistic theories.
- There is a Hamiltonian that governs time-dependence of the system and can be written as a sum of a free term describing free particles, and an interaction term. The Hamiltonian and Lagrangian can be obtained from one another according to a procedure known as a Legendre transformation, which involves the introduction of momentum functions.
- Thusfar, the theory is classical. The quantum theory is derived by Dirac's principles:
  - Promote the (field) generalized coordinates and momenta to operators on a (as-yet-unspecified) Hilbert space.

- Impose canonical commutation relations (CCR) between the generalized coordinate and momenta operators.
- Find a representation (i.e. a Hilbert space on which the operators act) on which the operators obey the CCR and the Euler-Lagrange equations for the free theory.

- **We impose one more requirement: The free theory (same Lagrangian as above without the interaction term) has a Hilbert space occupation-number representation**

$$|\psi_N\rangle = |N_1, N_2, ...\rangle \tag{1}$$

such that the free Hamiltonian has an energy spectrum

$$E_{\psi_N} = \sum_i N_i E_i + \textbf{constant} \tag{2}$$

where E_i is the energy of the i^{th} mode and N_i is the number of particles in the i^{th} mode.

I will illustrate how to do this both for a relativistic quantum field theory and for a non-relativistic many-body fluid theory. The critical distinction between these types of theories, is that the non-relativistic (NR) theories require particle conservation whereas the relativistic (R) theories require the ability to create and annihilate particles. I'll discuss this point and others in a later subsection.

2 Relativistic quantum field theory – example

- We'll start with the action S defined as a function of the complex field ψ.

$$S[\psi] = \int d^4x \mathcal{L}(\psi)(x) \tag{3}$$

where

$$\mathcal{L}(\psi) = \partial^\mu \psi^\dagger \partial_\mu \psi - m^2 \psi^\dagger \psi - g(\psi^\dagger \psi)^2. \tag{4}$$

The Action Principle states that we look for solution fields $\psi_S(x)$ so that S is an extremum, $\delta S[\psi_S] = 0$. These solution fields can be shown to satisfy the Euler-Lagrange equation

$$\left(\partial_\mu \partial^\mu + m^2 + 2g\psi_S^\dagger \psi_S\right) \psi_S = 0 \tag{5}$$

and its complex conjugate.

- Notice that if we define $\tilde{\psi}(x) = \psi(\Lambda x)$, where Λ is a Lorentz transformation matrix operating on the 4-vector x, then $S[\tilde{\psi}] = S[\psi]$. This expresses symmetry under Lorentz transformations (i.e., the theory is Lorentz invariant).

- The Hamiltonian for this theory can be obtained in the usual way from the Lagrangian[1]

$$\mathcal{H} = \partial_0\psi^\dagger(x)\partial_0\psi(x) + \boldsymbol{\nabla}\psi^\dagger(x)\boldsymbol{\nabla}\psi(x) + m^2\psi^\dagger(x)\psi(x) + g\left(\psi^\dagger(x)\psi(x)\right)^2 \tag{8}$$

The time t can be set to any value since H is time-independent. We've arbitrarily set $t = 0$. The free Hamiltonian H_0 is

$$H_0 = \int d^3x \left(\partial_0\psi^\dagger(x)\partial_0\psi(x) + \boldsymbol{\nabla}\psi^\dagger(x)\boldsymbol{\nabla}\psi(x) + m^2\psi^\dagger(x)\psi(x)\right), \tag{9}$$

and the interaction Hamiltonian $H_I$is

$$H_I = \int d^3xg\left(\psi^\dagger(x)\psi(x)\right)^2. \tag{10}$$

- To quantize the theory, we now declare ψ to be an operator. The (free, $g = 0$) E-L equation has a general solution

$$\psi_S(t, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}(2E_{\mathbf{p}})^{\frac{1}{2}}}\left(a_{\mathbf{p}}e^{-i(E_{\mathbf{p}}t-\mathbf{p}\cdot\mathbf{x})} + b_{\mathbf{p}}^\dagger e^{i(E_{\mathbf{p}}t-\mathbf{p}\cdot\mathbf{x})}\right) \tag{11}$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ and $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ are operators on the Hilbert space. This expression is often called the mode expansion, where the $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ are the mode operators. The momentum is $\dot{\phi}$, so the only nontrivial CCR is $[\psi(t, \mathbf{x}), \dot{\psi}(t, \mathbf{x}')] = i\delta^3(\mathbf{x} - \mathbf{x}')$. This can be applied to the above solution of the E-L equation.

[1]The momentum is defined by

$$\mathbf{\Pi}^0(x') \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\mathbf{\Psi})}(x') = \left(\partial^0\psi^\dagger(x'), \partial^0\psi(x')\right) \tag{6}$$

where the generalized coordinates and momenta $\mathbf{\Psi}$ and $\mathbf{\Pi}^0$ are defined by

$$\begin{aligned} \mathbf{\Psi}(x) &= \left(\psi(x), \psi^\dagger(x)\right) \\ \mathbf{\Pi}^0(x') &\equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\mathbf{\Psi})}(x') = \left(\partial^0\psi^\dagger(x'), \partial^0\psi(x')\right) \end{aligned} \tag{7}$$

and the Hamiltonian is related to the Lagrangian by $H = \int d^3x \left(\mathbf{\Pi}^0 \cdot \dot{\mathbf{\Psi}} - \mathcal{L}\right)$.

- By substituting the mode expansion into the CCR, we find that $[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] = [b_{\mathbf{p}}, b^\dagger_{\mathbf{p}'}] = \delta^3(\mathbf{p}-\mathbf{p}')$ and other equal-time commutators are 0. These are the defining equations for ladder operators. One consequence of this, is that we can construct our Hilbert space as follows:
  - Assume that there is a state $|0\rangle$ with the property that $a_{\mathbf{p}}|0\rangle = b_{\mathbf{p}}|0\rangle = 0$ for all momenta $\mathbf{p}$. This state is called *the vacuum*.
  - Then we can construct the normalized state $|n^{(a)}_{\mathbf{p}}\rangle = \frac{(a^\dagger_{\mathbf{p}})^{n^{(a)}_{\mathbf{p}}}}{\sqrt{n^{(a)}_{\mathbf{p}}!}}|0\rangle$ which has the property $a^\dagger_{\mathbf{p}} a_{\mathbf{p}} |n^{(a)}_{\mathbf{p}}\rangle = n^{(a)}_{\mathbf{p}}|n^{(a)}_{\mathbf{p}}\rangle$ and the normalized state $|n^{(b)}_{\mathbf{p}}\rangle = \frac{(a^\dagger_{\mathbf{p}})^{n^{(b)}_{\mathbf{p}}}}{\sqrt{n^{(b)}_{\mathbf{p}}!}}|0\rangle$ which has the property $a^\dagger_{\mathbf{p}} a_{\mathbf{p}} |n^{(b)}_{\mathbf{p}}\rangle = n^{(b)}_{\mathbf{p}}|n^{(b)}_{\mathbf{p}}\rangle$.
  - Our Hilbert space is the Fock space constructed from the basis set $|n^{(a)}_{\mathbf{p}_1}\rangle \otimes ... \otimes |n^{(a)}_{\mathbf{p}_m}\rangle \otimes |n^{(b)}_{\mathbf{p}'_1}\rangle \otimes ... \otimes |n^{(b)}_{\mathbf{p}'_m}\rangle$.
- The Fock space basis is an occupation number basis with $|n^{(a)}_{\mathbf{p}_1}, ..., n^{(a)}_{\mathbf{p}_m}, n^{(b)}_{\mathbf{p}'_1}, ..., n^{(b}_{\mathbf{p}'_m}\rangle \equiv |n^{(a)}_{\mathbf{p}_1}\rangle \otimes ... \otimes |n^{(a)}_{\mathbf{p}_m}\rangle \otimes |n^{(b)}_{\mathbf{p}'_1}\rangle \otimes ... \otimes |n^{(b)}_{\mathbf{p}'_m}\rangle$. If we substitute the mode expansion for the fields that appear in the expression Eq. (9) for the free Hamiltonian, we find that

$$H_0|n^{(a)}_{\mathbf{p}_1}, ..., n^{(a)}_{\mathbf{p}_m}, n^{(b)}_{\mathbf{p}'_1}, ..., n^{(b}_{\mathbf{p}'_m}\rangle = \left(\sum_{i=1}^{m}\left(n^{(a)}_{\mathbf{p}_i}E_{\mathbf{p}_i} + n^{(b)}_{\mathbf{p}'_i}E_{\mathbf{p}'_i}\right) + \text{constant}\right)|n^{(a)}_{\mathbf{p}_1}, ..., n^{(a)}_{\mathbf{p}_m}, n^{(b)}_{\mathbf{p}'_1}, ..., n^{(b)}_{\mathbf{p}'_m}\rangle. \tag{12}$$

3 Non-relativistic many-body theory of fluids

The idea is to cast NRMB (*non-relativistic many-body* theory) into the same formalism as quantum field theory. Here is a (hopefully) simple argument for why this can be done:

- We saw in the QFT discussion, that the Lagrangian formulation of quantum field theory leads directly to a Fock space with a momentum-state occupation-number representation that can be implemented with ladder operators.

- Conversely, if we start with such a representation and implementation, we can construct a Lagrangian quantum field theory for it.
- The resulting theory has the same free part as QFT. The interactive part is obtained from the interaction-Hamiltonian, noting that $\mathcal{L}_I = -\mathcal{H}_I$.[2]
- As for the question of non-relativistic versus relativistic, we can approach this in several ways but in principle it's simply a matter of taking the non-relativistic limit of the QFT theory.

3.1 Non-relativistic limit of QFT

I'll begin with the last bullet above by first discussing the non-relativistic limit of the free QFT theory discussed in the previous section.

We take the Lagrangian of Eq. (4), set $g = 0$ and reinstate the unit c.

$$\mathcal{L}(\psi) = \partial_0\psi^\dagger\partial_0\psi - c^2\boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi - m^2c^4\psi^\dagger\psi. \tag{13}$$

Then redefine the field in terms of a new field Ψ.

$$\psi(\mathbf{x}, t) = \frac{1}{\sqrt{2mc^2}}\Psi(x)e^{-imc^2}. \tag{14}$$

The transformed Lagrangian becomes[3]

$$\mathcal{L}(\Psi) = \frac{1}{2mc^2}\partial_0\Psi^\dagger\partial_0\Psi + i\Psi^\dagger\partial_0\Psi - \frac{1}{2m}\boldsymbol{\nabla}\Psi^\dagger \cdot \boldsymbol{\nabla}\Psi. \tag{15}$$

In the non-relativistic limit we can drop the first term because it is suppressed by a factor of c^2, thus leaving the non-relativistic Lagrangian as

$$\mathcal{L}_{NR}(\Psi) = i\Psi^\dagger\partial_0\Psi - \frac{1}{2m}\boldsymbol{\nabla}\Psi^\dagger \cdot \boldsymbol{\nabla}\Psi. \tag{16}$$

A general solution Ψ_{free} of the Euler Lagrange equation for this Lagrangian is the mode expansion

$$\Psi_{\text{free}}(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} a_{\mathbf{p}} e^{-i(Et-\mathbf{p}\cdot\mathbf{x})}, \tag{17}$$

[2] Strictly speaking, this equation is true only when the interaction Hamiltonian doesn't involve time-derivatives.

[3] See Lancaster Equation (12.22). Note that the action-term $\int d^3x \frac{-i}{2}\partial_0\Psi^\dagger\Psi$ can be integrated by parts (assuming the fields drop off quickly to 0 at infinity) to give $\int d^3x \frac{i}{2}\Psi^\dagger\partial_0\Psi$. As a result, the term in the Lagrangian proportional to i becomes $i\Psi^\dagger\partial_0\Psi$.

where $E = p_0 = \mathbf{p}^2/2m$, and $a_{\mathbf{p}}$ is time-independent. Note that this solution does not require the $b_{\mathbf{p}}$ modes. **All of this analysis can be applied when the Lagrangian has an interaction term, but there is a different time-dependence in the mode expansion. We can absorb all time-dependence into the mode coefficients $a_{\mathbf{p}}$. In expressions for the full Hamiltonian – which is time-independent – the mode time-dependence can be suppressed by arbitrarily setting $t = 0$.**

We now proceed to canonical quantization following the same procedure we used earlier. The generalized coordinates and momenta can be inferred from the non-relativistic Lagrangian as

$$\begin{aligned} \mathbf{\Psi}(x) &= \big(\Psi(x), \Psi^\dagger(x)\big) \\ \mathbf{\Pi}^0(x') &\equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \mathbf{\Psi}\right)}(x') = \big(i\Psi^\dagger(x'), 0\big)\,, \end{aligned} \tag{18}$$

leading to the canonical commutation relations

$$[\Psi(t,\mathbf{x}), i\Psi^\dagger(t,\mathbf{x}')] = i\delta^3(\mathbf{x}-\mathbf{x}'). \tag{19}$$

If we promote the coefficients $a_{\mathbf{p}}$ to operators and substitute the mode expansion into

$$[\Psi_{\text{free}}(t,\mathbf{x}), i\Psi^\dagger_{\text{free}}(t,\mathbf{x}')] = i\delta^3(\mathbf{x}-\mathbf{x}'), \tag{20}$$

we obtain

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] = \delta(\mathbf{p}-\mathbf{p}'). \tag{21}$$

If we follow the usual procedure for obtaining the Hamiltonian from the Lagrangian, we find that in the non-relativistic limit

$$\begin{aligned} H_{\text{NR}} &= \int d^3x \frac{1}{2m} \boldsymbol{\nabla}\Psi^\dagger \cdot \boldsymbol{\nabla}\Psi \\ &= \int d^3x \frac{d^3p d^3p'}{(2\pi)^3} \frac{\mathbf{p}\cdot\mathbf{p}'}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}'} e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \\ &= \int d^3p \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}. \end{aligned} \tag{22}$$

thus demonstrating that $a_{\mathbf{p}}$ are annihilation operators for particles of kinetic energy $\frac{\mathbf{p}^2}{2m}$.

3.2 Example: one-body forces

I'll illustrate the procedure for creating a field theory – in Lagrangian form – from a simple many-body theory, with an example where I begin by focusing on the interaction Hamiltonian. An excellent treatment of this is given in Lancaster 4.2.

The example will be a fluid whose interaction Hamiltonian is given by

$$H_I = \sum_{a=1}^{N} h_a \tag{23}$$

where the sum is over particles, and h_a is a single-particle Hamiltonian. We take N to be very large. Furthermore, h_a acts on the Hilbert space of a single particle, with momentum basis $|\mathbf{k}_i\rangle$, so that $\langle \mathbf{k}_i | h_a | \mathbf{k}_j \rangle$ is independent of the index a. This form of the interaction Hamiltonian assures that the Hamiltonian maintains the indistinguishability (e.g. symmetry) of particles.

Consider for example, single-particle momentum basis functions $\psi_{\mathbf{k}_1}(\mathbf{x}), \psi_{\mathbf{k}_2}(\mathbf{x}), ...$, which we can also write as $\langle \mathbf{x}|\mathbf{k}_1\rangle, \langle \mathbf{x}|\mathbf{k}_2\rangle, ...$ The multi-particle system has wavefunction $\psi\left(\mathbf{x}_1, \mathbf{x}_2, ...\right) = \psi_{\mathbf{k}_1}(\mathbf{x}_1)\psi_{\mathbf{k}_4}(\mathbf{x}_2)... + \psi_{\mathbf{k}_4}(\mathbf{x}_1)\psi_{\mathbf{k}_1}(\mathbf{x}_2)... + ...$ The sum is over all permutations of the single-particle states comprising the multi-particle wavefunction. Then let h_a act on a single-particle state as $V(\mathbf{x})\psi(\mathbf{x})$.

$$\begin{aligned} h_1\psi &= \left(V(\mathbf{x}_1)\psi_{\mathbf{k}_1}(\mathbf{x}_1)\right)\psi_{\mathbf{k}_4}(\mathbf{x}_2)... + \left(V(\mathbf{x}_1)\psi_{\mathbf{k}_4}(\mathbf{x}_1)\right)\psi_{\mathbf{k}_1}(\mathbf{x}_2)... \\ h_2\psi &= \psi_{\mathbf{k}_1}(\mathbf{x}_1)\left(V(\mathbf{x}_2)\psi_{\mathbf{k}_4}(\mathbf{x}_2)\right)... + \psi_{\mathbf{k}_5}(\mathbf{x}_1)\left(V(\mathbf{x}_2)\psi_{\mathbf{k}_1}(\mathbf{x}_2)\right)... \end{aligned} \tag{24}$$

etc. From this we immediately see that $H_I|\psi\rangle = \left(\sum_a V(\mathbf{x}_a)\right)|\psi\rangle$.

We begin our analysis by writing H_I using the momentum-state annihilation and creation operators which we define as the operators that create momentum states from the vacuum, namely $a_{\mathbf{k}}|0\rangle = |\mathbf{k}\rangle$. Following Lancaster Chapter 4.1, we can show that

$$H_I = \int d^3k_1 d^3k_2 V_{\mathbf{k}_1\mathbf{k}_2} a^\dagger_{\mathbf{k}_1} a_{\mathbf{k}_2}. \tag{25}$$

where

$$V_{\mathbf{k}'\mathbf{k}} = \frac{1}{(2\pi)^3}\int d^3x V(\mathbf{x}) e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}. \tag{26}$$

We also have, as usual, the free part of the Hamiltonian in the non-relativistic continuum limit, as

$$H_0 = \int d^3k \frac{\mathbf{k}^2}{2m} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}. \tag{27}$$

Setting $V_{\mathbf{k}_1\mathbf{k}_2} \to \tilde{V}(\mathbf{k}_1 - \mathbf{k}_2)$, and then changing the integration variables, the full Hamiltonian becomes

$$H = \int d^3p \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \int d^3p d^3q a^\dagger_{\mathbf{p}} a_{\mathbf{q}} \tilde{V}(\mathbf{p} - \mathbf{q}) \tag{28}$$

with

$$\tilde{V}(\mathbf{p} - \mathbf{q}) = \frac{1}{(2\pi)^3} \int d^3x V(\mathbf{x}) e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}. \tag{29}$$

We can use the mode expansion of Eq. (17) to show that the second term of the Hamiltonian is equivalent to

$$\int d^3x V(x) \Psi^\dagger(x) \Psi(x). \tag{30}$$

Note that we have replaced Ψ_{free} with Ψ because our theory is no longer free. See the discussion in the last section explaining that the mode expansion is still valid, and that when the mode operators (which, in the interacting theory, are time-dependent) appear in the full Hamiltonian, they can be taken at $t = 0$.

The above Hamiltonian consists of a free part which is the same as we obtained in Eq. (22) and which is derived from the non-relativistic free Lagrangian of Eq. (16) and an interactive part which is $-\int d^3x \mathcal{L}_{\mathcal{I}}$. We've therefore shown that our one-body-force many-body theory is obtained from the Lagrangian

$$\mathcal{L}(\Psi) = i\Psi^\dagger \partial_0 \Psi - \frac{1}{2m} \boldsymbol{\nabla}\Psi^\dagger \cdot \boldsymbol{\nabla}\Psi - V(x)\Psi^\dagger(x)\Psi(x). \tag{31}$$

3.3 Example: two-body forces

This section follows Lancaster chapter 4.4. The one-body example is unable to capture the effects of two-body interactions such as Coulomb interactions. Let the general two-body interaction be

$$H_I^{(2)} = \sum_{a<b}^{N} h_{ab}. \tag{32}$$

Following Lancaster, we can show that the general form of two-body interaction Hamiltonian is

$$H_I^{(2)} = \sum_{\alpha\beta\gamma\delta} \left(H_I^{(2)}\right)_{\alpha\beta\gamma\delta} a^\dagger_\alpha a^\dagger_\beta a_\gamma a_\delta. \tag{33}$$

An example two-particle operator acting on a two-particle state, would be $V(\mathbf{x}-\mathbf{y})\,(\psi_1(\mathbf{x})\psi_2(\mathbf{y})+\psi_1(\mathbf{y})\psi_2(\mathbf{x}))$. Lancaster shows for this case (in the continuum limit), that

$$H_V^{(2)} = \frac{1}{2}\int d^3p_1 d^3p_2 d^3q \tilde{V}(\mathbf{q}) a^\dagger_{\mathbf{p}_1+\mathbf{q}} a^\dagger_{\mathbf{p}_2-\mathbf{q}} a_{\mathbf{p}_2} a_{\mathbf{p}_1}, \tag{34}$$

where $\tilde{V}$ is defined by Eq. (29). Note that this can be described by the schematic

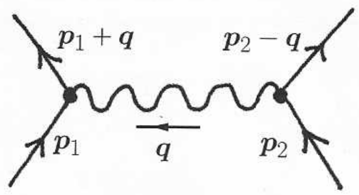

Fig. 4.8 A Feynman diagram for the process described in the text. Interpreting this diagram, one can think of time increasing in the vertical direction.

Through an argument similar to what we obtained starting with Eq. (**??**) the above theory can be obtained from a (non-relativistic) action

$$S(\Psi) = \int d^3x \left[i\Psi^\dagger \partial_0 \Psi - \frac{1}{2m} \boldsymbol{\nabla}\Psi^\dagger \cdot \boldsymbol{\nabla}\Psi \right] - \frac{1}{2}\int d^3x d^3y \Psi^\dagger(\mathbf{x})\Psi^\dagger(\mathbf{y}) V(\mathbf{x}-\mathbf{y}) \Psi(\mathbf{y})\Psi(\mathbf{x})). \tag{35}$$

4 Observations

- The usual implementation of the Principles in Section 1, involves an infinite number of particles. This is a natural outcome of theories whose boundaries are infinite or theories with periodic boundary conditions. Physical many-body theories have a large but finite number of particles but most results are insensitive to the finiteness or boundary effects.
- The occupation-number quantization construction both for NR and R theories, depends on the mathematical trick of ladder operators (from the CCR) combined with the solution of simple differential equations (the E-L equations).

- The ladder operators are useful for finding the spectrum of harmonic oscillators **but our many-body constructions have nothing to do with harmonic oscillators!**

- The occupation-number representation is required **because particles are indistinguishable**.

- For this kind of field theory formalism, the occupation-number representation is a *natural* outcome and therefore field theory is a *natural* way of describing the theory of many indistinguishable particles.

- In nature, what distinguishes relativistic and non-relativistic theories, is simply the typical speeds and energies of the systems being described. The non-relativistic theory is just the low-speed approximation of the relativistic theory.

- In non-relativistic theories, the energies are insufficient to allow the creation of massive particles. Thus the low-energy approximation for a theory of massive particles, will automatically lead to (heavy) particle conservation.