Superfluids Part I

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These notes follow Chapter 42 of Lancaster. Another reference I've found, which provides far more context and details, is `https://www-thphys.physics.ox.ac.uk/people/SteveSimon/QCM2020/QuantumMatterMostRecent.pdf`.

1 The superfluid Hamiltonian

From Lancaster Chapter 4 and my previous notes, we had the following action for a general non-relativistic bosonic many-body theory with only 2-body interactions.[1]

$$S(\phi) = \int d^3x \left[i\phi^\dagger \partial_0 \phi - \frac{1}{2m} \boldsymbol{\nabla}\phi^\dagger \cdot \boldsymbol{\nabla}\phi \right] - \frac{1}{2} \int d^3x d^3y \phi^\dagger(\mathbf{x}) \phi^\dagger(\mathbf{y}) V(\mathbf{x}-\mathbf{y}) \phi(\mathbf{y}) \phi(\mathbf{x}). \tag{1}$$

From this, we can derive the Hamiltonian

$$H = \frac{1}{2m} \int d^3x \boldsymbol{\nabla}\phi^\dagger \cdot \boldsymbol{\nabla}\phi + \frac{1}{2} \int d^3x d^3y \phi^\dagger(\mathbf{x}) \phi^\dagger(\mathbf{y}) V(\mathbf{x}-\mathbf{y}) \phi(\mathbf{y}) \phi(\mathbf{x}). \tag{2}$$

Our simplifying assumption, which we'll use for examining superfluids, is that the molecules aren't affected by any "background" potential (hence no 1-body interactions) and that the 2-body interaction is point-like – namely, $V(\mathbf{z}) = g\delta^{(3)}(\mathbf{z})$ where g is a coupling constant.

[1] In my previous notes I used the complex field ψ but for consistency with Lancaster Eq. (42.1) I have substituted the complex field ϕ

Then

$$H = \frac{1}{2m}\int d^3x \boldsymbol{\nabla}\phi^\dagger \cdot \boldsymbol{\nabla}\phi + \frac{g}{2}\int d^3x d^3y \phi^\dagger(\mathbf{x})\phi^\dagger(\mathbf{y})\phi(\mathbf{y})\phi(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}). \quad (3)$$

(Recall that $\tilde{V}(\mathbf{p}) = \frac{1}{(2\pi)^3}\int d^3x V(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{x}}$, so by plugging in the delta-function we get $\tilde{V}(\mathbf{p}) = \frac{g}{(2\pi)^3}$, which is independent of momentum.)

As previously shown (Lancaster 4.2 and my notes), this expression can be cast into a form using annihilation and creation operators, as

$$\begin{aligned} H &= \int d^3p \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{1}{2}\int d^3p_1 d^3p_2 d^3q \tilde{V}(\mathbf{q}) a^\dagger_{\mathbf{p}_1+\mathbf{q}} a^\dagger_{\mathbf{p}_2-\mathbf{q}} a_{\mathbf{p}_2} a_{\mathbf{p}_1} \\ &= \int d^3p \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{g}{2}\frac{1}{(2\pi)^3}\int d^3p_1 d^3p_2 d^3q a^\dagger_{\mathbf{p}_1+\mathbf{q}} a^\dagger_{\mathbf{p}_2-\mathbf{q}} a_{\mathbf{p}_2} a_{\mathbf{p}_1} \quad (4) \\ &= \int d^3p \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{g}{2}\frac{1}{(2\pi)^3}\int d^3k d^3p d^3q a^\dagger_{\mathbf{k}+\mathbf{q}} a^\dagger_{\mathbf{p}-\mathbf{q}} a_{\mathbf{p}} a_{\mathbf{k}}. \end{aligned}$$

As usual, we put the system in a box of volume $\mathcal{V}$, so that the momenta become discrete, and we get

$$H \approx \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{g}{2\mathcal{V}}\sum_{\mathbf{kpq}} a^\dagger_{\mathbf{k}+\mathbf{q}} a^\dagger_{\mathbf{p}-\mathbf{q}} a_{\mathbf{p}} a_{\mathbf{k}}, \quad (5)$$

in agreement with Lancaster Eq (42.1).

One might wonder about the normal ordering. Why aren't there terms of the form, for example, $\frac{g'}{2\mathcal{V}}\sum_{\mathbf{kpq}} a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{p}} a^\dagger_{\mathbf{p}-\mathbf{q}} a_{\mathbf{k}}$? Consider this term and apply the canonical commutation relations to the middle two factors in each summand – namely $a_{\mathbf{p}} a^\dagger_{\mathbf{p}-\mathbf{q}} = a^\dagger_{\mathbf{p}-\mathbf{q}} a_{\mathbf{p}} + \delta^{(3)}(\mathbf{q})$. We end up with (noting that H' refers to the Hamiltonian whose second term above isn't normal-ordered)

$$H' \approx \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{g'}{2\mathcal{V}}\sum_{\mathbf{kpq}} a^\dagger_{\mathbf{k}+\mathbf{q}} a^\dagger_{\mathbf{p}-\mathbf{q}} a_{\mathbf{p}} a_{\mathbf{k}} + \frac{g'}{2\mathcal{V}}\sum_{\mathbf{kp}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \quad (6)$$

The last term is a double sum. The sum over $\mathbf{k}$ is simply the total number operator and for a system with N particles, its value is N. So the last term ends up as $\sum_{\mathbf{p}} \left(\frac{g'N}{2\mathcal{V}}\right)$. This is a constant and can be ignored (even though it's an infinite constant).

Note that the usual order of things, is first the derivation of the Hamiltonian in terms of the annihilation and creation operators, and then the field description embodied in the action.

2 The dispersion relation and why we care

Once we have the Hamiltonian, we need to find its eigenstates and eigenvalues. Consider a Hamiltonian with eigenstates $|\mathbf{p}\rangle$ and energy-eigenvalues $E(\mathbf{p})$. If the energies are independent of the direction of $\mathbf{p}$, we write $E(p)$ and this function is known as a dispersion relation.[2] Here is an example of a dispersion relation which happens to correspond to Bogoliubov's approximation of our superfluid Hamiltonian.

$$E(p) = \sqrt{\frac{\mathbf{p}^2}{2m}\left(\frac{\mathbf{p}^2}{2m} + \frac{2Ng}{\mathcal{V}}\right)}. \tag{7}$$

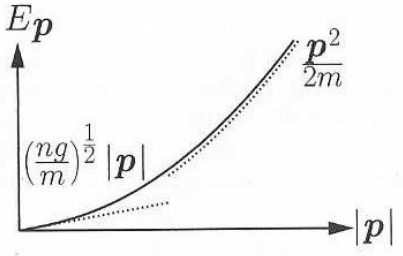
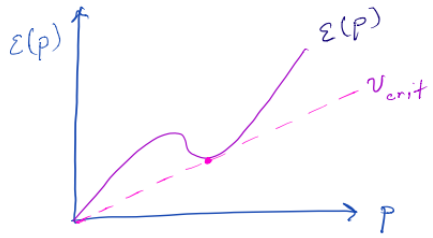


Fig. 42.2 The dispersion predicted by Bogoliubov's model. At low momentum the energy is linear in $|\boldsymbol{p}|$, at large momentum it is quadratic.

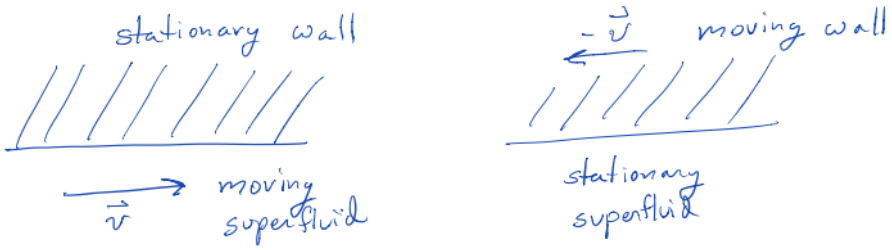
In this figure, the minimum slope of the curve (also known as the *critical velocity*) occurs at the origin and has the value $\sqrt{\frac{ng}{m}}$.

For a short-range potential $\mathcal{V}(\mathbf{k})$) which is more general than the delta-function, we can get a more general dispersion curve, for example

[2]Clearly there is some ambiguity in this definition, since I've said nothing about how one selects the variables used for identifying the energy eigenstates. Implicitly, these values are momenta or even more specifically, they are eigenvalues of momentum operators. It would be important to verify that those momentum operators implied in the dispersion relation, have the properties required for the discussion of the Landau criterion.



Landau proposed a criterion for superfluidity, based on the characteristics of the dispersion relation (I'm now going to quote liberally from `https://www-thphys.physics.ox.ac.uk/people/SteveSimon/QCM2020/QuantumMatterMostRecent.pdf`). We imagine flowing the superfluid past a stationary wall at velocity $\mathbf{v}$ as on the left of the figure below, and we would like to ask whether energy can be dissipated.



Another way to ask this is to ask whether quasiparticles can be excited. I believe that the complete argument goes like this: We want to know if a fluid can move by a wall without friction (more precisely, "viscosity"). In general, if the kinetic energy of fluid motion can be converted to heat energy – which we can think of as internal energy or particle-randomized kinetic energy (think of Brownian motion) – then the fluid will slow down. In other words, if the kinetic energy of the fluid can decrease by conversion to internal energy, then the fluid will slow down.

The argument is due to Landau and leads to a criterion known as Landau's criterion. Here I will follow Lancaster Example 42.5. A somewhat more traditional explanation – and more detailed – is given in the appendix to these notes.

Consider the fluid to be an object of mass M, initially flowing with a velocity $\mathbf{v}$ past a wall. The fluid has an initial energy E_0. After exciting a quasi-particle from its ground state, it can end up with a lower energy provided that the extra energy is transferred to the wall (as required by conservation of energy). Since the wall isn't moving, the energy would have to be transferred as heat – i.e., "the process would dissipate energy in the form of heat." That's all perfectly OK. Can the fluid end up with a larger energy? By the same kind of logic, this would require heat energy to be transferred from the wall and that's not OK.[3]

So the question becomes, "is the final fluid energy lower or higher than the initial fluid energy?" Assume the quasi-particle is excited from 0 relative-momentum (relative to the fluid motion) to a relative momentum $\mathbf{p}$. Then the fluid's velocity must change to $\mathbf{v}_f$ in order to conserve momentum, with

$$M\mathbf{v} = M\mathbf{v}_f + \mathbf{p}. \tag{8}$$

The internal energy due to the quasi-particle excitation is $E(p)$. Thus the final energy is $\frac{Mv_f^2}{2} + E(p) \approx \frac{Mv^2}{2} - \mathbf{v}\cdot\mathbf{p}$. We compare this to the initial energy $\frac{Mv^2}{2}$ and see that the final energy is less than the initial energy only if $E(p) - \mathbf{v}\cdot\mathbf{p} < 0$. In that case, the process can occur along with heat dissipation (and therefore a reduction in fluid speed). Otherwise the process can't occur and fluid velocity doesn't change.

Landau's criterion for superfluid flow is $|\mathbf{v}| < v_{\text{crit}}$ where $v_{\text{crit}} = \min_p \frac{\epsilon(p)}{p}$. See the dispersion graph above and note that $v_{\text{crit}} \neq 0$.

Suppose that the dispersion curve was the standard one for a massive nonrelativistic particle $\epsilon_m = \frac{p^2}{2m}$. Then $\frac{\epsilon_m(p)}{p} = \frac{p}{2m}$. When we take the minimum over all p, we obtain 0, therefore in that case, $v_{\text{crit}} = 0$ and there is no fluid flow. We see that the Bogoliubov dispersion relation Eq. (7) has a low-momentum limit of $\lim_{p\to 0} \frac{\epsilon_{\text{bog}}}{p} = \sqrt{\frac{Ng}{m\mathcal{V}}}$. However, we can see from the dispersion curve and the point on the curve that gives us v_{crit}, that the situation is a bit more complicated. Lancaster refers to these quasiparticles as Bogolons.

[3] I'm a bit fuzzy on the details of the logic here. I think the correct argument is that heat can't flow from colder to hotter, and that this follows from thermodynamics. But I'm not entirely sure that I can fill in all the dots. I like to relate these arguments back to statistical mechanics, and I haven't attempted to do that analysis – nor have I found a reference that appears to approach things just the way I want.

3 Wrestling the Hamiltonian into submission

Quartic terms are nasty. Bogoliubov proposed an approximation to Eq. (5), which leads to a quadratic Hamiltonian

$$H = \sum_{\mathbf{p}\neq 0} \left(\frac{\mathbf{p}^2}{2m} + ng \right) a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{1}{2} \sum_{\mathbf{p}\neq 0} ng \left(a^\dagger_{\mathbf{p}} a^\dagger_{\text{-}\mathbf{p}} + a_{\mathbf{p}} a_{\text{-}\mathbf{p}} \right), \tag{9}$$

where $n = \frac{N}{\mathcal{V}}$ is the number-density. This Hamiltonian can be diagonalized.

3.1 The Bogoliubov Hamiltonian

We now derive Eq. (9) starting from Eq. (5). Lancaster refers to this result as Bogoliubov's hunting license. Our derivation follows Lancaster 42.1, but also draws on Landau and Lifshitz Volume 9 section 25 which, in my opinion, is slightly clearer.

Since our experiments will be done at very low temperature, we can assume that they will only probe very low energy states – that is, states whose energy eigenvalues are close to the minimum. Recall that the lowest-energy state is called the ground state. We represent that state as $|\Omega\rangle$. Intuitively, since the lowest-energy free particles have 0 momentum, we assume that the ground state consists of N_0 particles of momentum $\mathbf{p} = 0$, where N_0 is a very large number (e.g. 10^{23}). [4]

We now wave our arms a bit, saying the magic words "all this is approximately true in first-order perturbation theory" – in other words, pretend for a moment that interactions are turned off. In that case, $|\Omega\rangle$ (the symmetrized N_0-body ground state) has the ladder-operator property that

$$a_{\mathbf{p}=0}|\Omega\rangle = \sqrt{N_0}|N_0 - 1\rangle. \tag{10}$$

Next, Bogoliubov says "the state $|N_0 - 1\rangle$ has almost as many particles as

[4] Strictly speaking, in an interacting theory, the ladder operator $a^\dagger_{\mathbf{p}}$ doesn't precisely increase the number of $\mathbf{p}$-momentum particles by 1. Rather, it creates a new basis state in Fock space. In first-order perturbation theory (the Born approximation), the energy eigenstates are approximately the free many-body states so our ground-state is only approximately described as N_0 0-momentum particles.

the state $|\Omega\rangle$ (e.g. compare $10^{23}-1$ to 10^{23}), so set $|N_0-1\rangle \approx |\Omega\rangle$", [5] thus

$$a_{\mathbf{p}=0}|\Omega\rangle \approx \sqrt{N_0}|\Omega\rangle. \tag{11}$$

Similarly,

$$a^\dagger_{\mathbf{p}=0}|\Omega\rangle \approx \sqrt{N_0}|\Omega\rangle. \tag{12}$$

Now recall that all states (in Fock space) can be generated by a sequence of creation operations acting on the ground state, so a general state can be written as

$$|s\rangle = \sum_{m=1}^{\infty} \sum_{n_1=0,\ldots,n_m=0}^{\infty} \left(a^\dagger_{\mathbf{p}_{n_1}}\right)^{n_1} \ldots \left(a^\dagger_{\mathbf{p}_{n_m}}\right)^{n_m} |\Omega\rangle. \tag{13}$$

Furthermore, since we are dealing with an N-body system, the above expression is constrained so that $\sum_{i=1}^{m} n_i = N$. If we act on this state with either the operator $a_{\mathbf{p}=0}$ or $a^\dagger_{\mathbf{p}=0}$, then because of the commutation relations, that operator will commute with all of the other creation operators that act on the vacuum, resulting for example in

$$\begin{aligned} a_{\mathbf{p}=0}|s\rangle &= a_{\mathbf{p}=0}\left(\sum_{m=1}^{\infty} \sum_{n_1=0,\ldots,n_m=0}^{\infty} \left(a^\dagger_{\mathbf{p}_{n_1}}\right)^{n_1} \ldots \left(a^\dagger_{\mathbf{p}_{n_m}}\right)^{n_m} |\Omega\rangle\right) \\ &= \left(\sum_{m=1}^{\infty} \sum_{n_1=0,\ldots,n_m=0}^{\infty} \left(a^\dagger_{\mathbf{p}_{n_1}}\right)^{n_1} \ldots \left(a^\dagger_{\mathbf{p}_{n_m}}\right)^{n_m}\right) a_{\mathbf{p}=0}|\Omega\rangle \\ &= \sqrt{N_0}\left(\sum_{m=1}^{\infty} \sum_{n_1=0,\ldots,n_m=0}^{\infty} \left(a^\dagger_{\mathbf{p}_{n_1}}\right)^{n_1} \ldots \left(a^\dagger_{\mathbf{p}_{n_m}}\right)^{n_m} |\Omega\rangle\right) \\ &= \sqrt{N_0}|s\rangle. \end{aligned} \tag{14}$$

Similarly,

$$a^\dagger_{\mathbf{p}=0}|s\rangle = \sqrt{N_0}|s\rangle. \tag{15}$$

Since these equations hold for every state, we simplify by saying that

$$a_{\mathbf{p}=0} = a^\dagger_{\mathbf{p}=0} = \sqrt{N_0}. \tag{16}$$

What is the relationship between N and N_0? [6] In general, because of the constraint $\sum_{i=1}^{m} n_i = N$, we know that when we operate on a state with the

[5] Frankly, other approximations would seem to me to be just as valid and would potentially lead to other results.

[6] The following paragraph follows the standard argument both in Lancaster and also, for example, in Landau and Lifshitz. However, approximations are being made for which

number operator $\hat{N}$ we'll get $\hat{N}|s\rangle = N|s\rangle$. We simplify this notation by saying $\hat{N} = N$. Since the number operator $\hat{N}$ is given by $\hat{N} = \sum_{\mathbf{p}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$ we then obtain

$$\begin{aligned} N = \hat{N} &= \sum_{\mathbf{p}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \\ &= a^\dagger_{\mathbf{p}=0} a_{\mathbf{p}=0} + \sum_{\mathbf{p}\neq 0} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \\ &\approx N_0 + \sum_{\mathbf{p}\neq 0} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}. \end{aligned} \tag{17}$$

Let's return to Eq. (5).

$$H \approx \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{g}{2\mathcal{V}} \sum_{\mathbf{kpq}} a^\dagger_{\mathbf{k+q}} a^\dagger_{\mathbf{p-q}} a_{\mathbf{p}} a_{\mathbf{k}}. \tag{18}$$

Our goal is to change the quartic term into a quadratic term, by using the Bogoliubov hunting license. We will expand the interaction part (proportional to g) of H in powers of N. The ladder operators, for $\mathbf{p} \neq 0$ are of order 1.

The leading term of order N^2 comes from the quartic sum, when $\mathbf{q}{+}\mathbf{k} = \mathbf{p}\text{-}\mathbf{q} = \mathbf{p} = \mathbf{k} = 0$. This contribution gives us $\frac{g}{2\mathcal{V}} a^\dagger_{\mathbf{0}} a^\dagger_{\mathbf{0}} a_{\mathbf{0}} a_{\mathbf{0}} \approx \frac{g}{2\mathcal{V}} N_0^2 \approx \frac{g}{2\mathcal{V}} N^2 - N \frac{g}{\mathcal{V}} \sum_{\mathbf{p}\neq 0} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$ where we have invoked Eq. (17).[6] Notice that the contribution from $a^\dagger_{\mathbf{0}} a^\dagger_{\mathbf{0}} a_{\mathbf{0}} a_{\mathbf{0}}$ involves both a constant (momentum-independent) term of order N^2 and a momentum-dependent term of order N. As usual when considering the spectrum of the Hamiltonian, we drop the constant term. We are then left with the sub-leading (order N) term $-N \frac{g}{\mathcal{V}} \sum_{\mathbf{p}\neq 0} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$. In fact, we see that order N is actually the leading non-trivial order.

There are also sub-leading terms (order N) which come from quartic terms where exactly two of the momenta are 0. For example, if $\mathbf{p} = \mathbf{k} = 0$, then the two remaining momenta are $\mathbf{k}{+}\mathbf{q} = \mathbf{q}$ and $\mathbf{p}\text{-}\mathbf{q} = \text{-}\mathbf{q}$, leading to a contribution of $N \frac{g}{\mathcal{V}} a^\dagger_{\mathbf{q}} a^\dagger_{\text{-}\mathbf{q}}$ (plus a sub-sub-leading term of order N^0 which

I haven't seen a full justification – despite my conviction that such a justification has been given in the literature. Here is my stab at a justification. I believe the key is to expand the momentum-0 ladder operators in powers of $1/\sqrt{N_0}$. Write $a_{\mathbf{p}=0} = \sqrt{N_0}\left(1 + \frac{\delta}{\sqrt{N_0}} P + ...\right)$ where P is an operator of order 1. Then $[a_{\mathbf{p}=0}, a^\dagger_{\mathbf{p}=0}] = 1$, so $|\delta|^2 [P, P^\dagger] = N_0$, thus δ is of order $\frac{1}{\sqrt{N_0}}$. In our expansions, we only go out to order N_0 and we also ignore terms that are momentum-independent. If we do this systematically, we'll discover that terms of order δ can be ignored.

we will ignore). There are five other possible combinations with exactly two momenta. These are shown in Lancaster Eq. (42.2).

We might expect that if a quartic term had exactly three momentum that are 0, then that term would be of order $N^{\frac{3}{2}}$. However, it's easy to see that the momenta are constrained so that it's impossible for exactly three to be 0. For example, suppose that $\mathbf{p} = \mathbf{k} = \mathbf{k+q} = 0$. Then this implies that $\mathbf{q} = 0$ and therefore $\mathbf{p\text{-}q} = 0$. So all 4 momenta are 0.

What about a quartic term where only one momentum is 0 or where no momenta are 0? In both those cases, the order of N is less than 1 (the leading non-trivial order) so can be dropped. [7]

Putting all of the terms together, we finally end up with Eq. (9)

$$H = \sum_{\mathbf{p}\neq 0} \left(\frac{\mathbf{p}^2}{2m} + ng \right) a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{1}{2} \sum_{\mathbf{p}\neq 0} ng \left(a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{\text{-}p}} + a_{\mathbf{p}} a_{\mathbf{\text{-}p}} \right),$$

where $n = \frac{N}{\mathcal{V}}$ is the number-density. Notice that the kinetic term is of order 0 in N. However, it's also of order 0 in g and so, in principle could be as large as the interaction terms which are proportional to Ng (said differently, our perturbation expansion is done separately for the two parameters g and N).

3.2 Diagonalizing the Hamiltonian

The idea is to redefine the annihilation and creation operators so that

- The new operators $b_{\mathbf{p}}$ also obey the ladder commutator rules
- Using the new operators, the Hamiltonian should be in the form $H = \sum_{\mathbf{p}} E(p) b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + \text{constant}$

[7] Lancaster, as well as some other references I've seen, claim that the the case with only one momentum is excluded because it gives a non-zero expectation value. I'm skeptical of that statement and besides, I'm not sure which expectation value is being considered or why it's relevant to the spectrum. Notably, Landau and Lifshitz do not make this claim. Since we don't actually care about terms of N-order less than 1, none of this matters anyway.

(Note that Lancaster uses the notation α instead of b, but I find that hard to read.) By redefining the ladder operators, we are effectively changing the basis states and therefore what we call the "particle states".

Two common terms used are

- *quasiparticles* (sometimes described as "excitations"), meaning the (families of) states created by applying the new creation operators $b^\dagger_{\mathbf{p}}$.
- *antiparticles* meaning the (families of) states created by applying the new annihilation operators $b_{-\mathbf{p}}$. Since all momenta are permitted (both positive and negative coordinates), the term "antiparticle" should really be "antiparticle with momentum $\mathbf{p}$".

Using this terminology, we'll see that the quasiparticles are linear combinations of particles and antiparticles of momentum $\mathbf{p}$.

The redefinition of the ladder operators is accomplished by the linear transformation (see Lancaster Eq. (42.7))

$$\begin{pmatrix} a_{\mathbf{p}} \\ a^\dagger_{-\mathbf{p}} \end{pmatrix} = \begin{pmatrix} u_{\mathbf{p}} & -v_{\mathbf{p}} \\ -v_{\mathbf{p}} & u_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} b_{\mathbf{p}} \\ b^\dagger_{-\mathbf{p}} \end{pmatrix}. \tag{19}$$

As Lancaster demonstrates, we can assure that the new operators obey the ladder commutation relations, provided that

$$u^2_{\mathbf{p}} - v^2_{\mathbf{p}} = 1, \qquad u_{\mathbf{p}} = u_{\mathbf{p}}*, \qquad v_{\mathbf{p}} = v_{\mathbf{p}}* \tag{20}$$

Rewrite Eq. (9) as

$$\begin{aligned} H &= \sum_{\mathbf{p}\neq 0} \begin{pmatrix} a^\dagger_{\mathbf{p}} & a_{-\mathbf{p}} \end{pmatrix} \begin{pmatrix} \epsilon_{\mathbf{p}} & \frac{1}{2}ng \\ \frac{1}{2}ng & 0 \end{pmatrix} \begin{pmatrix} a_{\mathbf{p}} \\ a^\dagger_{-\mathbf{p}} \end{pmatrix} \\ &= \sum_{\mathbf{p}\neq 0} \begin{pmatrix} b^\dagger_{\mathbf{p}} & b_{-\mathbf{p}} \end{pmatrix} \begin{pmatrix} u_{\mathbf{p}} & -v_{\mathbf{p}} \\ -v_{\mathbf{p}} & u_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} \epsilon_{\mathbf{p}} & \frac{1}{2}ng \\ \frac{1}{2}ng & 0 \end{pmatrix} \begin{pmatrix} u_{\mathbf{p}} & -v_{\mathbf{p}} \\ -v_{\mathbf{p}} & u_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} b_{\mathbf{p}} \\ b^\dagger_{-\mathbf{p}} \end{pmatrix} \end{aligned} \tag{21}$$

where $\epsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} + ng$. The matrix multiplication can be done and we have

$$H = \sum_{\mathbf{p}\neq 0} \begin{pmatrix} b^\dagger_{\mathbf{p}} & b_{-\mathbf{p}} \end{pmatrix} \mathcal{H} \begin{pmatrix} b_{\mathbf{p}} \\ b^\dagger_{-\mathbf{p}} \end{pmatrix} \tag{22}$$

where

$$\mathcal{H} = \frac{1}{2m}\begin{pmatrix} \left(\mathbf{p}^2 + 2gmn\right) u_{\mathbf{p}}^2 - 2gmnu_{\mathbf{p}}v_{\mathbf{p}} & -\left(\mathbf{p}^2 + 2gmn\right) u_{\mathbf{p}}v_{\mathbf{p}} + gmn\left(u_{\mathbf{p}}^2 + v_{\mathbf{p}}^2\right) \\ -\left(\mathbf{p}^2 + 2gmn\right) u_{\mathbf{p}}v_{\mathbf{p}} + gmn\left(u_{\mathbf{p}}^2 + v_{\mathbf{p}}^2\right) & \left(\mathbf{p}^2 + 2gmn\right) v_{\mathbf{p}}^2 - 2gmnu_{\mathbf{p}}v_{\mathbf{p}} \end{pmatrix}. \tag{23}$$

We can diagonalize $\mathcal{H}$ by solving for $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$, subject to the constraints of Eq. (20), so that the off-diagonal elements are 0. (See Lancaster p. 373). We obtain solutions

$$\begin{aligned} u_{\mathbf{p}} &= \frac{1}{2}\left(\frac{E(p)}{\epsilon_{\mathbf{p}}} + \frac{\epsilon_{\mathbf{p}}}{E(p)}\right) \\ v_{\mathbf{p}} &= \frac{1}{2}\left(-\frac{E(p)}{\epsilon_{\mathbf{p}}} + \frac{\epsilon_{\mathbf{p}}}{E(p)}\right) \end{aligned} \tag{24}$$

where

$$E(p) = \sqrt{\frac{\mathbf{p}^2}{2m}\left(\frac{\mathbf{p}^2}{2m} + 2ng\right)}. \tag{25}$$

This gives us

$$\mathcal{H} = \frac{E(p)}{4}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}. \tag{26}$$

We perform the matrix multiplications of Eq. (22) and obtain

$$H = \sum_{\mathbf{p}\neq 0} \frac{E(p)}{4}\left(3b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + b_{-\mathbf{p}}b_{-\mathbf{p}}^\dagger\right). \tag{27}$$

We note that in the second sum, we can change the summand from $\mathbf{p}$ to $-\mathbf{p}$ and then we can use the ladder operator identity $b_{\mathbf{p}}b_{\mathbf{p}}^\dagger = b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + 1$ to finally obtain

$$H = \sum_{\mathbf{p}\neq 0} E(p) b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + \sum_{\mathbf{p}\neq 0} \frac{E(p)}{4}. \tag{28}$$

The second term is a (infinite) constant so we can ignore it.[8] We now have the Hamiltonian in the diagonal form with eigenvalues obeying the dispersion relation $E(p) = \sqrt{\frac{\mathbf{p}^2}{2m}\left(\frac{\mathbf{p}^2}{2m} + 2ng\right)}$, which is the same as the dispersion relation Eq. (7), which we used to show that the Bogoliubov theory describes a superfluid.

[8] Lancaster doesn't show this constant.

4 Summary

Here is the sequence of steps for deriving the non-viscosity of super-cooled fluids with low fluid velocities.

- We consider a system consisting of N indistinguishable bosons.
- We hypothesize a many-body quantum Hamiltonian in which the dominant interactions are two-body short-range interactions.
- Because of the indistinguishability of particles, we can describe the Hilbert space by using the occupation-number representation (basis states can be described by stating how many particles are in each one-particle state). Two-body interaction operators look like sums of quartic terms that look like $a^\dagger_{\mathbf{p}_1} a^\dagger_{\mathbf{p}_2} a_{\mathbf{p}_3} a_{\mathbf{p}_4}$.
- The short-range interactions are approximated by a delta-function. If this were a classical system, it would describe pointwise collisions.
- The resulting Hamiltonian looks like

$$H \approx \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{g}{2\mathcal{V}} \sum_{\mathbf{kpq}} a^\dagger_{\mathbf{k+q}} a^\dagger_{\mathbf{p-q}} a_{\mathbf{p}} a_{\mathbf{k}}.$$

- We will want to diagonalize this Hamiltonian, but the quartic terms make that difficult. Bogoliubov used an approximation – valid when the temperature of the system is close to 0 and therefore most particles are in the ground state (of 0 energy) that allows us to approximate the above Hamiltonian with a quadratic (in ladder operators) form which we will be able to diagonalize. Bogoliubov's trick was to notice that

$$a_{\mathbf{p}=0}|s\rangle \approx \sqrt{N_0}|s\rangle.$$

  for all states (and therefore that we can write $a_{\mathbf{p}=0} = N_0$. By applying this trick, and then expanding the interaction term in powers of N, he ended up with

$$H = \sum_{\mathbf{p}\neq 0} \left(\frac{\mathbf{p}^2}{2m} + ng \right) a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{1}{2} \sum_{\mathbf{p}\neq 0} ng \left(a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{-p}} + a_{\mathbf{p}} a_{\mathbf{-p}} \right),$$

- If you introduce the operator $b_{\mathbf{p}}$ defined by

$$\begin{pmatrix} a_{\mathbf{p}} \\ a^{\dagger}_{-\mathbf{p}} \end{pmatrix} = \begin{pmatrix} u_{\mathbf{p}} & -v_{\mathbf{p}} \\ -v_{\mathbf{p}} & u_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} b_{\mathbf{p}} \\ b^{\dagger}_{-\mathbf{p}} \end{pmatrix},$$

where

$$u_{\mathbf{p}} = \frac{1}{2}\left(\frac{E(p)}{\epsilon_{\mathbf{p}}} + \frac{\epsilon_{\mathbf{p}}}{E(p)}\right)$$
$$v_{\mathbf{p}} = \frac{1}{2}\left(-\frac{E(p)}{\epsilon_{\mathbf{p}}} + \frac{\epsilon_{\mathbf{p}}}{E(p)}\right),$$

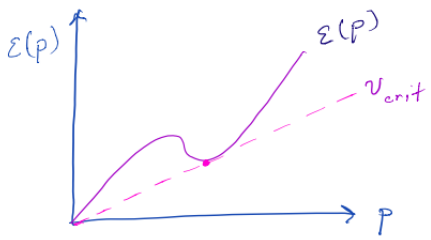
with $\epsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} + ng$, then we can show that

$$H = \sum_{\mathbf{p}\neq 0} E(p) b^{\dagger}_{\mathbf{p}} b_{\mathbf{p}} + \text{constant},$$

where

$$E(p) = \sqrt{\frac{\mathbf{p}^2}{2m}\left(\frac{\mathbf{p}^2}{2m} + 2ng\right)}.$$

- This relation between momentum and energy is known as the dispersion relation and is illustrated in the graph



In this figure, we have drawn a line from the origin through the local minimum. The slope of that line is known as the critical velocity and is found mathematically as $v_{\text{crit}} = \min_{p} \frac{E(p)}{p}$.

- Landau argued as follows: There is friction between the fluid and the walls **only** if heat is transferred to the wall by the moving fluid. That means there are some molecules in the moving fluid that have state-transitions where their energy is reduced. We examine state-transitions in the fluid rest-frame. These transitions obey the dispersion relation that connects rest-frame momentum p to energy $E(p)$. Then,

to obtain the dispersion relation relative to wall-frame momenta, we apply a Galilean transformation. If that transformed dispersion relation value, $E(p) - \mathbf{p} \cdot \mathbf{v}$ is negative, then the wall can dissipate energy (i.e., heat is transferred to the wall) which implies friction. Otherwise there is no friction which happens if the fluid velocity v satisfies $v < v_{\text{crit}} = \min_p \frac{E(p)}{p}$. That's the Landau criterion.

A Other (more traditional) derivation of the Landau criterion

This follows several treatments of the subject, but the one which I think is clearest is Fukuda et al. I would have preferred a treatment which explicitly discusses the thermodynamics, but I haven't found one to my satisfaction.

Since the temperature is almost 0, most quasiparticles (in the rest frame of the fluid) are in the ground state. Let us consider the possibility of exciting (in the fluid frame) one quasiparticle to a state with momentum $\mathbf{p}$ and energy $\epsilon_{\text{fluid}}(p)$. In the lab frame we obtain the energy of that quasiparticle by a Galilean transformation

$$\epsilon_{\text{lab}}(p) = \epsilon_{\text{fluid}}(p) - \mathbf{p} \cdot \mathbf{v} \tag{29}$$

I'll shortly give an argument for this. In the meantime, notice that if we weren't taking the walls into consideration, the fluid velocity would have no physical significance. Our analysis will therefore have to account for the possibility of energy transfer between the wall and the fluid.

Since the quasiparticle excitation changes the total system (including fluid and wall) energy by $\epsilon_{\text{lab}}(p)$, energy conservation requires that the rest of the system must experience a change of energy $\Delta = -\epsilon_{\text{lab}}(p)$. The only source for that energy is the wall – or more precisely, the interaction between the wall and the fluid. [9] In the lab frame, the wall is at rest so it cannot contribute mechanical energy. Only heat energy can change. If $\epsilon(p)$ is negative, then the changed heat energy Δ must be positive. [10] (We refer to that process

[9] You might object that there are other quasiparticles that could contribute that energy but if so, they would have to contribute via the same kind of excitations that we are considering. So in general, we can add up all the excitations in the fluid. At low temperature, there aren't likely to be many.

[10] I think a bit more discussion is required. As far as I can tell, many papers have

as "dissipation of energy"). In thermodynamics (or statistical mechanics), system energy is conserved so if heat energy increases, then system kinetic energy decreases – hence velocity decreases. Thus if $\epsilon(p)$ can be negative, then the fluid slows down – i.e., it has viscosity.

On the other hand, if $\epsilon(p)$ is positive, the wall is unable to impart a compensating negative energy. In that case, the excitation can't occur and as a result, the fluid doesn't slow down.

If $|\mathbf{v}| < \frac{\epsilon(p)}{p}$, then it is easy to show that ϵ_{lab} is always positive, in which case the fluid does not slow down. Landau's criterion for superfluid flow is $|\mathbf{v}| < v_{\text{crit}}$ where $v_{\text{crit}} = \min_p \frac{\epsilon(p)}{p}$. See the dispersion graph above and note that $v_{\text{crit}} \neq 0$.

Caveat emptor. The Landau criterion, as well as the remainder of Landau's theory of superfluidity and Bogoliubov's analysis (coming up soon), involves many approximations and assumptions. Some of these have been discovered to be highly consequential (for example, see Feynman's work c. 1955). Others not so much. Despite that, the Landau and Bogoliubov theories are beautiful explanations of rather mysterious phenomena, and have proven themselves to have great predictive power.

A.1 A loose end: Galilean transformation

Simon has a crisp explanation for the Galilean transformation given in Eq. (29). He asks us to recall that if a quantum state is an eigenvector of both energy and momentum, then its space and time dependence looks like

$$\Psi(t, \mathbf{x}) \propto e^{i(\mathbf{p}\cdot\mathbf{x} - Et)}. \tag{30}$$

In a moving frame, this becomes

$$\begin{aligned} \Psi'(t, \mathbf{x}) &\propto e^{i(\mathbf{p}\cdot(\mathbf{x}+\mathbf{v}t) - Et)} \\ &= e^{i(\mathbf{p}\cdot\mathbf{x} + (E - \mathbf{p}\cdot\mathbf{v})t)} \\ &= e^{i(\mathbf{p}\cdot\mathbf{x} + E')t}, \end{aligned} \tag{31}$$

where $E' = E - \mathbf{p}\cdot\mathbf{v}$ in agreement with Eq. (29).

been written exploring the nuances of this analysis, but what I refer to as heat energy is a collection of molecular excitations, and for "ordinary materials" these have positive energy. In ordinary English, this amounts to saying that the fluid heats up the wall through friction.