Superfluids Part II

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1 Introduction

We now turn to the field theoretic treatment of superfluids, where our fundamental quantities are fields rather than ladder operators. An extremely thorough online book on this subject, is Superfluid States of Matter, by Svistunov, Babaev and Prokof'ev.

From Lancaster Chapter 4 and my previous notes, we had the following approximate action for a non-relativistic bosonic many-body field theory with only very short-range 2-body interactions.

$$S(\phi) = \int d^3x \left[i\phi^\dagger \partial_0 \phi - \frac{1}{2m} \boldsymbol{\nabla}\phi^\dagger \cdot \boldsymbol{\nabla}\phi \right](\mathbf{x}) - \frac{g}{2}\int d^3x d^3y \phi^\dagger(\mathbf{x})\phi^\dagger(\mathbf{y})\delta^{(3)}(\mathbf{x}-\mathbf{y})\phi(\mathbf{y})\phi(\mathbf{x}). \tag{1}$$

This theory was taken as a model for superfluids.

From this, we can derive the Hamiltonian by following the usual transformation from Lagrangian to Hamiltonian.[1]

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[1] The second line below follows from the first only if we are cavalier about the ordering of operators. The canonical commutation relations imply that $\phi^\dagger(\mathbf{x})$ doesn't commute with $\phi(\mathbf{x})$ so an ordering has to be chosen. The usual ordering is normal-ordering. If we change the ordering, that would introduce a term quadratic in ϕ. However, we'll later also introduce another term quadratic in ϕ with a proportionality constant μ which is ultimately chosen by requiring the measured value of N. If we change the ordering of the factors in the quartic term, that will have the effect of redefining μ. This kind of procedure is known as renormalization and won't change the conclusions about superfluidity that we discuss later.

$$\begin{aligned} H &= \frac{1}{2m}\int d^3x \boldsymbol{\nabla}\phi^\dagger(\mathbf{x})\cdot\boldsymbol{\nabla}\phi(\mathbf{x}) + \frac{g}{2}\int d^3x d^3y \phi^\dagger(\mathbf{x})\phi^\dagger(\mathbf{y})\phi(\mathbf{y})\phi(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\ &= \frac{1}{2m}\int d^3x \left(\frac{1}{2m}\boldsymbol{\nabla}\phi^\dagger(\mathbf{x})\cdot\boldsymbol{\nabla}\phi(\mathbf{x}) + \frac{g}{2}\left(\phi^\dagger(\mathbf{x})\phi(\mathbf{x})\right)^2\right) \end{aligned} \tag{2}$$

We put the system in a box of volume $\mathcal{V}$, so that the momenta become discrete and then cast this expression into a form using annihilation and creation operators, using the mode expansion

$$\phi^\dagger(\mathbf{x}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}} a^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}. \tag{3}$$

This gives us

$$H \approx \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{g}{2\mathcal{V}} \sum_{\mathbf{kpq}} a^\dagger_{\mathbf{k}+\mathbf{q}} a^\dagger_{\mathbf{p}-\mathbf{q}} a_{\mathbf{p}} a_{\mathbf{k}}, \tag{4}$$

in agreement with Lancaster Eq (42.1). We started with this expression, used the Bogoliubov large-N ("hunting license") approximation and then went from there to an explanation for superfluidity. Now we want to see whether we can derive superfluidity directly using the action Eq. (1) and in the process, avoiding the Bogoliubov "hunting license" approximation (based on large N).

2 Obtaining the energy dispersion from field theory – broken symmetry

2.1 Statistical mechanics and the chemical potential

Previously, we proceeded by obtaining an energy dispersion relation and then using the Landau criterion to show that at small fluid velocities, the fluid has no viscosity. Our derivation relied on the fact that at low temperature, the N-molecule system mostly finds itself in the ground state. We might have thought that the ground state is essentially trivial, but a close examination of the Hamiltonian reveals that the interaction term acting on the ground state is of order N^2 hence the ground state is by no means a 0-energy state. We can then expand in inverse-powers of N to find an approximate dispersion relation that leads to the Landau criterion.

If we wished to proceed directly from the action (Eq. 1), we would immediately notice that this expression makes no direct reference either to N, the number of molecules, nor to the temperature. In other words, there's nothing in the Lagrangian or Hamiltonian that would lead us to conclude anything about the importance of the ground state at low temperature. Why not?

What's missing is the inclusion of statistical mechanics. Recall that any systematic study of many-body systems, must take into account the presence of some kind of "environment" – for example a container. We account for the environment with certain parameters that have been found to characterize constraints that are pertinent to the system of interest (in this case, the fluid). Those parameters often play the role of Lagrange multipliers whose values determine quantities like the average energy of the system or the average number of molecules in the system.

Following the usual arguments of statistical mechanics, if a system is in thermal equilibrium with a heat reservoir, and that system has a number N of particles, the probability of finding the system in a state with energy E can be derived to be

$$P = Ce^{-\beta(E-\mu N)} \tag{5}$$

where we have introduced the Lagrange multiplier $\beta\mu$. μ is known as the *chemical potential* and for a specific system can be inferred from a measurement of N. Recall that $\beta = \frac{1}{kT}$ so that the probability is high when the temperature is low . Furthermore, lower-energy states are more likely than higher-energy states.

Now let's apply this to our fluid system with Hamiltonian given by Eq. (2). Note that in the probability equation Eq. (5), the quantity μN can be written as $\mu \sum a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \to \mu \int d^3x \phi^\dagger(\mathbf{x})\phi(\mathbf{x})$. So if the eigenstates of the Hamiltonian H are denoted by E, then $E - \mu N$ are the eigenstates of $H - \mu \int d^3x \phi^\dagger(\mathbf{x})\phi(\mathbf{x})$ up to an arbitrary constant.

$$\begin{aligned}
H - \mu \int d^3x \phi^\dagger(\mathbf{x})\phi(\mathbf{x}) &= \int d^3x \left(\frac{1}{2m} \boldsymbol{\nabla}\phi^\dagger(\mathbf{x}) \cdot \boldsymbol{\nabla}\phi(\mathbf{x}) + \frac{g}{2}\left(\phi^\dagger(\mathbf{x})\phi(\mathbf{x})\right)^2 - \mu\phi^\dagger(\mathbf{x})\phi(\mathbf{x}) \right) \\
&= \int d^3x \left(\frac{1}{2m} \boldsymbol{\nabla}\phi^\dagger(\mathbf{x}) \cdot \boldsymbol{\nabla}\phi(\mathbf{x}) + \frac{g}{2}\left(\frac{\mu}{g} - \phi^\dagger(\mathbf{x})\phi(\mathbf{x})\right)^2 - \frac{\mu^2}{2g} \right).
\end{aligned} \tag{6}$$

The last term in the sum is a constant and can be dropped, leaving us with the Hamiltonian, H' whose eigenstates are of interest,

$$H' = \int d^3x \left(\frac{1}{2m} \boldsymbol{\nabla}\phi^\dagger(\mathbf{x}) \cdot \boldsymbol{\nabla}\phi(\mathbf{x}) + \frac{g}{2}\left(\frac{\mu}{g} - \phi^\dagger(\mathbf{x})\phi(\mathbf{x}) \right)^2 \right). \qquad (7)$$

2.2 A ground state of H'

In Eq. (7) each of the two terms of H' are positive, so the "minimum value" of H' occurs when ϕ is a constant where $\phi^\dagger\phi = \frac{\mu}{g}$.

The phrase "minimum value" is in quotation marks, because H' is an operator and not a real-valued function. If we were looking at a classical theory, it would indeed be true that we could find the minimum energy state by an appropriate choice of ϕ as described above. But what looking for in quantum mechanics, is the ground state (state with the smallest eigenvalue) of the operator H'. We can invoke a theorem about self-adjoint operators.

Theorem: For any self-adjoint operator A, and for any normalized state $|\psi\rangle$ which minimizes $\langle\psi|A|\psi\rangle$, $|\psi\rangle$ is an eigenstate of A whose eigenvalue is a minimum amongst eigenvalues of A.

We now introduce a connection between the classical and quantum theory. Let the normalized state $|\phi\rangle$ denote a state with the property $\hat{\phi}(\mathbf{x})|\phi\rangle = \phi(\mathbf{x})|\phi\rangle$. I've reintroduced the "hat" or "caret" mark to indicate an operator as opposed to a complex function. Previously, when writing field operators, I've dropped the "hat" for convenience but right now, we need to distinguish between operators and functions. So the state $|\phi\rangle$ has the property that when you act on it with the field operator $\hat{\phi}(\mathbf{x})$ it has the value $\phi(\mathbf{x})$ times itself. In other words, $|\phi\rangle$ is an eigenstate of the operator $\hat{\phi}(\mathbf{x})$ with eigenvalue $\phi(\mathbf{x})$. [2] We call such a state a "coherent state".

It remains to be seen whether there is such a state in the Hilbert space. Let us assume so for now. Then $\langle\phi|\hat{\phi}^\dagger(\mathbf{x}) = \phi^*(\mathbf{x})\langle\phi|$. As usual, let the Hamltonian be normal-ordered, so that the field operators $\hat{\phi}(\mathbf{x})$ are to the right of the field operators $\hat{\phi}^\dagger(\mathbf{x})$. Evaluate $\langle\phi|H'|\phi\rangle$ by applying $\hat{\phi}(\mathbf{x})$ to $|\phi\rangle$ on the right and $\hat{\phi}^\dagger(\mathbf{x})$ to $\langle\phi|$ on the left. Then (from Eq. (7), but explicitly

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[2] Perhaps things would be clearer if we used entirely different notation for the field operator and the state. So for example, we could say the field operator was $\mathcal{F}(\mathbf{x})$ and that a particular state $\phi\rangle$ was defined by the property that $\mathcal{F}(\mathbf{x})|\phi\rangle = \phi(\mathbf{x})|\phi\rangle$.

including the “hat” for the field operators)

$$\begin{aligned}\langle\phi|H'|\phi\rangle &= \langle\phi|\int d^3x\left(\frac{1}{2m}\boldsymbol{\nabla}\hat{\phi}^\dagger(\mathbf{x})\cdot\boldsymbol{\nabla}\hat{\phi}(\mathbf{x})+\frac{g}{2}\left(\frac{\mu}{g}-\hat{\phi}^*(\mathbf{x})\hat{\phi}(\mathbf{x})\right)^2\right)|\phi\rangle\\ &= \int d^3x\left(\frac{1}{2m}\boldsymbol{\nabla}\phi^*(\mathbf{x})\cdot\boldsymbol{\nabla}\phi(\mathbf{x})+\frac{g}{2}\left(\frac{\mu}{g}-\phi^*(\mathbf{x})\phi(\mathbf{x})\right)^2\right)\langle\phi|\phi\rangle\\ &= \int d^3x\left(\frac{1}{2m}\boldsymbol{\nabla}\phi^*(\mathbf{x})\cdot\boldsymbol{\nabla}\phi(\mathbf{x})+\frac{g}{2}\left(\frac{\mu}{g}-\phi^*(\mathbf{x})\phi(\mathbf{x})\right)^2\right),\end{aligned} \tag{8}$$

where the second line follows from the first noting that in the second line, the fields are complex functions, and the third line follows from the second noting that the coherent state is normalized.

Now choose $\phi(\mathbf{x}) = \sqrt{\frac{\mu}{g}}e^{i\theta}$. This is a constant, so the first term (involving gradients) is 0, and by construction, the second term is 0. Therefore the state $|\phi(\mathbf{x}) = \sqrt{\frac{\mu}{g}}e^{i\theta}\rangle$ has the property $\langle\phi|H'|\phi\rangle = 0$. Since H' is manifestly non-negative, the state $|\phi\rangle$ minimizes $\langle\phi|H'|\phi\rangle$ and by the theorem quoted above, $|\phi\rangle$ must be a ground state of H'.

This more or less completes the connection between the classical and quantum theories. In the classical theory, we saw the field satisfying $|\phi(\mathbf{x})| = \sqrt{\frac{\mu}{g}}$, was a minimum of the classical H'. In the quantum theory we saw that the coherent state $|\phi(\mathbf{x}) = \sqrt{\frac{\mu}{g}}e^{i\theta}\rangle$ is a ground state of the quantum operator H'. [3]

Finally, we note that for any coherent ground state with phase θ, we have the expectation value of the number density operator $\hat{\phi}^\dagger(\mathbf{x})\hat{\phi}(\mathbf{x})$

$$\langle\phi|\hat{\phi}^\dagger(\mathbf{x})\hat{\phi}(\mathbf{x})|\phi\rangle = \frac{\mu}{g}. \tag{9}$$

We therefore set $n = \mu/g$ as the ground-state number density, following Lancaster in section 42.3. In our treatment of all this, we’ve taken the theory in an infinite volume. However, that’s incompatible with our thermodynamic

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[3] Our argument hasn’t precluded the possibility that there could be a non-coherent state with the same energy as the ground state. I don’t know whether such a state is possible or what it would mean if it were, but I’ll ignore this possibility. In various treatments on the subject, I haven’t seen this discussed. However, the general theory has many complicated threads that I haven’t explored.

assumption of a fluid flowing through a finite volume bounded by walls. Nevertheless, for sufficiently large volumes we can add an external potential which is 0 inside the volume and infinite outside the volume, and our results will approximate those above. [4]

For a system with an experimentally expected value of N_0 particles, the chemical potential μ can be chosen to arrange that. Now we can ask what the variance is.

$$\begin{aligned}
\mathrm{Var}(N) &= \langle\phi|\hat{N}^2|\phi\rangle - \langle\phi|\hat{N}|\phi\rangle^2 \\
&= \langle\phi|\int d^3x d^3y \hat{\phi}(\mathbf{x})^\dagger\hat{\phi}(\mathbf{x})\hat{\phi}(\mathbf{y})^\dagger\hat{\phi}(\mathbf{y})|\phi\rangle - \langle\phi|\int d^3x \hat{\phi}(\mathbf{x})^\dagger\hat{\phi}(\mathbf{x})|\phi\rangle^2 \\
&= \langle\phi|\int d^3x d^3y \hat{\phi}(\mathbf{x})^\dagger\hat{\phi}(\mathbf{y})^\dagger\hat{\phi}(\mathbf{x})\hat{\phi}(\mathbf{y})|\phi\rangle + \langle\phi|\int d^3x \hat{\phi}(\mathbf{x})^\dagger\hat{\phi}(\mathbf{x})|\phi\rangle \\
&\qquad - \langle\phi|\int d^3x \hat{\phi}(\mathbf{x})^\dagger\hat{\phi}(\mathbf{x})|\phi\rangle^2 \\
&= \int d^3x d^3y \langle\phi|\hat{\phi}(\mathbf{x})^\dagger\hat{\phi}(\mathbf{y})^\dagger\hat{\phi}(\mathbf{x})\hat{\phi}(\mathbf{y})|\phi\rangle + N_0 - N_0^2 \\
&= \left(\int d^3x \left(\frac{\mu}{g}\right)\right)^2 + N_0 - N_0^2 \\
&= N_0^2 + N_0 - N_0^2 \\
&= N_0.
\end{aligned} \tag{10}$$

The third equality follows from the second by employing the canonical commutation relation $[\hat{\phi}(\mathbf{x}), \hat{\phi}^\dagger(\mathbf{y})] = \delta^{(3)}(\mathbf{x}-\mathbf{y})$, the fourth equality follows from acting on the left state with $\phi^\dagger$ and on the right with ϕ, and the fifth equality follows from the ground-state condition.

So although the ground state has N particles, there is uncertainty in that number with a dispersion of $\sqrt{N}$. This intuitively corresponds to Bogoliubov's assumption, where he used $a_{\mathbf{p}=0}|\Omega\rangle \approx \sqrt{N}|\Omega\rangle$.[5]

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[4] The effect will be to integrate the Hamiltonian density within the volume $\mathcal{V}$. In that case, we find that the expectation value of the total number of particles is $N = (\mu\mathcal{V})/g$.

[5] I haven't found any reference which has exactly the derivations or conclusions in my notes. There is a fair amount written about the uncertainty relation between phase angle and number operators, and I think my conclusions are somewhat similar. So ... caveat emptor.

2.3 Perturbations from the Ground State – breaking the U(1) symmetry

Recall our field Hamiltonian

$$H' = \int d^3x \left(\frac{1}{2m} \boldsymbol{\nabla}\phi^\dagger(\mathbf{x}) \cdot \boldsymbol{\nabla}\phi(\mathbf{x}) + \frac{g}{2} \left(\frac{\mu}{g} - \phi^\dagger(\mathbf{x})\phi(\mathbf{x}) \right)^2 \right). \tag{11}$$

This Hamiltonian is invariant under the U(1) transformation

$$\begin{aligned} \phi &\to \phi e^{i\theta} \\ \phi^* &\to \phi^* e^{-i\theta} \end{aligned} \tag{12}$$

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Now rewrite the field as $\phi = \phi_0 + \phi_1$ where ϕ_0 is a ground state. In general, $\phi_0 = \sqrt{\frac{\mu}{g}} e^{i\theta}$, where θ distinguishes one ground state from another. Let us pick $\theta = 0$ so that $\phi_0 = \sqrt{\frac{\mu}{g}}$. Expand the Hamiltonian to leading orders in ϕ_1 (or equivalenty, leading orders in $\sqrt{g}$). Then we get

$$H' \approx \int d^3x \left(\frac{1}{2m} \boldsymbol{\nabla}\phi_1^\dagger(\mathbf{x}) \cdot \boldsymbol{\nabla}\phi_1(\mathbf{x}) + \mu \int d^3x' \phi_1^*(\mathbf{x})\phi_1(\mathbf{x}) + \frac{\mu}{2} \int d^3x \left(\phi_1^2(\mathbf{x}) + \phi_1^{*2}(\mathbf{x}) \right) \right). \tag{13}$$

This Hamiltonian is **not** symmetric under a transformation $\phi_1 \to \phi_1 e^{i\theta}$ etc. The reason is that we chose one vacuum amongst all possible vacuums related by U(1) transformations. This choice of vacuum breaks the original symmetry. The physics we obtain by expanding around one vacuum will be identical to the physics we would obtain by expanding around another, but apart from that, we have "*broken the symmetry*".

This form of the Hamiltonian in terms of the field ϕ_1 is quadratic, and can therefore be diagonalized by standard methods of field theory [essentially, rewrite the Hamiltonian in the schematic form $\phi_{1i}^* Q^{ij} \phi_{1j}$ and then diagonalize Q. We can accomplish this most directly in momentum space (take the Fourier transforms of the fields).

Lo and behold, we end up with the Bogoliubov dispersion relation

$$E(p) = \sqrt{\frac{\mathbf{p}^2}{2m} \left(\frac{\mathbf{p}^2}{2m} + \frac{2Ng}{\mathcal{V}} \right)}. \tag{14}$$

3 Switching to the Lagrangian

I'm not going to get too philosophical about the definition of a particle. However, in a field theory, fields can be redefined and their corresponding mode expansions may describe ladder operators that annihilate or create states which differ from the original N-particle states. Some of these definitional ambiguities are due to choices of how to split a Hamiltonian into a free part and an interacting part.

For the time being, I'm going to follow a procedure based on the Lagrangian formulation of field theory. Here I'm assuming that H' describes the same theory as a Lagrangian $\mathcal{L}$ obtained from H' by the standard Legendre transformation.[6] I'll say more about that assumption in the next subsection.

In the meantime, we can see that the resulting theory has the (non-relativistic) Lagrangian (once again I'll revert to notation without "hats" on the operators)

$$\mathcal{L} = i\phi^\dagger(\mathbf{x})\partial_0\phi(\mathbf{x}) - \frac{1}{2m}\boldsymbol{\nabla}\phi^\dagger(\mathbf{x})\cdot\boldsymbol{\nabla}\phi(\mathbf{x}) - \frac{g}{2}\left(\frac{\mu}{g} - \phi^\dagger(\mathbf{x})\phi(\mathbf{x})\right)^2, \qquad (15)$$

where I've combined results from Eqs. (1) and (7). This is the same equation as Lancaster Eq. 42.23 with $n = \mu/g$. We'll use this description of the theory later.

3.0.1 From partition function to functional integral

There is a deep mathematical relationship between the fundamental dynamics of a system, and the thermodynamic properties of that system in equilibrium with an external environment. The thermodynamic properties can all be derived from derivatives (with respect to temperature, chemical potential or volume) of the grand canonical partition function, $\mathcal{Z}$:

> $\mathcal{Z}$ is a fundamental concept in statistical mechanics, particularly in the context of statistical thermodynamics. It's a mathemat-

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[6] In particular, the Lagrangian inherits the contribution from the chemical potential and thus includes the effect of thermal exchanges between the system and its environment. Ultimately, this becomes the mechanism whereby the number of particles is constrained and also which leads to a low-temperature theory dominated by a ground state populated mostly by 0-momentum particles.

> ical tool used to describe the statistical properties of a system in equilibrium with a reservoir, allowing for exchange of both energy and particles with the reservoir. This reservoir usually represents a much larger system with which the main system can exchange particles (like a gas exchanging molecules with a larger gas reservoir at a constant chemical potential) and energy (like a system in thermal contact with a larger heat bath at constant temperature).

$\mathcal{Z}$ is defined as

$$\mathcal{Z} = \sum_{\text{states s}} e^{-\beta(E_s - \mu N_s)} \tag{16}$$

As before, E_s is the energy of a state, where the set of states and energies define a Fock space where the states are generated by the eigenstates s of a Hamiltonian, whose eigenvalues are E_s. μ is the chemical potential and N_s is the number of particles in state s and is the eigenvalue of the number operator N acting on that state.

A somewhat similar-looking object, denoted Z and which we'll call the "field partition function", completely characterizes the (non-equilibrium) dynamics of the quantum system. All properties of interest can be derived from various functional derivatives of that partition function. Z is defined as

$$Z = \int \prod \mathcal{D}\phi \mathcal{D}\phi^* e^{iS[\phi,\phi^*]} \tag{17}$$

where the functional integral is over all values of the field ϕ and the action S is

$$\int dt d^3x \mathcal{L}(\phi, \phi^*)(t, \mathbf{x}) \tag{18}$$

for a Lagrangian $\mathcal{L}$.

As an interesting and important observation, we can (formally) analytically continue t to $-i\tau$ where τ is real. This is known as a Wick rotation, and the analytically continued spacetime is called Euclidean (instead of Minkowskian) spacetime.

I found a set of lectures by Gustavo Burdman (see especially Lecture 5 Lecture 6) which demonstrates how to transform the grand canonical partition function into a field partition function. In particular, he demonstrates the following:

$$\mathcal{Z} = \int \prod \mathcal{D}\phi \mathcal{D}\phi^* e^{-S_E[\phi,\phi^*]} \tag{19}$$

where

$$S_E[\phi,\phi^*] = \int_0^\beta d\tau d^3x\,(\phi^*\partial_\tau\phi + H(\phi^*,\phi) - \mu\phi^*\phi) \tag{20}$$

with the periodic boundary conditions $\phi(0) = \phi(\beta)$ and $\phi^*(0) = \phi^*(0)$.[7][8]

S_E is the analytic continuation to Euclidean space of the action S corresponding to the Lagrangian of Eq. (15), so we've more or less shown that the grand canonical partition function can be written as the field partition function – subject to the periodic boundary conditions in the Euclidean time variable. When $\beta \to \infty$, i.e., when the temperature goes to 0, the two partition functions are the same. Of course, they are used differently for dynamics than for equilibrium thermodynamics (at 0 temperature) but presumably one can employ this mathematical equivalence to gain new insights and compute interesting relationships.

A Principles of Statistical Mechanics

I've concluded that much of my confusion over Landau's criterion and, more generally, the formulation of superfluidity, has to do with my attempts to shortcut what we need to understand about statistical mechanics. In this section, I'll attempt to be a bit more thorough although I don't guarantee that I'll eliminate either my confusion or yours'. There's lots I've forgotten about statistical mechanics and thermodynamics, and some of that remains buried.

Some of this has to do with terminology and my use of it.

- *Internal Energy:* The following is from Wikipedia:

  > The internal energy of a thermodynamic system is the energy contained within it, measured as the quantity of energy necessary to bring the system from its standard internal state to its present internal state of interest, accounting for

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[7] If we are serious about treating ϕ^* as the complex conjugate of ϕ, then it's not necessary to separately show both ϕ and its conjugate in the various equations above. Under some circumstances, we think of these two objects as separate fields, only later related through conjugation. However, I can't think of any reason to do that in this context.

[8] I've made a few minor modifications to Burdman's equations. In particular, he appears to have dropped the spacial dependence of the fields so I've put those back in.

the gains and losses of energy due to changes in its internal state, including such quantities as magnetization.It excludes the kinetic energy of motion of the system as a whole and the potential energy of position of the system as a whole, with respect to its surroundings and external force fields. It includes the thermal energy, i.e., the constituent particles' kinetic energies of motion relative to the motion of the system as a whole. The internal energy of an isolated system cannot change, as expressed in the law of conservation of energy, a foundation of the first law of thermodynamics.

The internal energy cannot be measured absolutely. Thermodynamics concerns changes in the internal energy, not its absolute value. The processes that change the internal energy are transfers, into or out of the system, of matter, or of energy, as heat, or by thermodynamic work. These processes are measured by changes in the system's properties, such as temperature, entropy, volume, electric polarization, and molar constitution. The internal energy depends only on the internal state of the system and not on the particular choice from many possible processes by which energy may pass into or out of the system. It is a state variable, a thermodynamic potential, and an extensive property.

- The setup for statistical mechanics is described thoroughly and clearly in Reif's text Fundamentals of Statistical and Thermal Physics. For example, consider a total isolated system partitioned into 2 components A and B. Let's also imagine that the system Hamiltonian H can be written as

  $$H = H_A + H_B + H_{\text{int}}, \tag{21}$$

  where H_A is the Hamiltonian for system A, H_B is the Hamiltonian for system B and H_{int} is the extra term of interaction **and** is small relative to the Hamiltonians of the two subsystems.

  Define E to be the energy of subsystem A, and define $\bar{E}$ to be the mean energy amongst a collection (an *ensemble*) of systems whose macro-paramaters are all identical to those of A. We call $\bar{E}$ the **textitinternal energy** of A. Now we write

  $$\Delta\bar{E} = W + Q \tag{22}$$

  where $\Delta\bar{E}$ is the change of internal energy, W is the change in internal energy owing to a change in the external parameters (i.e., due to macroscopic **work** on A) and Q is called the **heat** absorbed by A. A similar equation can be written for system B, where by conservation of energy, its internal energy changes by the negative of the change of A's internal energy.

- Continuing in the footsteps of Reif, we now relate the above quantities to the Hamiltonian components. H_A is taken to be the Hamiltonian of system A prior to interaction and H'_A is taken to be the Hamiltonian after interaction. Similarly for H_B. We are picturing a process which starts and stops – and that both work and heat are exchanged only during the process. Sometimes the process is instigated by some purposeful activity, usually a change of external parameters such as the motion of a piston. Other times the process is instigated as a result of the fact that the two systems A and B are at different temperatures and therefore not in thermal equilibrium. The interaction Hamiltonian mediates the process.

  Reif asserts (reasonably, I think) that mechanical work W on system A changes the energy eigenvalues of H_A (so H'_A has different energy levels than H_A) but that heat absorption Q doesn't change the energy levels of H_A. In general, the interaction process will cause state transitions so that after the interaction, A' will have a different population of states than A and may even have different states (or different energy levels). In a statistical ensemble representing A and A', the mean population of states will thus change.

  Of particular importance to the superfluid discussion, is the system consisting of the fluid which we'll call A and the wall which we'll call B. In the wall's frame of reference, the wall stays fixed and motionless regardless of what the fluid does, so there are no "external" parameters that change. The internal energy of the wall can only change by heat absorption and thus by re-population of the original states.

- Now I'm going to stretch with an explanation that sounds right, but might be too simplistic to be trustworthy. Suppose that the fluid is at very low temperature and thus, in its rest frame, has most of its quasiparticles occupying its ground state with $\mathbf{p} = 0$. We ask the question "is it likely that a transition will occur where one of those ground-state quasiparticles will be excited to another state. All states are indexed by a momentum $\mathbf{p}$ and have a corresponding energy of $E(\mathbf{p})$. Suppose that in the wall's reference frame, the transition is positive (i.e., the change of fluid internal energy in the fluid is positive). This would imply that the change of energy in the wall is negative.

Is that impossible? No. At least, I don't think so. However, I think it turns out to be highly improbable. There is a quantitative analysis in Reif that I think is applicable. The key section is 15.2 and discusses what happens when a system (e.g. the fluid) is in contact with a heat reservoir (e.g. the wall), **even** when the two are possibly not in thermal equilibrium. First note that this is precisely the situation of interest to us: we want to show that

- above the critical velocity, the moving fluid and the wall remain in equilibrium – that is, their energies don't change,
- below the critical velocity, the moving fluid slows down and thus isn't in equilibrium until it stops moving.

The Reif analysis boils down to the equation

$$\frac{W_{rs}}{W_{sr}} = e^{-\beta(E_r - E_s)} \tag{23}$$

where, for our fluid example, r and s are two states of the fluid with energies E_r and E_s respectively, and W_{ab} is the transition probability from state a to state b. Consider the wall's frame of reference. The fluid's energy states are indexed by the quasiparticle momentum **$\mathbf{p}$ in the fluid's rest frame**. We have been looking at a transition, in the fluid rest frame, from the fluid's ground state $\mathbf{p} = 0$ to a state with momentum $\mathbf{p} \neq 0$ (more precisely, we have been considering a transition from where the occupation number of the $\mathbf{p}$ state goes from 0 to 1).In the wall frame, the state energies change by $E(\mathbf{p}) - \mathbf{v} \cdot \mathbf{p}$, so we have

$$\frac{W_{0,\mathbf{p}}}{W_{\mathbf{p},0}} = e^{-\beta(E(\mathbf{p}) - \mathbf{v}\cdot\mathbf{p})}. \tag{24}$$

Suppose $E(\mathbf{p}) - \mathbf{v}\cdot\mathbf{p} > 0$. Then the RHS above is extremely small when β is large, so a transition to an excited state is highly unlikely to happen (relative to a transition to the ground state). This demonstrates that the fluid does not lose energy (so doesn't slow down)[9] On the other hand, if $E(\mathbf{p}) - \mathbf{v} \cdot \mathbf{p} < 0$, the transition to the excited state (lower

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[9] Said differently, if all "excited states" have more energy than the states where all particles are traveling at velocity $\mathbf{v}$, then the latter state (which is the same as the ground state in the fluid rest-frame) is the ground state in the wall's rest frame. Therefore at 0 temperature, in equilibrium all particles are in that ground state – i.e., they don't slow down. On the other hand, if some "excited states" have lower wall-frame energy than the state where all particles move at $\mathbf{v}$, then at 0 temperature, in equilibrium the fluid will eventually settle into the state where all motion is at the critical velocity.

energy) is likely. This is the hallmark of energy-dissipation, and will have the effect of reducing the speed of the fluid.

- The way this is summarized in the literature, for example in the Landau and Lifshitz series, is to say that a negative change in the internal energy of A is "dissipated". What this means in our superfluid example, is that heat would be absorbed by the wall, allowing the fluid to lose energy, and ultimately slowing down as a consequence. On the other hand, a positive change in the internal energy of A could not occur as a result of dissipation since dissipation is strictly a positive increase of the wall's energy. Hence the fluid wouldn't slow down.

- There may be an alternative analysis following the logic used to derive the origin of friction of an object moving through a gas or fluid. Reif discusses this at length in his chapters 15.7 and 15.8 on the relation between dissipation, fluctuating forces and friction. In general, this is part of the subject of irreversible processes and in modern parlance is often subsumed as part of the topic of "coarse-graining". It should also be mentioned that Reif also discusses viscosity, as separate from friction. The same kinds of principle apply.