Topics prerequisite to Chapter 42

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1 Introduction

These notes supplement the set of notes "Basic formulation of Lagrangian and Hamiltonian classical and quantum field theory" as coverage of material in Lancaster prerequisite to Chapter 42. The following topics are covered (the Chapter numbers refer to Lancaster):

- Statistical mechanics (Chapter 21 – only a tiny bit)
- Noether's theorem (Chapter 10)
- Non-relativistic charge symmetry (end of Chapter 12.3)
- Broken symmetry (Chapter 26)
- Coherent states (Chapter 27)
- Topological objects (Chapter 29)

At some point, I'll try to insert a section explaining what a superfluid looks like and how we're aiming to characterize it.

2 Statistical Mechanics in Brief

THE PURPOSE OF THIS SECTION IS TO EXPLAIN WHY PARTICLES AT NEAR-ZERO TEMPERATURES, HAVE VERY LITTLE ENERGY.

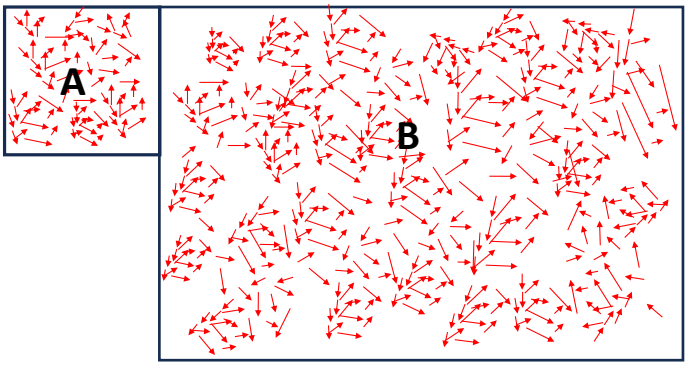
- Statistical mechanics describes the macro-dynamics of a large number, N, of objects constrained by the values of macro-observables such as pressure and temperature.
- A *state*:
  - Classical physics – specify the position and momentum of N particles. $(\vec{q}_1, \vec{p}_1, \vec{q}_2, \vec{p}_2, ..., \vec{q}_n, \vec{p}_n)$
  - Quantum physics – specify a state $|\Psi\rangle$ in the N-particle Hilbert space.
- **The postulate of equal a priori probabilities**: "An isolated system in equilibrium is equally likely to be in any of its accessible states." (F. Reif). How does this work?

Table 1: Total of three dice adding up to 7

Die 1	**Die 2**	**Die 3**
1	1	5
1	5	1
1	2	4
1	4	2
1	3	3
2	1	4
2	4	1
2	2	3
2	3	2
...	...	...
...	...	...

There are 15 possible configurations of three dice that add up to 7. In the absence of any other information, each such configuration is equally likely. Since there are 5 configurations where the first die is 1, then the probability of the first die = 1, is $5/15 = 33\%$. Similarly, the probability of the first die = 2 is $4/15 = 27\%$ and the probability of the first die = 5 is $1/15 = 7\%$.

The key takeaway: Imagine that "die 1" is a subsystem called "subsystem RED" and that "dice 1 & 2" make up a subsystem called "subsystem BLACK". For each total value of "subsystem RED", there is exactly one state. For each total value of "subsystem BLACK", the number of states depends on the value. For example, when the total value of "subsystem BLACK" is = 6, there are 5 states of "subsystem BLACK". When the total system value is fixed (e.g. = 7) then **the probability that "subsystem RED" has a particular value depends on the number of states for the two subsystems. In our example, lower values of "subsystem RED" have higher probabilities.**



- Imagine two subsystems in thermal interaction with one another and with a **total energy** of E^0. The total energy is obtained by adding up the energy of individual molecules (*each acts like a die*). System A is much smaller than System B and therefore has fewer states. (We call System B a "heat reservoir".) Then – assuming the postulate of equal a priori probabilities – we can compute the probability of finding a state s of System A with energy E.

$$P_s = Ce^{-\beta E} \tag{1}$$

  (See Lancaster page 196.) This is known as the Boltzmann factor. The parameter β is related to the number of states available to the reservoir (system B) and turns out to be inversely proportional to the temperature.

$$\beta = \frac{1}{kT} \tag{2}$$

  where k is the Boltzmann constant.

- **Therefore when β is large (temperature is small) energetic states are suppressed.**

3 Noether's theorem

Lancaster Chapter 10.

When there is a continuous symmetry, there is a divergenceless current and also a conserved charge.

Let $\phi_\alpha(x)$ be a family of fields parameterized by α so that $\phi_0(x) = \phi(x)$ **and** $\mathcal{L}(\phi_\alpha) = \mathcal{L}(\phi)$. This is a symmetry. For example consider the Lagrangian

$$\mathcal{L}(\phi) = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi - g(\phi^\dagger \phi)^2. \tag{3}$$

Let $\phi_\alpha = e^{i\alpha}\phi$. We see that if in Eq. (3) we substitute ϕ_α for ϕ, and $\phi_\alpha^\dagger$ for $\phi^\dagger$, then the Lagrangian doesn't depend on α.

Then define

$$D\phi = \left.\frac{\partial \phi_\alpha}{\partial \alpha}\right|_{\alpha=0}. \tag{4}$$

Noether's theorem says that there is a *current* known as the Noether current and defined as

$$J_N^\mu(x) = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}(x) \cdot D\phi(x), \tag{5}$$

and with the property that

$$\partial_\mu J_N^\mu(x) = 0. \tag{6}$$

Note that Lancaster presents a slightly more general version of Noether's theorem, but we won't require that. On the other hand, Lancaster has not explicitly referenced the situation where there are multiple fields that are transformed by the symmetry. I've chosen to remedy this by a slight abuse of notation. Extend ϕ to mean the vector $(\phi^1, \phi^2, ...)$. $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$ is also a vector $\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^1)}, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^2)}, ...\right)$, as is $D\phi = (D\phi^1, D\phi^2, ...)$.

Furthermore, as a consequence of the above, there is a conserved charge Q_N (again, this could be a vector) defined as

$$Q_N = \int d^3x J_N^0. \tag{7}$$

The conservation of Q_N means that it doesn't change with time. In other words,

$$\frac{dQ_N}{dt} = 0. \tag{8}$$

All of the above holds both in the classical theory and later, when we promote the classical fields to quantum operators, it also holds in the quantum theory.

In addition, in the quantum theory, we can derive

$$\left[\hat{Q}_N, \hat{\phi}\right] = -iD\hat{\phi}. \tag{9}$$

The "hat" denotes that the quantities are operators.

Let's apply all this to ϕ_α (see Lancaster 12.2).

$$\begin{aligned} D\phi(x) &= \left.\frac{\partial \phi_\alpha}{\partial \alpha}\right|_{\alpha=0} \\ &= \left.\frac{\partial \left(e^{i\alpha}\phi\right)}{\partial \alpha}\right|_{\alpha=0} \\ &= i\phi. \end{aligned} \tag{10}$$

We're not done. There is a second current arising from the transformation of $\phi^\dagger$. In just the way we obtained $D\phi$, we can also derive that $D\phi^\dagger = -i\phi^\dagger$.

Now we can construct the Noether current using Eqs. (5) and (3), and setting $\boldsymbol{\phi} = \left(\phi, \phi^\dagger\right)$.

$$\begin{aligned} J^\mu_N(x) &= \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \boldsymbol{\phi}\right)}(x) \cdot D\boldsymbol{\phi}(x) \\ &= \left(\partial^\mu \phi^\dagger, \partial^\mu \phi\right) \cdot \left(i\phi, -i\phi^\dagger\right) \\ &= i\phi\partial^\mu\phi^\dagger - i\phi^\dagger\partial^\mu\phi. \end{aligned} \tag{11}$$

Finally, we can construct the charge using Eq. (12).

$$\begin{aligned} Q_N &= \int d^3x J^0_N \\ &= i\int d^3x \left(\phi(x)\partial^0\phi^\dagger(x) - i\phi^\dagger(x)\partial^0\phi(x)\right). \end{aligned} \tag{12}$$

This quantity is time-independent, i.e., conserved.

4 Coherent States

4.1 A whirlwind tour of quantum mechanics

In classical particle mechanics, we have a quantity like "the position of the particle" $\mathbf{x}$ at "a time t". In quantum mechanics, we have a quantity like "the expectation value of a position operator for a particle-state $|s\rangle$ at "a time t". We

picture these two quantities as representing more or less the same kind of thing. However, in classical mechanics, when we say "$\mathbf{x}$(t)", we mean two things: (a) the particle position – measured from the origin – at time t is $\mathbf{x}$ and (b) the particle position – measured from ϵ – at time t is <u>not</u> $\mathbf{x}$. We can say that the position of the particle is "exact." The same might not be true in quantum mechanics. The expectation value of the position operator at time t is $\mathbf{x}$ but if you shift the position operator to an operator measuring the position displaced by ϵ, the expectation value may be non-zero. Or, another (intuitively easier) way of saying the same thing, is that the dispersion (after all, the term 'expectation value' tells us we are discussing probabilities) is non-zero.

The challenge in QM is how to picture all this. Sometimes we focus on the state, sometimes we focus on the observable, and sometimes we focus on the expectation value. These objects are intertwined in ways unlike the corresponding classical objects. Moreover, when we talk about time-evolution, sometimes we describe the state as evolving, sometimes we describe the observable as evolving and sometimes both.

One way of proceeding is to identify certain observable-state combinations that are "exact". As an example, consider the state $|x\rangle$ defined to be the state with the property that $\hat{x}|x\rangle = x|x\rangle$. Be careful with the overloading of the symbol "x". The object $\hat{x}$ should not be pronounced "hat x". Rather, it should be pronounced "the position operator". At the same time, recognize that the notation $|x\rangle$ is entirely ambiguous. It's meaning is clear only in context. In particular, in the context above, its definition is "a (normalized)[1] eigenstate of the position operator, with an eigenvalue of x". Just to appreciate the subtlety, consider the expectation value of the position operator for the state which is a position-eigenstate with eigenvalue "y". We would write this as $\langle y|\hat{x}|y\rangle$ and its value would be y. I recommend that you think this through until you are completely comfortable with its meaning. Finally note that the dispersion (of the probability distribution of $|x\rangle$ with respect to the position operator) is 0. Hence we have a "natural" way of dealing with an "exact" position in quantum mechanics.

Of course, what goes along with this formalism is that states that are position-exact, are incapable of being momentum-exact. That's Heisenberg's uncertainty principle. It's also what makes it very difficult to have an intuitively

[1]For the purists amongst you, the position eigenstates, at least in the case where position lies on a continuum, aren't actually normalizable. We need to treat things more rigorously but that can be done.

clear picture of states – even position eigenstates. Be that as it may, QM proceeds by noting that any state can be written as a superposition of position eigenstates. Sometimes the position isn't really important, but the energy is. In that case, we might be interested in states of "exact energy". Since such states are superpositions of position eigenstates, we tend to picture them in terms of a weighted-graph of the position-eigenvalues, where the weighting is proportional to the amplitudes of the corresponding position-eigenstates in the superposition. At best, this is just a heuristic picture.

Here's where we're going. Just like our definition of position eigenstates, we can define field eigenstates by $\hat{\phi}(\mathbf{x})|\phi\rangle = \phi(\mathbf{x})|\phi\rangle$. Just as for the position operator, we shouldn't pronounce $\hat{\phi}$ as "hat ϕ" but rather as "the field operator". Again, the state notation $|\phi\rangle$ is ambiguous and must be understood from context. In this case, the context involves the field operator and the definition of the state is as "the eigenstate of the field operator, which has the eigenvalue(s) $\phi(\mathbf{x})$. Just as before, these states are "exact" with respect to the field operators.

And just as before (more or less), these states – known as coherent states – can be superimposed to form any other state of interest in our theory. Much of our experience in fundamental-particle field theory is centered around scattering experiments and in particular, scattering at relatively large momenta. The momentum eigenstates can be written in terms of coherent states, but this doesn't typically lead to useful insights. In many-body theory, our experiences are more varied. In particular, at very low temperatures and very large systems (e.g. a fluid), the states of interest are the field eigenstates. These are most readily visualized as the field solutions to classical equations of motion.

4.1.1 Proof that the sum $\sum_{n=1}^{\infty} n = -\frac{1}{12}$.

One way of understanding the relationship between quantum and classical many-body theories, is to note that (a) classical theories tend to agree with quantum theories when most particles have low momenta (b) large-momenta behavior modifies the classical theories. The exact way in which all this happens can be quite subtle, especially because the unmodified classical theory often has divergences when extended to include arbitrarily large momenta. In fact, the classical statistical theory of electromagnetism in the late 19th century led to an energy-divergence known as the ultraviolet catastrophe, a problem that was solved by Planck's discovery of quantum mechanics. I

think that one interesting way to illustrate how this kind of thing works, is to examine the "classical problem" of summing integers.

$$\begin{aligned} c &= 1+2+3+4+5+6+... \\ 4c &= \quad 4 \quad +8 \quad +12+... \\ c-4c &= 1-2+3-4+5-6+... \\ -3c &= \lim_{x\to 1}\frac{d}{dx}\frac{-1}{(1+x)} = \frac{1}{(1+1)^2} = \frac{1}{4} \\ c &= -\frac{1}{12} \end{aligned} \tag{13}$$

Clearly this is nonsense. And yet ...

See Wikipedia under "$1+2+3+4..$". This summation has a famous history and the above trick was even given by the famous mathematician Ramanujan. It's an example of the kind of practice engaged by field theorists and which goes under the heading of "renormalization theory" (sometimes things are less abstruse – for example, Planck's original cure of the ultraviolet catastrophe). The point is that there is a way in which the sum can be modified to something that converges but depends on a parameter s whose limit is taken after the sum, but not before. For example, consider the function

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + ... \tag{14}$$

When $s \to -1$, the series looks the same as before and has the same issues as before. However, if we follow the steps in the trick-solution, we end up with an expression that can be analytically continued in s from regions where each sum converges, to $s = -1$, and which leads to the same result. One might object to using the trick of analytic continuation, but there are other regularization methods, including one devised by Ramanujan, that lead to sensible answers.

A physicist has the obligation to justify the use of regularization methods, and more importantly has to assure that different "justified" methods lead to the same answers. That has been the focus of research of many physicists and mathematical physicists for decades, with lots of progress both theoretically and with computer simulations.

The particular series above appears in string theory and with accepted regularization methods leads ultimately to the famous result that certain string theories are only consistent in 26 dimensions.

4.2 Motivation: connecting classical EM to QED

What is the relationship between a classical field like the electric field $\mathbf{E}$, and the quantum field operator $\hat{\mathbf{E}}$? It seems plausible that in some limit, the two quantities are related. In the usual correspondence between quantum and classical physics, we'd expect that the classical field $\mathbf{E}$ is the "measured value of the field observable in some state of the system". We'd write this as

$$\langle s|\hat{\mathbf{E}}|s\rangle. \tag{15}$$

How would we characterize that state? Let's imagine that the state describes a single photon of momentum $\mathbf{p}'$. Then we would write

$$\langle \mathbf{p}'|\hat{\mathbf{E}}|\mathbf{p}'\rangle. \tag{16}$$

Actually, for the electric field, we'd also want to specify the polarization of the photon but let's ignore that for simplicity. We'll now show this expectation value is 0.

In the spirit of simplicity, let's pretend that the electric field is a scalar $\hat{E}(\mathbf{x})$ and that we can do the usual mode expansion so that, up to normalization,

$$\hat{E}(\mathbf{x}) = \hat{E}_C(\mathbf{x}) + \hat{E}_C^\dagger(\mathbf{x}) \tag{17}$$

where

$$\hat{E}_C(\mathbf{x}) = \int \frac{d^3p}{p} \hat{a}_{\mathbf{p}} e^{-i(pt-\mathbf{p}\cdot\mathbf{x})}. \tag{18}$$

Since we plan to evaluate $\langle \mathbf{p}'|\hat{E}_C|\mathbf{p}'\rangle$, let's consider the term in the mode expansion, $\langle \mathbf{p}'|\hat{a}_{\mathbf{p}}|\mathbf{p}'\rangle$. Recall that $|\mathbf{p}'\rangle = \hat{a}^\dagger_{\mathbf{p}'}|0\rangle$. Also recall that

$$\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}'} = \hat{a}^\dagger_{\mathbf{p}'}\hat{a}_{\mathbf{p}} + \delta^{(3)}(\mathbf{p}-\mathbf{p}'). \tag{19}$$

Therefore

$$\begin{aligned} \hat{a}_{\mathbf{p}}|\mathbf{p}'\rangle &= \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}'}|0\rangle \\ &= \hat{a}^\dagger_{\mathbf{p}'}\hat{a}_{\mathbf{p}}|0\rangle + \delta^{(3)}(\mathbf{p}-\mathbf{p}')|0\rangle \\ &= \delta^{(3)}(\mathbf{p}-\mathbf{p}')|0\rangle, \end{aligned} \tag{20}$$

where the last line follows from the fact that $\hat{a}_{\mathbf{p}}|0\rangle = 0$.

Now we are ready to evaluate $\langle \mathbf{p}'|\hat{a}_{\mathbf{p}}|\mathbf{p}'\rangle$. From the last equalities, we get

$$\begin{aligned} \langle \mathbf{p}'|\hat{a}_{\mathbf{p}}|\mathbf{p}'\rangle &= \delta^{(3)}(\mathbf{p}-\mathbf{p}')\langle \mathbf{p}'|0\rangle \\ &= \delta^{(3)}(\mathbf{p}-\mathbf{p}')\langle 0|\hat{a}_{\mathbf{p}'}|0\rangle \\ &= 0, \end{aligned} \tag{21}$$

where the second line follows from the fact that $\langle \mathbf{p}'| = \langle 0|\hat{a}_{\mathbf{p}'}$ and the third line follows from the fact that $\hat{a}_{\mathbf{p}'}|0\rangle = 0$.

We've shown that the mode expansion annihilation operators contribute 0 to the expectation value of the electric field in a single photon state. The same kind of argument shows that the mode creation operators act the same way, and therefore that the electric field has an expectation value of 0.

Although it might not be immediately obvious, it turns out that the field expectation value in an N-photon state is also 0. The reason is that the ladder operators change the photon number on the right by either plus or minus 1, and therefore we end up with the state on the left having a different number of photons than the state on the right – which means the states are orthogonal.

This result is a bit strange. How can our classical limit make sense if the electric field has an expectation value of 0 for any N-photon state? The answer is that the appropriate states to be used in the classical limit, are states with an inexact number of photons. More precisely, the states must be superpositions of differing numbers of photons. For example, a one-photon state plus a two-photon state etc. This turns out to be the key insight for the construction of coherent states. The development of quantum optics by the Nobel laureate Roy Glauber, made extensive use of these coherent states (which were given their name by Glauber, but were first discovered by Schrodinger).

What we want, in connecting quantum to classical electromagnetism, is a way to define a state whose field expectation value is $\mathcal{E}(\mathbf{x})$, for any choice of real $\mathcal{E}$. One way to achieve this, would be to find a (normalized) state, which we'll call $|\mathcal{E}\rangle$, with the property that $\hat{E}_C(\mathbf{x})|\mathcal{E}\rangle = \mathcal{E}_C(\mathbf{x})|\mathcal{E}\rangle$ and where $\mathcal{E}_C$ is defined, nonuniquely, by $\mathcal{E} = \mathcal{E}_C + \mathcal{E}_C^*$.[2] That is, the state $|\mathcal{E}\rangle$ is simultaneously an eigenvector of the operator $\hat{E}_C(\mathbf{x})$ **for every value of x.** Notice that $\langle\mathcal{E}|\hat{E}_C^\dagger(\mathbf{x}) = \langle\mathcal{E}|\mathcal{E}_C^*(\mathbf{x})$. From these observations, we see that $|\mathcal{E}\rangle$

[2]It would seem that we are adding some unnecessary complexity (no pun intended) by rewriting the real field $\mathcal{E}$ as a sum of two complex fields whose imaginary parts add up to 0. I believe we need to do this because of properties of the ladder operators that we'll encounter later, although I haven't tracked down the culprit that forces us, in this treatment of coherent states, to deal with non-real fields.

has the property that

$$\begin{aligned}\langle\mathcal{E}|\hat{E}(\mathbf{x})|\mathcal{E}\rangle &= \langle\mathcal{E}|\hat{E}_C(\mathbf{x}) + \hat{E}_C^\dagger(\mathbf{x})|\mathcal{E}\rangle \\ &= (\mathcal{E}_C(\mathbf{x}) + \mathcal{E}_C^*(\mathbf{x}))\,\langle\mathcal{E}|\mathcal{E}\rangle \\ &= \mathcal{E}(\mathbf{x})\end{aligned} \tag{22}$$

where the last line follows from the normalization of the state. So now we have found a state whose field expectation value is whatever we want it to be. We call such a state a *coherent state*.

Since the fields obey the Euler-Lagrange and other equations, we can see that the expectation values will obey those equations, so that gives us a clear connection between classical EM and QED.

4.3 Constructing the coherent states

We continue to work backwards. The classical electric field (simplified to be a scalar field) can be written (up to normalization) as

$$\mathcal{E} = \mathcal{E}_C + \mathcal{E}_C^* \tag{23}$$

where $\mathcal{E}_C$ can be written as a Fourier transform (a.k.a. mode expansion)

$$\mathcal{E}_C(\mathbf{x}) = \int \frac{d^3p}{p}\alpha(\mathbf{p})e^{-i(pt-\mathbf{p}\cdot\mathbf{x})}. \tag{24}$$

I'll use the notation $\mathcal{E}_C$ rather than $\hat{E}$ to make it clear that we're talking about an ordinary (complex) function. The coefficients $\alpha(\mathbf{p})$ are complex functions and are essentially the Fourier coefficients of the electric field.

Now suppose we can find a state which we'll call $|\alpha\rangle$, with the property that $a_\mathbf{p}|\alpha\rangle = \alpha(\mathbf{p})|\alpha\rangle$. Then, plugging in Eq. (18).

$$\begin{aligned}\hat{E}_C(\mathbf{x})|\alpha\rangle &= \int \frac{d^3p}{p}\hat{a}_\mathbf{p}e^{-i(pt-\mathbf{p}\cdot\mathbf{x})}|\alpha\rangle \\ &= \int \frac{d^3p}{p}\alpha(\mathbf{p})e^{-i(pt-\mathbf{p}\cdot\mathbf{x})}|\alpha\rangle \\ &= \mathcal{E}_C(\mathbf{x})|\alpha\rangle.\end{aligned} \tag{25}$$

If we compare this to the previous section, we see that the way to construct an electric-field eigenstate, is to construct an annihilation-operator eigenstate characterized by the Fourier coefficients of the electric field.

So our problem has reduced to the problem of solving $\hat{a}_{\mathbf{p}}|\alpha\rangle = \alpha(\mathbf{p})|\alpha\rangle$. We make one further observation which simplifies our task. Recall that states in the Hilbert space are constructed from the Fock state basis, where basis vectors are of the form $|n_{\mathbf{p}_1}\rangle \otimes |n_{\mathbf{p}_2}\rangle ... \otimes |n_{\mathbf{p}_m}\rangle$, with $n_{\mathbf{p}}$ denoting the number of particles having momentum $\mathbf{p}$. The Fock space is written formally as $\bigotimes_{\mathbf{p}} H_{\mathbf{p}}$ or suggestively as $H_{\mathbf{p}_1} \otimes H_{\mathbf{p}_2} \otimes ....$

We employ this Fock-space decomposition to further reduce our coherent-state construction to the simpler problem of $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. In other words, construct a state which is an eigenvector of the annihilation operator, with the eigenvalue α. This formally becomes an exercise in simple harmonic oscillator quantum mechanics and is Lancaster's starting point. The simplest thing to do is to write down the answer and then show that it's correct.

$$|\alpha\rangle = c_0 \left(1 + \frac{\alpha}{1!}\hat{a}^\dagger + \frac{\alpha^2}{2!}\left(\hat{a}^\dagger\right)^2 + \frac{\alpha^3}{3!}\left(\hat{a}^\dagger\right)^3 + ...\right)|0\rangle, \tag{26}$$

where c_0 is chosen so that $\langle\alpha|\alpha\rangle = 1$. Let's see what happens when we apply $\hat{a}$ to one of the terms in the sum, for example the term $c_0\frac{\alpha^2}{2!}\left(\hat{a}^\dagger\right)^2|0\rangle$. We use the identity, to be proven shortly,

$$\hat{a}\left(\hat{a}^\dagger\right)^n = \left(\hat{a}^\dagger\right)^n \hat{a} + n\left(\hat{a}^\dagger\right)^{n-1}. \tag{27}$$

Then

$$\begin{aligned}\hat{a}\left(c_0\frac{\alpha^2}{2!}\left(\hat{a}^\dagger\right)^2\right)|0\rangle &= c_0\frac{\alpha^2}{2!}\hat{a}\left(\hat{a}^\dagger\right)^2|0\rangle \\ &= c_0\frac{\alpha^2}{2!}\left(\left(\hat{a}\right)^2\hat{a} + 2\hat{a}^\dagger\right)|0\rangle \\ &= \alpha c_0\left(\frac{\alpha}{1!}\hat{a}^\dagger\right)|0\rangle,\end{aligned} \tag{28}$$

where the last line follows from the fact that $\hat{a}|0\rangle = 0$. We see that by applying the annihilation operator to the third term in the series for $|\alpha\rangle$, that we get the second term in the series, but multiplied by α. It turns out that this pattern holds for all terms in the series – namely that the annihilation operator changes the n^{th} term into α times the $(n-1)^{st}$ term. This demonstrates that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, as claimed.

The only remaining part of the proof is to show Eq. (27). We proceed by induction. Suppose we've shown that $\hat{a}\left(\hat{a}^\dagger\right)^n = \left(\hat{a}^\dagger\right)^n\hat{a} + n\left(\hat{a}^\dagger\right)^{n-1}$. Now

consider $\hat{a}\left(\hat{a}^\dagger\right)^{n+1}$.

$$\begin{aligned}\hat{a}\left(\hat{a}^\dagger\right)^{n+1} &= \hat{a}\hat{a}^\dagger\left(\hat{a}^\dagger\right)^n \\ &= \left(\hat{a}^\dagger\hat{a}+1\right)\left(\hat{a}^\dagger\right)^n \\ &= \hat{a}^\dagger\hat{a}\left(\hat{a}^\dagger\right)^n + \left(\hat{a}^\dagger\right)^n \\ &= \hat{a}^\dagger\left(\left(\hat{a}^\dagger\right)^n\hat{a} + n\left(\hat{a}^\dagger\right)^{n-1}\right) + \left(\hat{a}^\dagger\right)^n \\ &= \left(\hat{a}^\dagger\right)^{n+1}\hat{a} + (n+1)\left(\hat{a}^\dagger\right)^n.\end{aligned} \tag{29}$$

This proves the induction step (and it's a simple matter to show that the theorem is true for $n = 1$, starting off the induction sequence).

Now rewrite the series for the momentum-$\mathbf{p}$ coherent state $|\alpha\rangle_{\mathbf{p}}$, as an exponential (from which we can easily derive c_0), so that

$$|\alpha\rangle_{\mathbf{p}} = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger_{\mathbf{p}}}|0\rangle. \tag{30}$$

Finally we deal with the coherent states of the multiparticle Fock space. This is just a product-state so can be written as

$$\begin{aligned}|\alpha\rangle &= \prod_{\mathbf{p}} e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger_{\mathbf{p}}}|0\rangle \\ &= e^{-\frac{N|\alpha|^2}{2}} e^{\alpha\int d^3p\hat{a}^\dagger_{\mathbf{p}}}|0\rangle\end{aligned} \tag{31}$$

where N is the total number of particles in the system (in elementary particle field theory, N is an infinite constant, but which may factor out in applications.)

4.4 Some properties of coherent states

The usual way to find properties of a state is to compute the expectation value of observables as measured in that state. So if the state is $|\alpha\rangle$ and if $\mathcal{O}$ is an observable, we would compute the expectation value $\langle\alpha|\mathcal{O}|\alpha\rangle$.

Another popular way to find properties (which turns out to be equivalent to the above) is to compute the probability amplitude that the state of interest, in this case $|\alpha\rangle$, is found to be in a known state $|s\rangle$. We would compute $\langle s|\alpha\rangle$. Another way of thinking of this, is that $|\alpha\rangle$ is a superposition of orthonormal states, one of which is $|s\rangle$. The coefficient of $|s\rangle$ in that superposition, is $\langle s|\alpha\rangle$.

4.4.1 Number of quanta

We start by considering states that describe particles of momentum $\mathbf{p}$. The math is just the same as for a single linear harmonic oscillator. How many particles are there in a coherent state? There are several ways of approaching that question. But what's critical to note, is that there isn't a precise answer. In fact, the situation is reminisce of the Heisenberg Uncertaintly Principle. If we know that a state is coherent, then it isn't certain how many particles are in that state. Instead, we can ask what the probability is of finding n particles.

$$P_n = |\langle n|\alpha\rangle|^2. \tag{32}$$

This can be easily computed by noting that

$$|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|0\rangle. \tag{33}$$

Then use the expansion of $|\alpha\rangle$ of Eq. (26), and the fact that $\langle n|m\rangle = \delta_{n,m}$, to get

$$P_n = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \tag{34}$$

It is easy to compute the average and standard deviation of this distribution. Equivalently, we note that the average number of particles is just the expectation of the number observable $\hat{a}^\dagger\hat{a}$ so we compute

$$\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2\langle\alpha|\alpha\rangle = |\alpha|^2, \tag{35}$$

where we've invoked the definition of the coherent state, namely $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$.

What we've shown is that we can construct a coherent state with an average of N particles, by simply taking $\alpha = \sqrt{N}$.

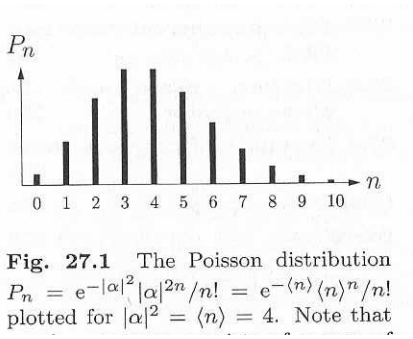


Fig. 27.1 The Poisson distribution $P_n = \mathrm{e}^{-|\alpha|^2}|\alpha|^{2n}/n! = \mathrm{e}^{-\langle n\rangle}\langle n\rangle^n/n!$ plotted for $|\alpha|^2 = \langle n\rangle = 4$. Note that

Figure 1: Lancaster Fig. 27.1

Lancaster also derives the fractional uncertainty (standard deviation of the random variable $\hat{n}$ divided by its mean) as

$$\frac{\Delta\hat{n}}{\langle\hat{n}\rangle} = \frac{1}{\sqrt{N}}. \tag{36}$$

This can be interpreted by saying that for a coherent state with a very large average number of particles, the uncertainty (a.k.a. dispersion) is small. Often we are interested in situations where the number of particles is "macroscopic" – of the order of 10^{23}.

As a reminder, this distribution is for a specific momentum. When describing a field, each momentum is associated with an "α" that is the field's Fourier coefficient – so the average number of particles of momentum $\mathbf{p}$ is the modulus-squared of the Fourier coefficient.

4.4.2 Time evolution

The following pertains to the free multiparticle Hamiltonian. Recall that the free Hamiltonian acts on number eigenstates by

$$\hat{H} = \sum_{\mathbf{p}} \left(\hat{n}_{\mathbf{p}} + \frac{1}{2}\right) \sqrt{p^2 + m^2} \tag{37}$$

where $\hat{n}_{\mathbf{p}}$ is the number operator which acts on the state $\left(|n_{\mathbf{p}_1}\rangle \otimes ...|n_{\mathbf{p}}\rangle... \otimes |n_{\mathbf{p}_n}\rangle...\right)$ as $\hat{n}_{\mathbf{p}} \left(|n_{\mathbf{p}_1}\rangle \otimes ...|n_{\mathbf{p}}\rangle... \otimes |n_{\mathbf{p}_n}\rangle...\right) = n_{\mathbf{p}} \left(|n_{\mathbf{p}_1}\rangle \otimes ...|n_{\mathbf{p}}\rangle... \otimes |n_{\mathbf{p}_n}\rangle...\right)$. We've shown that $\hat{n}_{\mathbf{p}} = \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}$. Also, the number $\frac{1}{2}$ which appears in the Hamiltonian is just a constant (called the "0-point energy") that can be dropped.

The state $|\alpha\rangle$ has a time-evolution governed by Schrodinger's equation and which can be written as

$$\begin{aligned} |\alpha\rangle(t) &= e^{-i\hat{H}t}|\alpha\rangle \\ &= e^{-i\left(\sum_{\mathbf{p}}\left(\hat{n}_{\mathbf{p}}+\frac{1}{2}\right)\right)\sqrt{p^2+m^2}\Big)t}|\alpha\rangle \\ &= \prod_{\mathbf{p}} e^{-i\left(\left(\hat{n}_{\mathbf{p}}+\frac{1}{2}\right)\right)\sqrt{p^2+m^2}\Big)t}|\alpha\rangle. \end{aligned} \tag{38}$$

Sometimes, instead of the sum in the exponent, we write an integral to indicate that continuum limits are being taken. However, I don't know what

the appropriate continuum-limit "product" notation would be, so I've left things as sums and products. Once again, we expand $|\alpha\rangle_{\mathbf{p}}$ with Eq. (26)

$$\begin{aligned} |\alpha\rangle_{\mathbf{p}} &= c_0 \left(1 + \frac{\alpha}{1!}\hat{a}^\dagger_{\mathbf{p}} + \frac{\alpha^2}{2!}\left(\hat{a}^\dagger_{\mathbf{p}}\right)^2 + \frac{\alpha^3}{3!}\left(\hat{a}^\dagger_{\mathbf{p}}\right)^3 + ...\right)|0\rangle \\ &= c_0 \left(|0\rangle + \frac{\alpha}{\sqrt{1!}}|1\rangle + \frac{\alpha^2}{\sqrt{2!}}|2\rangle + ...\right). \end{aligned} \tag{39}$$

I've been slightly sloppy with notation. A different value of α can be used for each momentum, so we should write $\alpha(\mathbf{p})$.

When we apply the time evolution operator as in Eq. (38), we obtain

$$|\alpha(\mathbf{p})\rangle(t) = |\alpha(\mathbf{p})e^{-i\sqrt{p^2+m^2}t}\rangle. \tag{40}$$

The notation is interpreted as follows: The state $|\alpha(\mathbf{p})\rangle$ is a multiparticle state represented in momentum Fock space and the state $|\alpha\rangle$ is $|\alpha\rangle = \bigotimes_{\mathbf{p}} |\alpha(\mathbf{p})\rangle$. On the LHS, we have the state which has evolved from the coherent state $|\alpha(\mathbf{p})\rangle$ after a time t. In this notation, $|\alpha(\mathbf{p})\rangle(0) = |\alpha(\mathbf{p})\rangle$. On the RHS, we have a phase factor multiplied by a time-dependent coherent state where $\alpha(\mathbf{p}) \to \alpha(\mathbf{p})e^{-i\sqrt{p^2+m^2}t}$.

The key observation here, is that a coherent state evolves into a new coherent state.

4.4.3 Completeness

Start by considering a single mode Hilbert space – i.e., the space of states of particles with momentum $\mathbf{p}$, and an arbitrary state $|s\rangle$ in that space. Then it can be shown (the proof is a bit involved) that we can write

$$|s\rangle = \frac{1}{\pi}\int d^2\alpha |\alpha\rangle\langle\alpha|s\rangle \tag{41}$$

where the integral is over both the real and imaginary part of α. This identity can be summarized as a "partition of unity"

$$\frac{1}{\pi}\int d^2\alpha |\alpha\rangle\langle\alpha| = \mathbb{1} \tag{42}$$

Although this identity can be very useful, it's important to note that the coherent states aren't orthogonal.

Similarly, for fields, we can show a functional integral partition of unity[3]

$$\int \mathcal{D}\psi |\psi\rangle\langle\psi|. \tag{43}$$

4.5 Phase and number

Lancaster, in chapter 27.3, discusses the idea of a phase operator, and points out some issues with attempting to define such an object. I think the discussion is interesting and motivational, so read it for background. [4] I'll cover part of the discussion.

First notice that the idea of a phase operator is related to coherent states. A coherent state for a single mode, is characterized by a complex number α (more generally, for a field, the state is characterized by $\alpha(\mathbf{p})$). We write $\alpha = |\alpha|e^{i\theta}$. θ is known as the phase of the coherent state. Recall that coherent states are used when describing states that behave somewhat classically. So, for example, when discussing electromagnetic fields, these can behave somewhat classically when we take the expectation value of the electromagnetic field operators in a coherent state characterized by a complex variable $\mathbf{E}(\mathbf{p})$. The phase is important because it accounts for classical interference phenomena. On the other hand, when using n-particle states, the phase is dispersed (that is, the phase is essentially unknown) and we regard those as incoherent.

The phase-number relationships are a way of quantifying these ideas. We start by defining operators $\hat{e}_-$ and $\hat{e}_+$. These are the same operators that Lancaster defines respectively as $\widehat{e^{i\phi}}$ and $\widehat{e^{-i\phi}}$, but I want to avoid the temptation of regarding these objects as composite-operators of some kind. The operators are defined by

$$\begin{aligned} \hat{e}_-|n\rangle &= |n-1\rangle \\ \hat{e}_+|n\rangle &= |n+1\rangle \end{aligned} \tag{44}$$

First let's see what these operators have to do with the phase. Using the

[3]Functional integrals require some rigor in their definition. One approach is to generate the space of functions from some basis, such as plane waves. If we were using the plane-wave basis, then a linear superposition would be described by a Fourier transform, and the functional integral would be formally described as $\mathcal{D}\psi = \prod_{\mathbf{p}} \pi^{-1} d\alpha_{\mathbf{p}}$.

[4]One caveat: Lancaster mentions an operator $(\hat{n})^{-\frac{1}{2}}$. I don't believe such an operator can be defined, since $\hat{n}|0\rangle = 0$ and therefore $\hat{n}$ isn't invertible, and a fortiori, its square-root isn't invertible.

representation of the coherent state as given by Eq. (39),

$$\begin{aligned}
\hat{e}_-|\alpha\rangle &= c_0\left(\frac{\alpha}{\sqrt{1!}}|0\rangle + \frac{\alpha^2}{\sqrt{2!}}|1\rangle + ...\right) \\
&= c_0\alpha\left(\frac{1}{\sqrt{1}\sqrt{0!}}|0\rangle + \frac{\alpha}{\sqrt{2}\sqrt{1!}}|1\rangle + ...\right) \\
&= c_0\alpha\widehat{\frac{1}{\sqrt{n+1}}}\left(\frac{1}{\sqrt{0!}}|0\rangle + \frac{\alpha}{\sqrt{1!}}|1\rangle + ...\right) \\
&= \alpha\widehat{\frac{1}{\sqrt{n+1}}}|\alpha\rangle.
\end{aligned} \tag{45}$$

I've introduced the operator $\widehat{\frac{1}{\sqrt{n+1}}}$. This is defined by

$$\widehat{\frac{1}{\sqrt{n+1}}}|n\rangle = \frac{1}{\sqrt{n+1}}|n\rangle. \tag{46}$$

It's not hard to show that this operator is well-defined and is the inverse of the (invertible operator) $\widehat{\sqrt{n+1}}$ defined by

$$\widehat{\sqrt{n+1}}|n\rangle = \sqrt{n+1}|n\rangle. \tag{47}$$

Returning to Eq. (45), compute the expectation value of $\hat{e}_-$.

$$\langle\alpha|\hat{e}_-|\alpha\rangle = \alpha\langle\alpha|\widehat{\frac{1}{\sqrt{n+1}}}|\alpha\rangle. \tag{48}$$

I'm now going to wave my arms slightly. We know that $\langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2$ and that when $|\alpha|$ is large, the dispersion of $\hat{n}$ is small. In other words, when you superpose n-particle states to form the coherent state $|\alpha\rangle$, the dominant contributions come from the states with approximately $|\alpha|^2$ particles. Therefore, whenever we see an expectation value of a function of $\hat{n}$, we can replace that expectation value by the same function of $|\alpha|^2$. For example, when $|\alpha|$ is large, $\langle\alpha|\widehat{\frac{1}{\sqrt{n+1}}}|\alpha\rangle \approx \frac{1}{\sqrt{|\alpha|^2+1}} \approx \frac{1}{|\alpha|}$.

Then Eq. (48) becomes

$$\begin{aligned}
\langle\alpha|\hat{e}_-|\alpha\rangle &\approx \frac{\alpha}{|\alpha|} \\
&= \frac{|\alpha|e^{i\theta}}{|\alpha|} \\
&= e^{i\theta}.
\end{aligned} \tag{49}$$

At long last! We see that in a coherent state $\hat{e}_-$ has (approximately) the value of the exponential of the phase (thus justifying the Lancaster notation $\widehat{e^{i\phi}} \equiv \hat{e}_-$).

Similarly, we can show that when $|\alpha|$ is large, $\langle\alpha|\hat{e}_+|\alpha\rangle \approx e^{-i\theta}$. Lancaster then defines

$$\begin{aligned} \widehat{\cos\phi} &= \frac{1}{2}\left(\widehat{e^{i\phi}} + \widehat{e^{-i\phi}}\right), \\ \widehat{\sin\phi} &= \frac{1}{2i}\left(\widehat{e^{i\phi}} - \widehat{e^{-i\phi}}\right), \end{aligned} \tag{50}$$

and the expectation values of these operators are respectively approximately $\cos\theta$ and $\sin\theta$. Finally, we have operators that can be used as proxies for the phase-operators that extract the phase of coherent states as described earlier.

Our next step is to derive an uncertainty relationship, where we find that an exact knowledge of the phase implies an inexact knowledge of the particle-number and vice versa.

From above, it can be derived[5]

$$\begin{aligned} [\hat{n}, \widehat{\cos\phi}] &= -i\widehat{\sin\phi} \\ [\hat{n}, \widehat{\sin\phi}] &= i\widehat{\cos\phi}, \end{aligned} \tag{51}$$

and in the usual way one derives uncertainty relations from commutation relations, we get

$$\begin{aligned} \Delta n \Delta\cos\phi &\geq \frac{1}{2}|\langle\widehat{\sin\phi}\rangle| \\ \Delta n \Delta\sin\phi &\geq \frac{1}{2}|\langle\widehat{\cos\phi}\rangle|. \end{aligned} \tag{52}$$

4.6 Relationships between classical and quantum field theories on N-particle systems

So far, the best exposition I've found on this, is in Superfluid States of Matter, by Svistunov, Babaev and Prokof'ev (SBP) https://people.umass.edu/bvs/Book.pdf. SBP adopt an approach to superfluids, wherein much of their analysis is done

[5]Frankly, I haven't come up with an entirely straightforward derivation but for a complete discussion of all this, see the lengthy review by Carruthers and Nieto in Reviews of Modern Physics: https://astro.swarthmore.edu/comps/phase.pdf

using classical Hamiltonian theory. They point out in the introduction to their book, that Planck's constant always appears in the quantum theory via the ratio $\frac{\hbar}{m}$, and that classical results can be obtained by substituting a parameter γ wherever in the quantum Hamiltonian, we see the ratio $\frac{\hbar}{m}$. Until page 305, the authors proceed to examine superfluidity using purely classical theory. They cover the Landau criterion, Bogoliubov dispersion, phonons, rotons, topology, vortices, etc. Starting on page 305, the authors introduce the theory of coherent states and cover lots of interesting material. Rather than get hung up on the various problems associated with the proper definition of phase, they focus on an approach known as "coarse-graining". Roughly speaking, this is an approximation procedure which smooths over short distances. Then in the appropriate limits (which admittedly I haven't worked through) one can treat coherent states as a proper orthogonal basis with a well-defined phase operator (don't worry if this sentence is confusing – I haven't mastered any of the details of what this actually means and I don't think the book fills in all the necessary details).

In summary, the authors conclude that the classical-quantum correspondence can be used to justify all of the classical-superfluid results provided one restricts attention to low-temperature systems where small-momentum states are highly populated. The quantum-specific effects only show up for low-population momentum states. For Chapter 42, those effects won't concern us.

Here are a few points to help flesh out the classical-quantum connections.

- SBP derives path integrals on pages 297 - 329. These are used both for quantum scattering (time-evolution) as well as for quantum statistical mechanics (the statistical-mechanics "partition function"). For concreteness, here are the results.
  - The evolution operator is

$$e^{-iHt} = \int \mathcal{D}\psi e^{iS} |\psi(t)\rangle\langle\psi(0) \tag{53}$$

    where S is the classical action. For the non-relativistic many-body theories we've been looking at,

$$S[\psi] = -\int_0^t dt \int d^3x \mathrm{Im}\left(\psi^* \frac{\partial\psi}{\partial t}\right) + H[\psi] \tag{54}$$

– The statistical-mechanics partition function is

$$S_E[\psi] \tag{55}$$

where S_E is the classical-field Euclidean action

$$S_E[\psi] = -\int_0^\beta d\tau \int d^3x \left(\psi^* \frac{\partial \psi}{\partial \tau}\right) + H[\psi] \tag{56}$$

The integrals are formally over the "paths" ψ and are derived using coherent states represented as $|\psi\rangle$ (think of this field-based path integral as a generalization of the single-particle path integral whose classical path is described as $\mathbf{x}(t)$ and whose basis states are represented as $|\mathbf{x}\rangle$).

In the path integral formalism, the dominant contributions come from the paths for which the action is an extremum ($\frac{\delta S}{\delta \psi} = 0$). These paths are the solutions of the Euler-Lagrange equation. Ultimately, that is the reason why the classical solutions of the E-L equations are also relevant in QM.

- For a free theory, a coherent state consisting of free particles of momentum $\mathbf{p}$ evolves into another coherent state whose phase factor is $e^{-i\omega_{\mathbf{p}}t}$ times the original state. This is proven in subsection (4.4.2) above and is the same evolution as we'd see for a complex free classical field whose energy is $\omega_{\mathbf{p}}$. In other words, if we solve for the time-evolution $[\phi(\mathbf{x})](t) \equiv \phi(\mathbf{x}, t)$ of the free-field classical Euler-Lagrange equation with initial condition $\phi(\mathbf{x,0}) = \phi_0(\mathbf{x})$, then in the corresponding quantum theory (in the Schrodinger representation), coherent states will evolve from $|\phi_0\rangle$ to $|[\phi](t)\rangle$.

- When the theory isn't free, we can still solve (in principle) the Euler-Lagrange equations to find the time-evolution of the classical field. It's also true that in the quantum theory an initial coherent state will evolve in time – in the Schrodinger representation. Just to avoid notational saturation, call the quantum field operators $\hat{\mathcal{F}}(\mathbf{x}, t)$. These satisfy the (operator) Euler-Lagrange equations. Now we have time-dependence of the operators, so we are in the Heisenberg representation. In that representation, the initial coherent state wouldn't evolve. However, consider the expectation value $\langle\phi_0|\hat{\mathcal{F}}(\mathbf{x}, t)|\phi_0\rangle$, where (by definition) we've defined the coherent state to have the eigenvalue property $\hat{\mathcal{F}}(\mathbf{x}, 0)|\phi_0\rangle = \phi(\mathbf{x})|\phi_0\rangle$. Since the field operator satisfies the Euler-Lagrange equation, then so does the expectation value. So the expectation value must have the same solution as the classical field.

However, there is nothing we've shown that guarantees that the initial coherent state will, over time, remain an eigenstate of the field (either in the Schrodinger or Heisenberg picture). So we no longer have the simple correspondence between coherent state and classical field, that we saw in the free theory.

- SBP pages 301-304 demonstrate that for states with high occupation number (for low temperatures, these are low-momentum states) coherent states evolve into new coherent states that follow the evolution of the classical field. Here are some of the highlights of this derivation.
  - With the effective Hamiltonian that we have previously hypothesized for our many-body theory, the Euler-Lagrange equation for an extremum ψ_0 looks like

    $$i\frac{\partial\psi_0(\mathbf{x},t)}{\partial t} = \left[\frac{-\nabla^2}{2m} - \mu(\mathbf{x}) + \int d^3x' \left[\psi_0(\mathbf{x}',t)\right]^2 U(\mathbf{x}-\mathbf{x}')\right]\psi_0(\mathbf{x},t),$$

    or

    $$\psi_0(\mathbf{x},t+dt) = \psi_0(\mathbf{x},t) - idt\left[\frac{-\nabla^2}{2m} - \mu(\mathbf{x}) + \int d^3x' \left[\psi_0(\mathbf{x}',t)\right]^2 U(\mathbf{x}-\mathbf{x}')\right]\psi_0(\mathbf{x},t). \tag{57}$$

    This is known as the Gross-Pitaevskii equation and looks like a nonlinear extension of the Schrodinger equation (but remember, ψ is a field and not a wavefunction).
  - Then SBP go on to examine how the coherent state $|\phi_t\rangle$ evolves as a function of t. The objective is to (a) demonstrate that the state maintains its coherence going from time t_0 to time t and (b) evolves according to the same differential equation as the classical ψ_0. Notice that we have implicitly taken the Schrodinger representation where the state evolves, but the field $\hat{\psi}(\mathbf{x})$ does not.
  - We use the notation $|s(t)\rangle$ to denote a generic state at time t and if the state evolves according to the Hamiltonian evolution operator we have $|s(t_0+dt)\rangle = \left(1 - i\hat{H}dt\right)|s(t_0)\rangle$.
  - We also use the notation $|\Phi(\mathbf{x})\rangle$ to denote the eigenstate (simultaneous for all $\mathbf{x}$) of $\hat{\psi}(\mathbf{x})$ with an eigenvalue $\Phi(\mathbf{x})$. By definition, $|\Phi(\mathbf{x})\rangle$ is a coherent state. Let us take $|s(t_0)\rangle \equiv |\Phi(\mathbf{x})\rangle$. Then

SBP show that

$$\hat{\psi}(\mathbf{x})|s(t_0+dt)\rangle \approx \left\{\Phi(\mathbf{x}) - idt\left[\frac{-\nabla^2}{2m} - \mu(\mathbf{x}) + \int d^3x' \left[\Phi(\mathbf{x}')\right]^2 U(\mathbf{x}-\mathbf{x}')\right]\Phi(\mathbf{x})\right\}|s(t_0+dt)\rangle \tag{58}$$

- This equation tells us that $|s(t_0+dt)\rangle$ is an eigenstate of the field operator (i.e. a coherent state), with the eigenvalue in curly braces. But by comparing that eigenvalue to the RHS of Eq. (57), we see that we can replace it by the LHS of Eq. (57), but with $\psi_0(\mathbf{x},t)$ replaced by $\Phi_t(\mathbf{x}) \equiv \Phi'(\mathbf{x},t)$. That is, the curly braces can be replaced by $\Phi_{t_0+dt}(\mathbf{x})$, with the understanding that $\Phi'(\mathbf{x},t)$ is a solution of the Gross-Pitaevskii equation.
- The net result of all this is that if we take a coherent state (with large occupation numbers in the lower momentum states) at time t, and if that state is represented by $|\Phi(\mathbf{x},t)\rangle$, then at time $t+\delta t$, the evolved state will be the coherent state $|\Phi(\mathbf{x},t+\delta t)\rangle$, where Φ is precisely the same function of t that would be obtained by solving the classical field equation for Φ.

- Many of the intuitions about symmetry-breaking, superfluidity and vortices, come from considerations of the phase (we haven't encountered this yet, but we will). In the preceding section on coherent states, we've discussed the phase operator which gives us a way of discussing phase even for states that aren't coherent (remember, for a coherent state, the "phase" is simply the phase of the "eigenvalue" $\phi(x)$.) Based on the above correspondence between classical fields and coherent states, we see that if the classical phase evolves like $\theta(t)$, then the phase of the corresponding coherent state evolves the same way.