# Superfluids – Slides I

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September 10, 2024

# Outline

Many Body Physics – Statistical Mechanics

Many Body Physics – Second Quantization

Quantum Field Theory

Short-distance interaction

Landau criterion

# Stat Mech: The Law of Equal A Priori Probability

**The postulate of equal a priori probabilities**: An isolated system in equilibrium is equally likely to be in any of its accessible states.

Table: Total of three dice adding up to 7

| Die 1 | Die 2 | Die 3 |
|---|---|---|
| 1 | 1 | 5 |
| 1 | 5 | 1 |
| 1 | 2 | 4 |
| 1 | 4 | 2 |
| 1 | 3 | 3 |
| 2 | 1 | 4 |
| 2 | 4 | 1 |
| 2 | 2 | 3 |
| 2 | 3 | 2 |
| ... | ... | ... |
| ... | ... | ... |

3 dice add up to a total of 7.
15 possible configurations.

- 5 configs where the *red system* is 1, so $P(1) = 5/15 = 33\%$.
- Similarly, $P(2) = 4/15 = 27\%$.
- Etc. until $P(5) = 1/15 = 7\%$.

# Stat Mech: Thermal Distribution and Chemical Potential

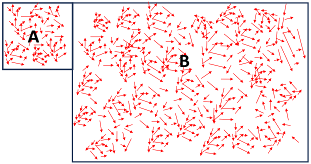



Figure: Total energy is $E_{\text{tot}}$. Total particles = $N_{\text{tot}}$.

- System B has many more states than system A.
- Thermal equilibrium implies Law of Equal a Priori Probability $\implies$ Thermal distribution
- A and B are free to exchange energy but not particles $\implies$ $P_A(E) = Ce^{-\beta E}$ ($\beta$ = inverse temperature).
- A and B are also free to exchange particles $\implies$ $P_A(E, N) = Ce^{-\beta(E-\mu N)}$ ($\mu$ = chemical potential).

# Stat Mech: Low Temp

At low temperature, the probability of A being in the ground state is much larger than in any other state.

**PROOF**

- Call $E - \mu N$ the "effective energy" (EE) of the state.
- $\mathcal{E}_0 =$ lowest EE (ground state(s)).
- Suppose $\mathcal{E} > \mathcal{E}_0$.
- If $\beta$ is very large, $e^{-\beta\mathcal{E}} << e^{-\beta\mathcal{E}_0}$.
- Low temp $\implies$ high $\beta$
- So at low temp, $P(\mathcal{E}_0, N) >> P(\mathcal{E}, N)$.

So in low-temperature many-body physics, focus on the ground state and lowest excited states.

# Second-Quantization: Number Representation

- Basis states $|\psi_{\mathbf{p}}\rangle = |N_{\mathbf{p}_1}, N_{\mathbf{p}_2}, \ldots\rangle$ ($N_{\mathbf{p}_i}$ free particles with $\mathbf{p}_i$).
- $|N_{\mathbf{p}_1}, N_{\mathbf{p}_2}, \ldots\rangle \equiv |N_{\mathbf{p}_1}\rangle \otimes |N_{\mathbf{p}_2}\rangle \otimes \ldots$
- For bosons, the particles with equal momenta are indistinguishable.
- Creation operators:

$$|N_{\mathbf{p}}\rangle = \frac{\left(a_{\mathbf{p}}^{\dagger}\right)^{N_{\mathbf{p}}}}{\sqrt{N_{\mathbf{p}}!}}|0\rangle$$

  where $[a_{\mathbf{p}}, a_{\mathbf{p}'}^{\dagger}] = \delta(\mathbf{p} - \mathbf{p'})$.
- More generally, states are indexed by $\alpha$ rather than $\mathbf{p}$.

$$|N_{\alpha}\rangle = \frac{\left(a_{\alpha}^{\dagger}\right)^{N_{\alpha}}}{\sqrt{N_{\alpha}!}}|0\rangle$$

# Second-Quantization: Two-body

- The free Hamiltonian $H_{\text{free}}$ is

$$H_{\text{free}} = \int d^3p \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}.$$

where we can show that $a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$ is the number operator $N_{\mathbf{p}}$.

- General two-body interaction is

$$H_I^{(2)} = \sum_{a<b}^{N} h_{ab}.$$

- In the number representation, two particles change states.

$$H_I^{(2)} = \sum_{\alpha\beta\gamma\delta} \left(H_I^{(2)}\right)_{\alpha\beta\gamma\delta} a^\dagger_\alpha a^\dagger_\beta a_\gamma a_\delta.$$

- For example,

$$H_V^{(2)} = \frac{1}{2} \int d^3p_1 d^3p_2 d^3q \tilde{V}(\mathbf{q}) a^\dagger_{\mathbf{p}_1+\mathbf{q}} a^\dagger_{\mathbf{p}_2-\mathbf{q}} a_{\mathbf{p}_2} a_{\mathbf{p}_1}.$$

# QFT – Equivalence to Second Quantization

- Define a field $\psi$ by $\psi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} a_{\mathbf{p}} e^{-i(Et - \mathbf{p}\cdot\mathbf{x})}$, where $E = \frac{\mathbf{p}^2}{2m}$.
- $\psi^\dagger(\mathbf{x})\psi(\mathbf{x})$ is the number density operator.
- The effective Hamiltonian can be written as

$$H_{\text{eff}} = H - \mu N = H_{\text{free}} + H_V^{(2)} - \mu N = \int d^3x \left( \frac{1}{2m} \nabla\psi^\dagger \cdot \nabla\psi - \mu\psi^\dagger\psi \right) + \frac{1}{2} \int d^3x d^3y \psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{y}) V(\mathbf{x} - \mathbf{y})\psi(\mathbf{y})\psi(\mathbf{x}))$$

where

$$\tilde{V}(\mathbf{q}) = \frac{1}{(2\pi)^3} \int d^3x V(\mathbf{x}) e^{-i(\mathbf{q})\cdot\mathbf{x}}.$$

- Can derive from an action for a non-relativistic quantum field theory

$$\mathcal{L}_{\text{eff}}(\psi) = \int d^3x \left[ i\psi^\dagger\partial_0\psi - \frac{1}{2m} \nabla\psi^\dagger \cdot \nabla\psi + \mu\psi^\dagger\psi \right] - \frac{1}{2} \int d^3x d^3y \psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{y}) V(\mathbf{x} - \mathbf{y})\psi(\mathbf{y})\psi(\mathbf{x})).$$

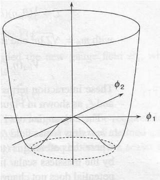
# Short-distance interaction – Hamiltonian

- Bogoliubov's model. Set $V(\mathbf{x}-\mathbf{y}) = g\delta(\mathbf{x}-\mathbf{y})$.
- Then

$$H_{\text{eff}} = \int d^3x \left( \frac{1}{2m} \nabla\psi^\dagger \cdot \nabla\psi - \mu\psi^\dagger\psi + \frac{g}{2}\left(\psi^\dagger\psi\right)^2 \right)$$

$$= \int d^3x \left( \frac{1}{2m} \nabla\psi^\dagger(\mathbf{x}) \cdot \nabla\psi(\mathbf{x}) + \frac{g}{2}\left(\frac{\mu}{g} - \psi^\dagger(\mathbf{x})\psi(\mathbf{x})\right)^2 - \frac{\mu^2}{2g} \right).$$

- "Mexican Hat" potential for the classical field theory




$\frac{\mu}{g} - \psi^\dagger\psi$ with $\phi_1 \equiv \psi_R$ and $\phi_2 \equiv \psi_I$

# Short-distance interaction – Low temperature states

- Coherent states are field eigenstates $\psi(\mathbf{x})|\text{state}\rangle = \tilde{\psi}(\mathbf{x})|\text{state}\rangle$.
  - Only study low-energy high occupation-number states.
    - They are most probable at low temperature.
  - $\psi$ is an operator, $\tilde{\psi}$ is a complex valued function.
  - For readability, we identify the state $|\text{state}\rangle$ as $|\tilde{\psi}\rangle \equiv |\text{state}\rangle$.
  - $\tilde{\psi}$ satisfies the E-L equation.
  - $\tilde{\psi}$ is a classical field corresponding to $\psi$. Interchangeable.
    - Going forward, identify $\psi$ with $\tilde{\psi}$, etc.
- Define the number-density field $\rho(\mathbf{x}) = \psi^*(\mathbf{x})\psi(\mathbf{x})$.
- "Polar coordinates"

$$\psi(\mathbf{x}) = \sqrt{\rho(\mathbf{x})}e^{i\theta(\mathbf{x})}.$$

- $\Theta$ is called the "phase field". Very important!
- Then the Hamiltonian can be written as

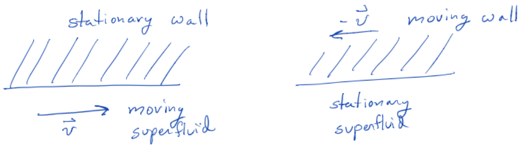
$$H_{\text{eff}} = \int d^3x \left\{ \frac{1}{2m} \left[ \frac{(\nabla\rho)\cdot(\nabla\rho)}{4\rho} + \rho\,(\nabla\theta)\cdot(\nabla\theta) \right] + \frac{g}{2}\left(\frac{\mu}{g} - \rho\right)^2 \right\}.$$

# Short-distance interaction – Dispersion relation

- The value of $\rho$ at the energy-minimum, is $\rho = \frac{\mu}{g}$.
- States at low temp have near-minimum energy, so $n(= \rho) \approx \frac{\mu}{g}$.
- Find $\approx$ eigenvalues by writing $\sqrt{\rho} = \sqrt{n} + h$ and expand to order $h^2$.
- Energy eigenstates are parameterized by the momentum $\mathbf{p}$.

$$E_{\mathbf{p}} = \sqrt{\frac{p^2}{2m}\left(\frac{p^2}{2m} + 2ng\right)}.$$

# Landau criterion – Superfluidity at low temperature




- Wall friction excites a state with momentum $\mathbf{p}$ and energy $\epsilon_{\text{fluid}}(p)$.
- Galilean transformation from fluid to lab frame.

$$\epsilon_{\text{lab}}(p) = \epsilon_{\text{fluid}}(p) - \mathbf{p} \cdot \mathbf{v}.$$

- By conservation of energy, $\Delta E_{\text{wall}} = -\epsilon_{\text{lab}}(p)$.
- Only $\Delta E_{\text{wall}} \geq 0$ is thermodynamically possible (by stat. mech.)
  - So if $\epsilon_{\text{lab}}(p) < 0$ the wall heats up and the fluid slows down (dissipation).
- $|\mathbf{v}| < \min_p \frac{\epsilon(p)}{p} \implies \epsilon_{\text{lab}}(p) > 0$; **excitation is not possible**

# Landau criterion – critical velocity

$$v_{\text{crit}} \equiv \min_p \frac{\epsilon(p)}{p}$$

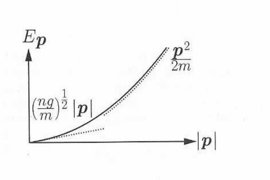
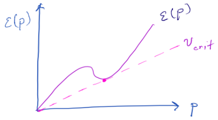



**Fig. 42.2** The dispersion predicted by Bogoliubov's model. At low momentum the energy is linear in $|p|$, at large momentum it is quadratic.

$$v_{\text{crit}}^{\text{bog}} = \left(\frac{ng}{m}\right)^{\frac{1}{2}}$$




Experimental dispersion curve

- Dispersion curve is different for large $p$.
- Larger $p$ = smaller distances.
- Hypothesis– the dip is caused by a roton.
- Rotons are collective excitations
- Experimental critical v $<<$ $v_{\text{crit}}$