Superfluids – Slides II: Vortices

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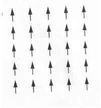
Outline

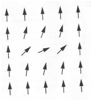
What is a Vortex? – Phase Current

- Lowest-energy state has $\nabla\theta = 0$.
- So we have to pick a vacuum with fixed $\theta = \theta_0$.



In the vacuum state, the phases at different positions are aligned($\nabla\theta = 0$)

- Non-vacuum states have a phase current $\jmath = \frac{1}{m}(\rho\nabla\theta)$.

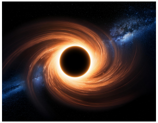


Non-vacuum states may have a phase current ($\nabla\theta \neq 0$)

- Continuity equation (Noether) $\dot{\rho} = \nabla \cdot \mathbf{j}$.

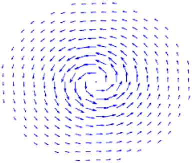
What is a Vortex? – Phase Discontinuity

- A vortex requires a spacial region of discontinuous / undefined $\nabla\theta$.
  - Mathematically is an excluded region or point.
- For example, flow around a torus.



Black hole – center missing

- Other example, flow around a drain.



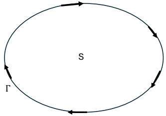
Vortex where $\nabla\theta$ blows up at center

What is a Vortex? – Mathematical origin

- Our *delta*-function interaction Hamiltonian is only an approximation.
- Mathematically, exclude the immediate region around particles.
- Replace unknown region with conditions observed experimentally.
  - Key example: a *vortex core* (**coming soon**)
- Has the effect of modifying the Hilbert space.
  - Extends the spectrum (new excitations).

What is a Vortex? – Circulation

- (Noether) $\mathbf{p} = \nabla\theta$, so $\mathbf{v} = \frac{1}{m}\nabla\theta$.
- "Circulation" $\mathcal{C}$ is the integral of $\mathbf{v}$ around a contour.



Full contour starts at Γ_i and ends at Γ_f

$$\mathcal{C} = \oint_\Gamma \mathbf{v}\cdot d\mathbf{s} = \frac{1}{m}\oint_\Gamma \nabla\theta\cdot\mathbf{s} = \frac{1}{m}\left(\theta(\Gamma_\mathrm{f}) - \theta(\Gamma_\mathrm{i})\right).$$

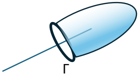
- $\psi(\mathbf{x}) = e^{i\theta(\mathbf{x})}\sqrt{\rho(\mathbf{x})}$ is single-valued $\implies \theta(\Gamma_\mathrm{f}) - \theta(\Gamma_\mathrm{i}) = 2\pi n$.
- **If** $\nabla\theta$ is differentiable in S, then Stoke's theorem $\implies n = 0$.

$$\mathcal{C} = \oint_\Gamma \mathbf{v}\cdot d\mathbf{s} = \int_S (\nabla\times\mathbf{v})\cdot\mathbf{dA} = \frac{1}{m}\int_S (\nabla\times\nabla\theta)\cdot\mathbf{dA} = 0.$$

- Circulation is "quantized" with $\mathcal{C} = \frac{2\pi}{m}n$
  - $n = 0$ if phase is differentiable
  - $n \neq 0$ is a **vortex core** $\implies$ non-differentiable
  - A vortex core is a topological defect

What is a Vortex? – Vortex Cores

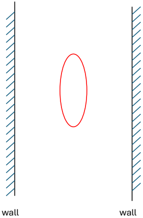
- Feynman calls them "vortex lines" because width is atomic ($< 1\ \mathring{A}$).
- A vortex line cannot have free ends. Suppose otherwise.

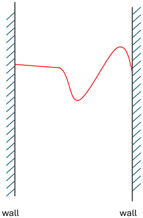


The contour γ goes around the core, which is a line segment. Pick a bounding surface not crossed by the core.

  - By assumption, θ is differentiable away from the line.
  - $\nabla \times \mathbf{v} = 0$ on bounding surface.
  - By Stokes theorem, $\mathcal{C} = 0 \implies$ segment isn't a vortex core.

- There can be core lines closed in a loop or between walls.





What is a Vortex? – Gross-Pitaevskii equation

- Recall, our superfluid Lagrangian is

$$\mathcal{L} = i\psi^*(\mathbf{x})\partial_0\psi(\mathbf{x}) - \frac{1}{2m}\boldsymbol{\nabla}\psi^*(\mathbf{x})\cdot\boldsymbol{\nabla}\psi(\mathbf{x}) - \frac{g}{2}\left(\frac{\mu}{g} - \psi^*(\mathbf{x})\psi(\mathbf{x})\right)^2 - V(\mathbf{x}).$$

  which now includes an external potential $V(\mathbf{x})$ for the walls.

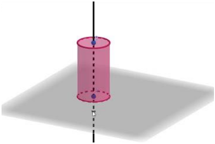
- Euler-Lagrange equation is

$$i\dot{\psi}(t,\mathbf{x}) = \left[\frac{-\nabla^2}{2m} + V(\mathbf{x}) - \mu + g|\psi(\mathbf{x}')|^2\right]\psi(\mathbf{x})$$

  Called Gross-Pitaevskii equation. Generalizes Schrodinger equation.

Cylindrical Vortex – Cylindrical stationary GP equation

- Superfluid in a cylinder, with a vortex core through center.



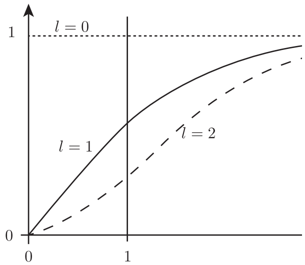
Take V to be 0 inside and ∞ outside the cylinder. Radius R, length $L >> R$.

- Stationary solution has $\dot{\psi} = 0$ and also $\frac{\partial H}{\partial \psi} = 0$.
  - Ground state in Hilbert space with topological defect
- Cylindrical coordinates: r, ϕ, z. Ansatz $\psi = \sqrt{\rho} e^{in\phi}$.
- Scaled variables: $\tilde{\rho} = \frac{g}{\mu}\rho$, $\alpha = \sqrt{m\mu}r$. Define $\rho' = \frac{d\rho}{d\alpha}$, etc.
- Stationary GP equation for ansatz with scaled variables

$$\tilde{\rho}'' + \frac{\tilde{\rho}'}{\alpha} - 2\left(\tilde{\rho}^2 - 1 + \frac{n^2}{2\alpha^2}\right)\tilde{\rho} = 0.$$

Cylindrical Vortex – Stationary Solution

- Numerical solution



Density of cylindrical vortices for winding numbers $n = 0$, 1 and 2. The x-axis is the dimensionless distance α. The curves asymptote to 1 as $\alpha \to \infty$. The variable l is equivalent to n in the text.

- $\lim_{\alpha \to 0} \tilde{\rho}(\alpha) \propto \alpha^n$.
- $\lim_{(\text{large } \alpha)} \tilde{\rho}(\alpha) \to 1 - \frac{n^2}{4\alpha^2}$.
- Derivation of the velocity:
  - From the definition of circulation, $\mathcal{C} = 2\pi v r$.
  - We also showed $\mathcal{C} = \frac{1}{m}\left(\theta(\Gamma_f) - \theta(\Gamma_i)\right) = \frac{2}{m}\pi n$.
  - So $v = \frac{n}{rm}$.
  - Note that the velocity diverges towards the core.

Cylindrical Vortex – Cutoff Kinetic Energy

- Large-r approximation: Take integrals from $r = r_c$ to $r = R$.
- Cutoff kinetic energy = integral from r_c of mass density $\times v^2/2$.
  - Change variables from r, ρ to $\alpha, \tilde{\rho}$, and r_c, R to $\bar{r}_c = \sqrt{m\mu}r_c$, $\bar{R} = \sqrt{m\mu}R$, use a cylindrical measure and set $v = \frac{n}{rm}$.

$$KE_{>r_c} = \frac{2\pi}{mg} L \int_{\bar{r}_c}^{\bar{R}} m\tilde{\rho}(\alpha)\frac{v^2}{2}\alpha d\alpha$$
$$= \frac{\pi\mu}{mg} L \int_{\bar{r}_c}^{\bar{R}} \tilde{\rho}(\alpha)\frac{n^2}{\alpha^2}\alpha d\alpha$$

  - Use the large-α approximation of $\tilde{\rho}$.

$$KE_{>r_c} = \frac{\pi L n^2 \mu}{mg} \int_{\bar{r}_c}^{\bar{R}} \left(1 - \frac{n^2}{4\alpha^2}\right) \frac{1}{\alpha^2}\alpha d\alpha$$
$$= \frac{\pi L \rho_0 n^2}{m} \left(\log(R/r_c) + \mathcal{O}(\frac{1}{r_c^2})\right),$$

Cylindrical Vortex – Total Energy

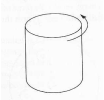
- Total E also requires $KE_{\leq r_c}$ and also PE. We show both are small.
  - Total KE (cylindrical coords.) is $\tilde{\rho}'' + \frac{\tilde{\rho}'}{\alpha} - n^2 \frac{\tilde{\rho}}{\alpha^2}$
  - $KE_{\leq r_c}$
    - Integral near $\alpha = 0$ has $\rho(\alpha) \propto \alpha^n$ so KE $\to 0$.
    - Integral approaching $\bar{r}_c$ from below is dominated by $\log(\bar{r}_c)$.
    - So $KE_{\leq r_c} \approx \mathcal{O}(\frac{L\rho_0}{m} \log(\bar{r}_c))$.
  - PE is a sum of terms below cutoff and above cutoff
    - Above cutoff

$$PE_{>\bar{r}_c} = \frac{\pi L \mu}{mg} \int_{\bar{r}_c}^{\bar{R}} \alpha d\alpha \, (1 - \bar{\rho})^2$$
$$\approx \frac{\pi L n^2 \mu}{8mg} \left(\frac{1}{\bar{r}_c^2} - \frac{1}{\bar{R}^2} \right).$$

    - Similar to discussion of $KE_{\leq r_c}$, $PE_{\leq r_c} \approx \mathcal{O}\left(\frac{1}{\bar{r}_c^2}\right)$

- $E \approx \frac{\pi L \rho_0 n^2}{m} \log(\frac{R}{r_c}) \left(1 + \frac{\mathcal{O}(\log(\bar{r}_c))}{\log(\frac{R}{r_c})} + \mathcal{O}\left(\frac{1}{\bar{r}_c^2}\right) \right)$.
- Set r_c to atomic length. For $R >> r_c$ results are insensitive to r_c.

Rotating cylinder – Initial state



Rotating can of helium of radius R_0

- Solid helium created near 0°K under pressure > 25 atm.
- The container is rotated at angular velocity ω.
  - Total angular momentum

$$J = 2\pi L \int_0^{R_0} \left(m\rho_0 r^2 \omega\right) rdr$$
$$= \pi L m \rho_0 \omega \frac{R_0^4}{2}.$$

  - Kinetic energy of rigid motion (doesn't include binding energy)

$$E^K_{R_0} = 2\pi L \int_0^{R_0} \left(m\rho_0 \frac{(r\omega)^2}{2}\right) rdr$$
$$= \pi L m \rho_0 \omega^2 \frac{R_0^4}{4}.$$

- Then the pressure is released to melt the helium.

Rotating cylinder – Single vortex energy

- Conserve angular momentum by picking the winding number n.
  - Using $v = \frac{n}{rm}$

$$J_{\text{vortex}} = 2\pi L \int_0^{R_0} \left(m\rho_0 r \frac{n}{rm} \right) r dr$$
$$= \pi L \rho_0 n R_0^2,$$

  - Assume one vortex so set $J_{\text{vortex}} = J$. Then

$$n = m\omega \frac{R_0^2}{2}.$$

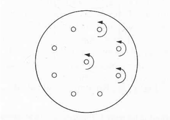
- Vortex energy

$$E \approx \frac{\pi L \rho_0 n^2}{m} \left(\log(R_0/r_c) + \mathcal{O}(\frac{1}{\tilde{r}_c^2}) \right)$$
$$= \frac{\pi L m \omega^2 \rho_0 R_0^4}{4} \log(R_0/r_c)$$
$$= \log(R_0/r_c) E_R^K.$$

- The single vortex energy is much greater than initial kinetic energy.
  - So isn't thermodynamically favored.

Rotating cylinder – Vortex array hypothesis

- Hypothesize that a vortex array has lower energy.
  - Each vortex has winding number $n = 1$.
  - Vortices are closely spaced.
  - Vortices have uniform density.



A circular portion of the helium container of radius R, seen from above

  - $n(r)$ = number of vortices within r of the core.
  - $n(r)/n(R_0) = r^2/R_0^2$
- Winding number at r is $n(r)$, so $v(r) = \frac{n(r)}{rm} = \left(\frac{n(R_0)}{mR_0^2}\right) r$.
  - Looks like rigid rotation with $\omega' = \frac{n(R_0)}{mR_0^2}$.
  - Explanation: between vortices, velocities tend to cancel.

Rotating cylinder – Vortex array energy

- For conservation of angular momentum, $\omega' = \omega$.
  - Therefore $n(R_0) = mR_0^2\omega$.
  - The line density is $\tilde{n} = \frac{n(R_0)}{\pi R_0^2} = \frac{m\omega}{\pi}$.
  - Distance between cores $\approx \sqrt{\tilde{n}}$.
  - If $\omega = 1\,\text{rad/sec}$, then cores are about $2mm$ apart.
- For total energy
  - Per vortex use energy formula with $r_c = 4.0\AA$ and $R = 2\,\text{mm}$.
  - Then multiply by number of vortices $n(R_0) = mR_0^2\omega$.
  - Total array energy

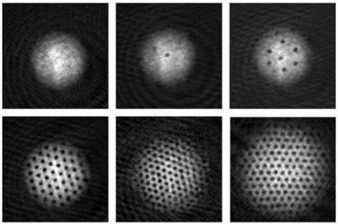
$$
\begin{aligned}
E_A &= (m\omega R_0^2)\frac{\pi L\rho_0}{m}\log(R/r_c) \\
&= \omega L\pi R_0^2\rho_0\log(R/r_c) \\
&= 14\rho_0\omega\pi LR_0^2.
\end{aligned}
$$

Rotating cylinder – Ground state

- Compare the array energy E_A to the original solid cylinder $E_{R_0}^K$.

$$\frac{E_A}{E_R^K} = \frac{14\rho_0\omega\pi L R_0^2}{\pi L m \rho_0 \omega^2 \frac{R_0^4}{4}} = \frac{64}{m\omega R_0^2}$$

- Set $R_0 = 1\,\text{cm}$, $\omega = 1\,\text{rad}/\text{sec}$, and $\frac{1}{m} = 0.00015\,\text{cm}^2/\text{sec}$.
  - The mass value is for helium-4, with $\hbar = 1$.
- Then $\frac{E_A}{E_R^K} \approx 10^{-2}$. The vortex array is thermodynamically favored.



Vortices in a Bose Einstein Condensate. The dark spots are the cores of the vortices.

Ring vortices – Outline of discussion

A smoke ring. The core is the center and all around it, there is a vortex flow that circulates around the core.

- We'll treat a ring like an excitation of the fluid ground state.
- The most symmetric closed loop is circular.
- Generalize the previous vortex velocity equation $v = \frac{n}{rm}$.
- Lowest energy excitation has $n = 1$.
- Each point on the ring influences $\mathbf{v}$ at every other point.
- We'll show that $\mathbf{v}$ is perpendicular to the ring.
- Then we compute the momentum and energy of the ring.
- From dispersion relation, find critical velocity for superfluidity.
- This critical velocity is much small than the Bogoliubov v_c.

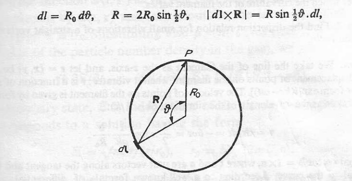
Ring vortex – velocity

- Vorticity is related to velocity, as current is related to magnetic field.
- By analogy with magnetostatics Biot Savart law,

$$\mathbf{v}_{\mathrm{P}} = \frac{1}{2m} \int \frac{d\mathbf{l} \times \mathbf{r}}{r^3}$$

where

  - the integral ranges over positions on the core
  - r is the displacement from point P to the core position
  - $d\mathbf{l}$ is the differential line segment at the point



Derivation of vortex ring velocity. P. 115 of Landau and Lifshitz Vol. 9.

- $\mathbf{v}_P$ is perpendicular to the ring. Compute the integral with cutoff.

$$\mathbf{v}_{\mathrm{P}} = \left(\frac{1}{8mR_0}\right) 2 \int_{\frac{r_c}{R_0}}^{\pi} \frac{d\theta}{\sin \frac{1}{2}\theta} \approx \frac{1}{2mR_0} \log \frac{R_0}{r_c}.$$

Ring vortex – energy and momentum

- Recall energy of a linear vortex of length L and max radius R

$$E \approx \frac{\pi L \rho_0 n^2}{m} \log(\frac{R}{r_c})$$

- Picture smoke ring.
  - Length is $L = 2\pi R_0$.
  - Vortex lines are 0 at center, so max radius $R \approx R_0$.
  - $n = 1$ is lowest-energy excitation
- Substitute for L, n and R to get ring vortex energy.

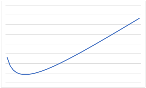
$$E \approx \frac{2\pi^2 R_0 \rho_0}{m} \log(\frac{R_0}{r_c})$$

- $\frac{dE}{dp} = v \implies dp = \frac{dE}{v} \implies dp \approx \left(4\pi^2 R_0 \rho_0\right) dR_0$
- Integration $\implies p \approx 2\pi^2 R_0^2 \rho_0$
- Dispersion relation is

$$E \approx \frac{\pi\sqrt{\rho_0}}{\sqrt{2}m}\sqrt{p} \log\left(\frac{p}{2\pi^2 \rho_0 r_c^2}\right) \quad (1)$$

Ring vortex – critical velocity

- Dispersion in thin aperture



Vortex ring dispersion curve

- $v_{\text{crit}} = \min_p \frac{E(p)}{p}$.
- For large p, $\frac{E(p)}{p} \to 0$.
- But $R_0 < D$ where D is radius of the enclosure, so $p < 2\pi^2 D^2 \rho_0$.
  - Since $p \approx 2\pi^2 R_0^2 \rho_0$, then $p < 2\pi^2 D^2 \rho_0$.
  - So $v_{\text{crit}} = \frac{\hbar}{mD} \log \frac{D}{r_c}$. (*reinstate* $\hbar$ previously set to 1)
- Use Feynman values: $\frac{\hbar}{m} = 1.5 \times 10^{-4}\text{cm}/\text{s}, D = 10^{-5}\text{cm},\ r_c = 4\mathring{A}$
  - Leads to $v_{\text{crit}} \approx 80\,\text{cm}/\text{sec}$.
  - Experimental value is about $20\,\text{cm}/\text{sec}$.
  - Much closer than Bogoliubov prediction.