Superfluids Part III

Bill Celmaster

September 2, 2024

1 Review of Bogoliubov dispersion

1.1 Review of the Bogoliubov method

We began our treatment of superfluidity with an approach due to Bogoliubov, based on second quantization of a many-body system. In that approach, we represented the Hamiltonian as a symmetrized sum of terms involving annihilation and creation operators acting on single-particle states. The Hamiltonian describes a system with very short-range interactions (we choose a delta-function interaction). We then converted to an occupation-number description of the system, where there is a separate annihilation and creation operators for each momentum, and those operators change the occupation number of the occupation state for that momentum.

$$H \approx \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{g}{2\mathcal{V}} \sum_{\mathbf{kpq}} a^\dagger_{\mathbf{k+q}} a^\dagger_{\mathbf{p-q}} a_{\mathbf{p}} a_{\mathbf{k}}, \tag{1}$$

This Hamiltonian isn't obviously diagonalizable, but Bogoliubov came up with a useful approximation (his so-called "hunting license") that can be applied for a system whose ground state has a very high population number (as would be the case near a temperature of absolute zero.) He ends up with

$$H \approx \sum_{\mathbf{p}\neq 0} \left(\frac{\mathbf{p}^2}{2m} + ng \right) a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{1}{2} \sum_{\mathbf{p}\neq 0} ng \left(a^\dagger_{\mathbf{p}} a^\dagger_{\text{-}\mathbf{p}} + a_{\mathbf{p}} a_{\text{-}\mathbf{p}} \right), \tag{2}$$

where $n = \frac{N}{\mathcal{V}}$ is the number-density.

Since the terms are all quadratic, it's possible to do a straightforward diagonalization and we find that

$$E(p) = \sqrt{\frac{\mathbf{p}^2}{2m}\left(\frac{\mathbf{p}^2}{2m} + \frac{2Ng}{\mathcal{V}}\right)}. \tag{3}$$

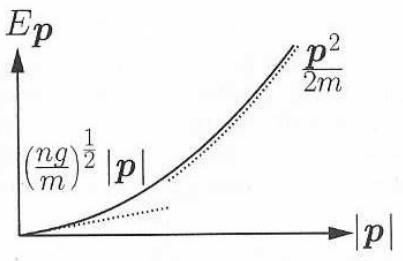
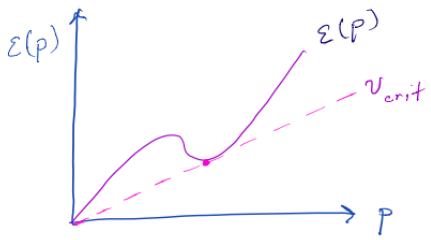


Fig. 42.2 The dispersion predicted by Bogoliubov's model. At low momentum the energy is linear in $|\boldsymbol{p}|$, at large momentum it is quadratic.

In this figure, the minimum slope of the curve (also known as the *critical velocity*) occurs at the origin and has the value $\sqrt{\frac{ng}{m}}$.

For a short-range potential $\mathcal{V}(\mathbf{k})$) which is more general than the delta-function, we can get a more general dispersion curve, for example



1.2 Review of the field theory method

The second-quantization technique can be rewritten using the formalism of field theory. We had shown

$$\begin{aligned} H - \mu \int d^3x \phi^\dagger(\mathbf{x})\phi(\mathbf{x}) &= \int d^3x \left(\frac{1}{2m} \boldsymbol{\nabla}\phi^\dagger(\mathbf{x}) \cdot \boldsymbol{\nabla}\phi(\mathbf{x}) + \frac{g}{2} \left(\phi^\dagger(\mathbf{x})\phi(\mathbf{x})\right)^2 - \mu\phi^\dagger(\mathbf{x})\phi(\mathbf{x}) \right) \\ &= \int d^3x \left(\frac{1}{2m} \boldsymbol{\nabla}\phi^\dagger(\mathbf{x}) \cdot \boldsymbol{\nabla}\phi(\mathbf{x}) + \frac{g}{2} \left(\frac{\mu}{g} - \phi^\dagger(\mathbf{x})\phi(\mathbf{x}) \right)^2 - \frac{\mu^2}{2g} \right). \end{aligned} \tag{4}$$

where μ was introduced as a Lagrange multiplier for a statistical partition function describing a system that can exchange particles with a reservoir. Dropping the constant term, we have

$$H' = \int d^3x \left(\frac{1}{2m} \boldsymbol{\nabla}\phi^\dagger(\mathbf{x}) \cdot \boldsymbol{\nabla}\phi(\mathbf{x}) + \frac{g}{2} \left(\frac{\mu}{g} - \phi^\dagger(\mathbf{x})\phi(\mathbf{x}) \right)^2 \right). \tag{5}$$

We see that the classical theory has a minimum for $\phi_0 = \sqrt{\frac{\mu}{g}} e^{i\theta}$, where θ is a constant that can be chosen arbitrarily. We rewrite $\phi = \phi_0 + \phi_1$ then expand H' in ϕ_1 through quadratic order. If we set $\theta = 0$, then we obtain

$$H' \approx \int d^3x \left(\frac{1}{2m} \boldsymbol{\nabla}\phi_1^\dagger(\mathbf{x}) \cdot \boldsymbol{\nabla}\phi_1(\mathbf{x}) + \mu \int d^3x' \phi_1^*(\mathbf{x})\phi_1(\mathbf{x}) + \frac{\mu}{2} \int d^3x \left(\phi_1^2(\mathbf{x}) + \phi_1^{*2}(\mathbf{x}) \right) \right). \tag{6}$$

Note that by picking $\theta = 0$, we have chosen a direction in θ-space and therefore have broken a symmetry that was manifest in the original Lagrangian ($\phi \to \phi e^{i\theta}$).

Since H' is quadratic, it can easily be diagonalized in terms of eigenstates parametrized by $\mathbf{k}$ with the resulting dispersion shown in Eq. (3).

Arguably, this entire procedure is classical. We haven't integrated over path space, nor have we introduced operators. However, as we've demonstrated previously, the classical results can be carried over to the quantum theory provided we restrict our attention to many-body states of low momentum at low temperature, i.e., states of large occupation number. The quantum ground state (i.e., minimum energy eigenstate) has been shown to be the coherent state ϕ_0.

It's relatively easy to compare the Bogoliubov and field methods, and see the relationship between them. In fact, there is a fairly rigorous equivalence

between the methods. However, the field method has both notational and conceptual advantages. I find that particularly true as we turn to some of the topological features of superfluidity.

2 The superfluid Lagrangian and currents

2.1 Caveat emptor

Going forward, I'm going to be a bit sloppy in distinguishing classical and quantum physics. Although Lancaster isn't clear about this, the remaining discussion (and much of the previous discussion) of superfluids can all be done within the context of classical field theory. As explained previously, the quantum-classical equivalence is permitted here by identifying ϕ in the designation of the coherent state $|\phi\rangle$, with the classical field ϕ, provided that we identify the operator Hamiltonian function $H(\hat{\phi})$ with the classical Hamiltonian function $H(\phi)$. Note that I may use the notation $\phi^\dagger$ intermittently with the notation ϕ^*. This first of these refers to the adjoint of an operator, and the second to the complex conjugate of a number.

While I'm at it, I've just noticed that for most of these notes on many-body physics and superfluids, I've suppressed the time-coordinate. You might not have noticed, since the notation x is used in field theory to indicate the space-time 4-vector. However, I've used the notation $\mathbf{x}$. The boldface type generally denotes spacial coordinates. My usage follows Lancaster and many other sources, and is typical of non-relativistic physics. However, even in non-relativistic physics there is time-dependence which, in many cases, isn't explicitly shown here.

2.2 Polar coordinates – phase and density fields

Elsewhere (see notes called "Condensed Matter I"), we've discussed the non-relativistic limit of field theories, and we've obtained both the Hamiltonian

discussed above, and the Lagrangian (see notes called "Superfluids II")

$$\begin{aligned}\mathcal{L} &= i\phi^\dagger(\mathbf{x})\partial_0\phi(\mathbf{x}) - H' \\ &= i\phi^\dagger(\mathbf{x})\partial_0\phi(\mathbf{x}) - \frac{1}{2m}\boldsymbol{\nabla}\phi^\dagger(\mathbf{x})\cdot\boldsymbol{\nabla}\phi(\mathbf{x}) - \frac{g}{2}\left(\frac{\mu}{g} - \phi^\dagger(\mathbf{x})\phi(\mathbf{x})\right)^2 .\end{aligned} \tag{7}$$

It turns out to be convenient to write the field in polar coordinates. The angular coordinate – which we'll call ***the phase field*** is of fundamental importance in discussing superfluids, since the key properties of superfluids have to do with the angular current which we will define shortly. We write

$$\phi(\mathbf{x}) = \sqrt{\rho(\mathbf{x})}e^{i\theta(\mathbf{x})}, \tag{8}$$

where $\rho(\mathbf{x}) = \phi^\dagger(\mathbf{x})\phi(\mathbf{x})$ which we've previously identified as the number-density operator (classically, we'd call this the matter density). We call ρ ***the density field***

Then

$$\mathcal{L} = -\rho\partial_0\theta - \frac{1}{2m}\left[\frac{1}{4\rho}(\nabla\rho)^2 + \rho(\nabla\theta)^2\right] - \frac{g}{2}\left(\frac{\mu}{g} - \rho\right)^2 . \tag{9}$$

We have dropped a term $\frac{1}{2}\partial_0\rho$ since this is a total derivative of a function presumed to vanish at $\pm\infty$. This is the same as Lancaster Eq. (42.24), although Lancaster has noted that in the ground state, the number density, n_0 is $\frac{\mu}{g}$.

2.2.1 The spectrum – rederiving Bogoliubov's dispersion relation

It should be possible to start with field theory and obtain the Bogoliubov dispersion relation.Surprisingly, I haven't been able to google any work that does this, although I'm pretty sure it must be a relatively well-known procedure. I've come up with something that I'll present here. But it's worth noting that today (July 28) when I queried ChatGPT, I got a very nice derivation, different from mine. The derivation has a few logic errors but I'm pretty sure it can be corrected to give the right answer. My query was "show how to obtain the Bogoliubov dispersion relation starting from field theory".

My approach (not ChatGPT's) closely follows Lancaster starting the middle of page 376. Let $\sqrt{\rho(x)} = \sqrt{n} + h$ where $n = N/\mathcal{V}$. Then expand the Lagrangian Eq. (9) up to quadratic order in h.

$$\mathcal{L} = -\frac{1}{2m}(\nabla h)^2 - 2gnh^2 - \left(2\sqrt{n}\partial_0\Theta\right)h - \frac{n}{2m}(\nabla\Theta)^2 + ... \tag{10}$$

Since this Lagrangian is quadratic in the fields, the theory can be solved exactly (to this order in n). Lancaster, in example 42.4, shows a trick for eliminating the h fields and obtaining an effective Lagrangian in terms of the Θ fields:

$$\mathcal{L}_{\text{eff}} = n\partial_0\Theta\frac{1}{2gn - \frac{1}{2m}\nabla^2}\partial_0\Theta - \frac{n}{2m}\left(\nabla\Theta\right)^2 + ... \tag{11}$$

The denominator term $2gn - \frac{1}{2m}\nabla^2$ can be expanded in momentum space (i.e., transform the phase field to momentum space, then $\nabla^2 \to \mathbf{p}^2$. Then expand for small p or alternatively large n:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2g}\left(\partial_0\Theta\right)^2 - \frac{n}{2m}\left(\nabla\Theta\right)^2 + \frac{1}{8mg^2n}\partial_0\Theta\nabla^2\partial_0\Theta + ... \tag{12}$$

This resembles Lancaster Eq. (42.30) but includes one more term.

Apply the Euler-Lagrange equation

$$\partial_0\frac{\partial\mathcal{L}_{\text{eff}}}{\partial\left(\partial_0\phi\right)} + \nabla\cdot\frac{\partial\mathcal{L}_{\text{eff}}}{\partial\left(\nabla\phi\right)} = 0. \tag{13}$$

to obtain

$$\frac{1}{g}\partial_0^2\Theta - \frac{n}{m}\nabla^2\Theta + \frac{1}{4mg^2n}\partial_0\nabla^2\partial_0\Theta = 0. \tag{14}$$

Hypothesize a solution of the form

$$\Theta = Ce^{-i(Et-\mathbf{p}\cdot\mathbf{x})}. \tag{15}$$

Our differential equation becomes

$$-\frac{1}{g}E^2 + \frac{n}{m}p^2 + \frac{1}{4mg^2n}E^2p^2 = 0. \tag{16}$$

Then solve for E in terms of p.

$$\begin{aligned} E &= \sqrt{\frac{\frac{ng}{m}p^2}{1-\frac{1}{4mgnp^2}}} \\ &\approx \sqrt{\frac{ng}{m}p^2\left(1+\frac{p^2}{4mgn}\right)} \\ &= \sqrt{\frac{p^2}{2m}\left(\frac{p^2}{2m}+2ng\right)}. \end{aligned} \tag{17}$$

The second line is an expansion in $\frac{1}{n}$, and the third line is the Bogoliubov dispersion relation.

2.3 Noether's theorem and currents

In the notes "Topics in Field Theory", we've also how to obtain currents and conserved charges using Noether's theorem. Let's apply this to various symmetries of our problem.

2.3.1 U(1) symmetry

Noether's theorem applies even though we may choose to rewrite the fields by expanding around one of the vacua (thereby breaking the manifest symmetry). The superfluid Lagrangian, Eq. (9) has the U(1) symmetry $\theta \to \theta_\alpha = \theta + \alpha$. Define $D\theta = \left.\frac{d\theta_\alpha}{d\alpha}\right|_{\alpha=0}$. Then we have $D\theta = 1$. The Noether current is

$$\begin{aligned} J^\mu_{U(1)} &= D\theta \cdot \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \theta\right)} \\ &= \begin{cases} -\rho & \text{if } \mu = 0 \\ -\frac{\rho}{m}\partial_i \theta & \text{if } \mu = i. \end{cases} \end{aligned} \tag{18}$$

Note that in the ground state, ρ is non-zero (i.e, its expectation value in the ground state) because of symmetry breaking so the current is non-zero. However, if the Hamiltonian had looked a bit different (for example, suppose g were negative), the ground state might have been chosen to preserve the symmetry – namely $\rho = 0$. In that case, the ground-state current would have been 0.

By Noether's theorem, $\partial_\mu J^\mu_{U(1)} = 0$, so

$$\dot{\rho} = \frac{1}{m}\nabla \cdot \left(\rho \nabla \theta\right). \tag{19}$$

Following Superfluid States of Matter, by Svistunov, Babaev and Prokof'ev (SBP) https://people.umass.edu/bvs/Book.pdf, we set

$$\mathbf{j} \equiv \frac{1}{m}\left(\rho \nabla \theta\right) \tag{20}$$

and therefore we have $\dot{\rho} = \nabla \cdot \mathbf{j}$. This is a critical equation because, by analogy with a similar equation in hydrodynamics, it describes a continuity equation for matter (particles).

Lancaster, following a slightly different path arrives at a related equation (you would need to take an approximation where the density is close to

the ground state density, and you'd also need to invoke the Euler-Lagrange equation for the density).

The phase is often referred to as an *order parameter*, since has long-range correlations (as a result of the broken symmetry).

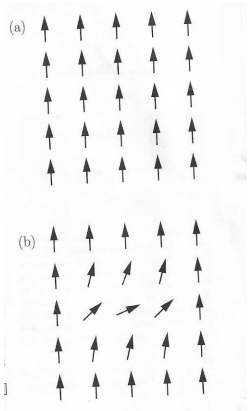


Figure 1: (a) Uniform phase angle in the ground state (b) In a slightly excited state, the density is approximately constant and the current is proportional to the gradient of the phase $\nabla\theta$

Because of the continuity equation, we see that the phase gradient shows the flow of matter in the fluid.

2.3.2 Time translation symmetry

The Noether theorem can also be applied to the symmetry resulting from $t \to t+\alpha$. See section 10.3 in Lancaster. We obtain the Noether current and from that, we derive the conserved "charge" which, for time translation, is the Hamiltonian. (Note that in principle, the Hamiltonian defined in this way, might be different than the Hamiltonian defined as in Hamiltonian mechanics. Nevertheless, they are related.)

$$\begin{aligned} H &= \int d^3x \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_t\theta\right)} \partial_t\theta - \mathcal{L} \right] \\ &= \int d^3x \left\{ \frac{1}{2m} \left[\frac{(\nabla\rho)\cdot(\nabla\rho)}{4\rho} + \rho\left(\nabla\theta\right)\cdot\left(\nabla\theta\right) \right] + \frac{g}{2}\left(\frac{\mu}{g} - \rho\right)^2 \right\}. \end{aligned} \tag{21}$$

2.3.3 Space translation symmetry

Again, refer to section 10.3 in Lancaster. The conserved "charge" for space translations, is the momentum.

$$\begin{aligned} P^i &= \int d^3x \left[\frac{\partial \mathcal{L}}{\partial \left(\partial_0 \theta\right)} \partial^i \theta \right] \\ &= \int d^3x \rho \partial_i \theta. \end{aligned} \tag{22}$$

Notice from Eq. (20) that the momentum density is the mass density times the U(1) current j.

2.4 Superfluid velocity and other fields

Since classical momentum of a fluid is total mass times velocity, and since (Eq.(22)) the superfluid momentum density is particle-mass times the current $\mathbf{j}$, it seems reasonable to conclude that the fluid velocity $\mathbf{v}$ is just $\mathbf{v} = \frac{1}{\rho}\mathbf{j} = \frac{1}{m}\nabla\theta$.

There are more formal ways of arriving at that conclusion. These rely on the observation that the superfluid velocity should be the velocity of a moving (relative to the walls) frame of reference in which the fluid is at rest. Several references which discuss this are https://physics.stackexchange.com/questions/292152/galilean-transformation-in-non-relativistic-quantum-mechanics and Fonda & Ghirardi's Symmetry Principles in Quantum Physics, Sec. 2.5, pgs.83-89:. scribd.com/doc/30539019/, and SBP page 50. (See Stefanovich's derivation of Eq. 1.80[1] https://billcelmaster.com/wp-content/uploads/2024/06/Stefanovich-Galilean-transformation.pdf

SBP also derive the Galilean transformations in changing to the "primed" frame moving at velocity $\mathbf{v}_0$ relative to a unprimed frame.

$$\begin{aligned} \mathbf{j}' &= \mathbf{j} + \rho \mathbf{v}_0 \\ \mathbf{P}' &= \mathbf{P} + Nm\mathbf{v}_0 \\ E' &= E + \mathbf{P}\cdot\mathbf{v}_0 + Nm\frac{v_0^2}{2} \end{aligned} \tag{23}$$

[1]For wavefunctions rather than fields

3 Vortices

We now begin a section which is way out of my comfort zone. I think it would help if one had a good intuition about magnetostatics, especially about the interactions of magnetic field carried by multiple wires. It would also help if one had a good intuition about hydrodynamics, since vortices show up in the classical theory and in particular, are a hallmark of turbulence. Whatever intuition I might once have had from these topics (maybe none?) is gone now – except for recognizing that water draining in a bathtub tends to form a vortex! I also believe that there is an area of mathematics which is directly applicable to the subject matter in the context of solutions to differential equations. Again, if I ever knew this, I've forgotten it, although in this instance I feel reasonably comfortable with the specific mathematical requirements for dealing with vortices in superfluids.

For a treatment of vortices in superfluids, using pictures and intuitions, see Feynman Statistical Mechanics: A Set of Lectures towards the end of chapter 11. Most of my notes will follow material from SBP, although from time to time I'll take inspiration from Lancaster section 42.4.[2] I've also found it useful to refer to the Landau and Lifshitz book Statistical Physics Volume 9 Part II (by Lifshitz and Pitaevskii) around page 112.

3.1 Why do we need to treat vortices separately from quasiparticle excitations?

Since we supposedly have a Lagrangian/Hamiltonian for the superfluid many-body system, why aren't vortices already part of that theory? As far as I can tell, the answer partly has to do with experimental observations. Here's what I think: a variety of theoretical constructs were explored, both to explain some features of rotating superfluids and also to model vortex behavior long known in hydrodynamics physics. What emerged from all this was an acknowledgment of the incompleteness of the superfluid Lagrangian. Clearly no attempt had been made to include, in the Lagrangian, interaction

[2]I'm not entirely convinced that Lancaster's treatment is completely correct. Like many other treatments, Lancaster builds his arguments based on some simplifications and analogies. I think the subject matter suffers from great complexity, and that the physics has progressed by a sequence of credible approximations. To my taste, I would have liked a treatment which is explicit about the details of what is being approximated and what isn't, but maybe it's assumed that I have a richer understanding of fluids or electromagnetism.

terms describing phenomena occurring on the scale of atomic sizes. One phenomenological way to model such phenomena, is to hypothesize that lines, which we'll call *vortex lines* or *vortex cores*, can form with atom-size radii, such that in those cores, the particle density is 0 (and as we'll see later, the flow velocity is singular on those vortex lines).

There are some immediate consequences – to be demonstrated over the course of the next few sections – of such an assumption.

- The vortex lines either form closed loops or else their ends terminate on the walls containing the superfluid.
- The vortex lines become boundary conditions ($\rho = 0$) for solutions of the Euler-Lagrange equations. In the vicinity of each vortex line, the fluid becomes a vortex.
- Just as we can treat the quasiparticle energy-eigenstates as excitations of the ground state, we can also treat the vortices as excitations of the ground state, with energies that are calculated by evaluating the Hamiltonian in the presence of these vortices.
- The vortices can be described as excitations of a Hamiltonian system. That is, vortices can be regarded as microstates orthogonal to the pressure-states of the Bogoliubov theory. They can also be treated using the rules of statistical mechanics based on equal a priori probability of a system finding itself in any one of its vortex microstates.[3] See Onsager, "Statistical Hydrodynamics". A more modern review of this topic appears in the Ph.D. thesis by Thomas Ashbee.https://discovery.ucl.ac.uk/id/eprint/1427632/1/TLA_thesis_final.pdf

3.2 Vortex cores

In general, the fluid vorticity $\mathbf{w}$ is defined as

$$\mathbf{w} = \nabla \times \mathbf{v}. \tag{24}$$

[3]To be honest, I haven't managed to put together a set of references that address, to my satisfaction, the details of this kind of thing. Certainly, the standard literature treats the subject matter as though vortices are quite separate from quasi-particles. I think some of this approach originates with classical hydrodynamics.

Now recall that $\mathbf{v} = \frac{1}{m}\nabla\theta$. This has the consequence that $\mathbf{w} = 0$ since the curl of a gradient is 0. **HOWEVER, THAT IS ONLY TRUE WHEN θ AND ITS GRADIENT ARE DEFINED!!** When a fluid has $\mathbf{w} = 0$, it is called *irrotational*. We'll return to this critical issue, but first, note from Stoke's theorem that

$$\oint_\Gamma \mathbf{v} \cdot d\mathbf{s} = \int_S (\nabla \times \mathbf{v}) \cdot \mathbf{dA} = \int_S \mathbf{w} \cdot \mathbf{dA}. \tag{25}$$

Γ is a contour and S is a surface bounded by the contour, with $\mathbf{ds}$ a vector line element of the contour, and $\mathbf{dA}$ a normal area element of the surface. If θ and its gradient are defined, then the vorticity $\mathbf{w}$ is 0 and the contour integral of $\mathbf{v}$ is 0. We call that contour integral the *circulation*.

In summary, we seem to have demonstrated that an irrotational fluid has no regions where the circulation is nonzero. Yet, it has been known experimentally for a long time, that vortices (which have nonzero circulation) can appear in apparently-irrotational fluids. This can be explained by hypothesizing that in the fluid, there are lines or "cylinders" (which we call cores), of either undefined or non-zero vorticity. We call these "vortex lines" or "vortex cores".

Mathematically the situation for superfluids is reasonably straightforward. Recall that the angle θ is the phase angle of the wave function ϕ, namely $\phi = e^{i\theta}|\phi|$. Even though the wavefunction can be defined everywhere, the same can't be said of θ. When $\phi = 0$ (and therefore $\rho = 0$), the angle becomes ill-defined. This is a bit reminiscent of the change from rectilinear to polar coordinates, where one of the spacial points must be excluded. Consider a situation (Fig. 2) where we have a fluid with a hole in it. We'll refer to this hole as "the core".

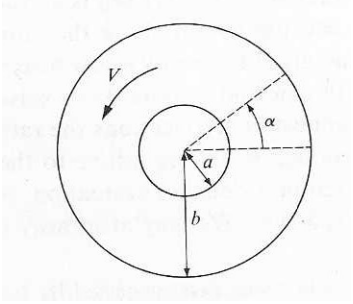


Figure 2: Fluid showing (interior circle) cross-section of a cylindrical vortex core

Surrounding the core is some fluid moving counterclockwise with speed v. Let Γ be a circular contour of radius r surrounding the core Since $v = \frac{1}{m}\nabla\theta$, and since the single-valuedness of ϕ requires that when θ varies along the contour, it must travel an integer multiple of 2π, we can write

$$\oint_\Gamma \mathbf{v} \cdot d\mathbf{s} = \frac{1}{m}\int_0^{2l\pi} d\theta = \frac{2l\pi}{m}. \tag{26}$$

The number l is called the *winding number*. We see that the contour integral is effectively quantized, although we've used a classical rather than quantum-mechanical argument.

How can the integral be nonzero? We previously had concluded from Stokes' theorem that the line integral had to be 0 on account of the vorticity being 0 on a bounding surface. However, as previously noted, the theorem is only applicable when θ is defined everywhere on the bounding surface – that is, when $\rho \neq 0$ everywhere on the interior of Γ. In our example, $\rho = 0$ in the core, and therefore θ is undefined in the core. If the core crosses every bounding surface, then Stokes' theorem doesn't apply. This condition requires that the core either forms a loop or else its two ends must terminate on the boundaries (the walls of the container). To see this, suppose the core is simply a line segment (or rather, a cylindrical segment) as in Fig. 3.

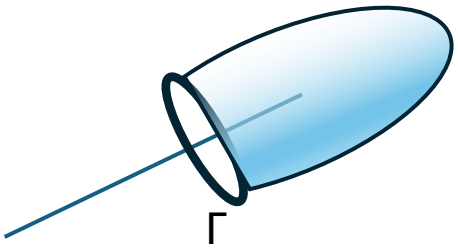


Figure 3: The contour γ goes around the core, which is a line segment. However, we can pick a bounding surface which is not crossed by the core.

Since the core doesn't intersect the bounding surface in this example, ϕ is defined everywhere on the surface thus the vorticity is 0 on the surface and the line integral has to be 0.

Where do vortex cores come from? I don't happen to have a good answer to that question, although Feynman (pages 339-342) provides a motivation that I haven't bothered to sort through. Vortex cores are thin cylinders where the particle density becomes 0. On page 336, Feynman argues based on surface-tension, that the radius of a core is approximately 0.4 Å, thus negligible, so he refers to the cores as "vortex lines".

3.3 Solving the Gross-Pitaeskii equation for a cylindrical vortex

We have previously run into the Gross-Pitaeskii (GP) equation as the Euler-Lagrange equation of the non-relativistic limit of a super-cooled fluid with a delta-function interaction potential. In general, one also adds to the potential and external potential of the form $V(\mathbf{x})$ (which one obtains by adding that potential term to the Lagrangian). The GP equation is a generalization of the Schrodinger equation.

Of particular interest is the time-independent (aka stationary) solution of the GP equation. This can be shown to be a solution for the minimum of H' as a function of the field and therefore a candidate for the ground-state (recall the correspondence between the coherent state and the field). The GP equation we'll study is

$$\left[\frac{-\nabla^2}{2m} + V(\mathbf{x}) - \mu(\mathbf{x}) + g|\phi_0(\mathbf{x}')|^2\right]\phi_0(\mathbf{x}) = 0 \tag{27}$$

where ϕ_0 denotes a solution to the equation.

We'll use the external potential V as a way of including the walls. Since we want to solve the GP equation for a vortex, we'll set up a simple situation.

- The container of helium is a cylinder (similar to Fig. 5) of radius R and length L. To describe this, use variables θ, d and z where d is the distance from the center of the cylinder.
- To implement the effect of the walls we'll set V to

$$V(\mathbf{x}) = \begin{cases} 0 & \text{if } s < R \\ +\infty & \text{if } |s \geq R \end{cases} \tag{28}$$

- To implement the cylindrical symmetry, set $\phi_0(\mathbf{x}) = \sqrt{\rho_0(d)}e^{il\theta}$, noting that the phase must be quantized by the condition above for the contour integral on circular contours around the vortex core.
- As a constraint (after all, we're studying a vortex) assume the vortex core is a straight line through the center of the cylinder going from one end-wall to the other. The boundary condition is $\rho(0) = 0$.
- Take L to be very large so that we can ignore boundary effects at the end-walls.

We rewrite the GP equation (Eq. (27)) in cylindrical coordinates and we also simplify it by changing fields and variables to

$$\begin{aligned} \tilde{\rho} &= \frac{g}{\mu}\rho, \\ \alpha &= \sqrt{m\mu}d. \end{aligned} \tag{29}$$

The resulting equation is

$$\tilde{\rho}'' + \frac{\tilde{\rho}'}{\alpha} - 2\left(\tilde{\rho}^2 - 1 + \frac{l^2}{2\alpha^2}\right)\tilde{\rho} = 0, \tag{30}$$

where an expression f' is the derivative of f with respect to α.

The numerical solution is shown in Fig. 4 (taken from `https://ethz.ch/content/dam/ethz/special-interest/phys/theoretical-physics/cmtm-dam/documents/qg/Chapter_04.pdf`).

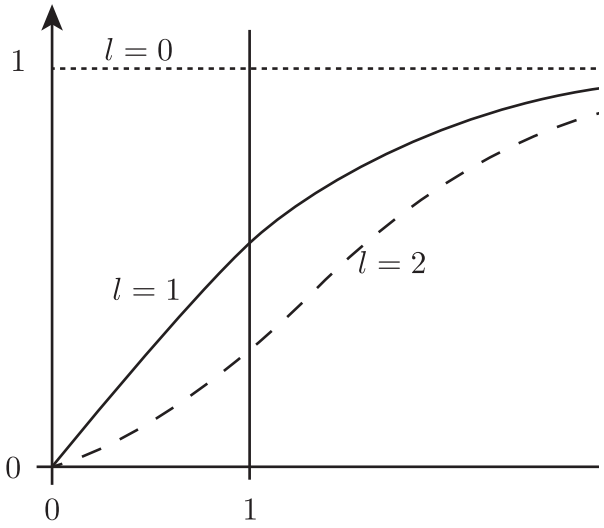


Figure 4: Density of cylindrical vortices for winding numbers 0, 1 and 2. The x-axis is the dimensionless distance α defined in the text. The curves asymptote to 1 as $\alpha \to \infty$.

It is easy to approximately solve the GP equation, Eq. (30) near $\alpha = 0$ and also asymptotically for large values of α (but recall that the solution only applies within the walls of the container).

Near the vortex core, the density is $\tilde{\rho}(\alpha) \propto \alpha^l$.

It's also easy to compute the velocity of the fluid circling the core. From Eq. (26), we obtain

$$2\pi v d = \frac{2l\pi}{m}$$

so

$$v = \frac{l}{dm}. \tag{31}$$

Near the center of the vortex (as the density goes to zero), the velocity grows in inverse proportion to the distance from the core. This is familiar from our experience of water draining out of the bathtub.

Far from the core, we can derive that the density becomes $\tilde{\rho}(\alpha) \approx 1 - \frac{l^2}{4\alpha^2}$.

3.4 Energy of a cylindrical vortex

We can compute the kinetic energy of the vortex as $\frac{-\nabla^2}{2m}\phi_0 = \tilde{\rho}'' + \frac{\tilde{\rho}'}{\alpha} - l^2\frac{\tilde{\rho}}{\alpha^2}$. Far from the core, this is $m\tilde{\rho}(\alpha)\frac{v^2}{2}$ noting that $m\tilde{\rho}$ is the mass density. Using Eq. (31) for the velocity, and working in cylindrical coordinates where the volume element (for a cylindrically symmetrical integrand) is $\frac{1}{m\mu}2\pi\alpha d z d\alpha$,[4] we have

$$\begin{aligned} KE_{>r_c} &= \frac{2\pi}{m\mu}L\int_{\tilde{r_c}}^{\tilde{R}} m\rho(\alpha)\frac{v^2}{2}\alpha d\alpha \\ &= \frac{2\pi}{mg}L\int_{\tilde{r_c}}^{\tilde{R}} m\tilde{\rho}(\alpha)\frac{v^2}{2}\alpha d\alpha \\ &= \frac{2\pi}{mg}L\int_{\tilde{r_c}}^{\tilde{R}} m\tilde{\rho}(\alpha)\frac{l^2}{2m^2d^2}\alpha d\alpha \\ &= \frac{\pi L l^2\mu}{mg}\int_{\tilde{r_c}}^{\tilde{R}}\left(1-\frac{l^2}{4\alpha^2}\right)\frac{1}{\alpha^2}\alpha d\alpha \\ &= \frac{\pi L\rho_0 l^2}{m}\left(\log(R/r_c) + \mathcal{O}(\frac{1}{r_c^2})\right), \end{aligned} \tag{32}$$

where $\tilde{R} = \rho_0 R$, $\tilde{r_c} = \rho_0 r_c$ and $\rho_0 = \frac{\mu}{g}$. The cutoff value r_c is picked arbitrarily, somewhere around the point that the density curve begins to flatten out in Fig. 4. (Sometimes we set $\tilde{r_c}$ to 1 which we call *the healing length* lies about halfway up the hill of the curve as shown in Fig. 4.) Since the energy grows logarithmically out to the cylindrical walls containing the Helium, the precise location of r_c doesn't matter provided it doesn't become too small.

Let's confirm that the above-computed kinetic energy is larger than the kinetic energy from the region between the vortex core and the cutoff r_c. In Eqs.(32), replace the integrand by the full KE. The KE is $\tilde{\rho}'' + \frac{\tilde{\rho}'}{\alpha} - l^2\frac{\tilde{\rho}}{\alpha^2}$. Its value near the core using $\tilde{\rho}(\alpha) \approx k\alpha^l$ is 0. The calculation leaves out corrections as α approaches r_c from below, but based on the numerical results shown in Fig. 4, we can see that the approximation should be reasonably useful and makes clear that the kinetic energy is dominated by $\frac{\pi L\rho_0 l^2}{m}\log(\tilde{R}) = \frac{\pi L\rho_0 l^2}{m}\log(R\sqrt{m\mu})$.

So far, I've only talked about the kinetic energy. But the fluid has potential energy, to be computed from the potential term in the Hamiltonian. What is its contribution to the total energy? I'll proceed to show that its contribution is approximately 0. The reason for this, is that by definition, our vortex

[4] I've converted from the coordinate d to $\alpha = \sqrt{m\mu}d$.

is a solution of the equation which minimizes the Hamiltonian subject to a boundary condition where the density is 0 along the vortex core. That minimum evidently is achieved when the potential energy is 0 (which occurs in the valley of the Mexican hat).

The energy is obtained by computing the Hamiltonian given in Eq. (5) ,

$$H' = \int d^3x \left(\frac{1}{2m} \boldsymbol{\nabla}\phi^\dagger(\mathbf{x}) \cdot \boldsymbol{\nabla}\phi(\mathbf{x}) + \frac{g}{2} \left(\frac{\mu}{g} - \phi^\dagger(\mathbf{x})\phi(\mathbf{x}) \right)^2 \right).$$

Change to cylindrical coordinates, write $\phi = \rho e^{il\theta}$.

$$\begin{aligned} E_{>r_c} &= \frac{\pi L \mu}{mg} \int_{\tilde{r_c}}^{\tilde{R}} \left[\left(\frac{d\sqrt{\tilde{\rho}}}{d\alpha} \right)^2 + \frac{l^2}{\alpha^2}\tilde{\rho} + \frac{1}{2}\left(\tilde{\rho} - 1\right)^2 \right] \alpha d\alpha \\ &= \frac{\pi L \rho_0 l^2}{m} \left(\log(R/r_c) + \mathcal{O}(\frac{1}{\tilde{r}_c^2}) \right). \end{aligned} \tag{33}$$

We arrive at the last line of the equation by taking the large α approximation and then expanding in powers of $\frac{1}{\alpha}$.

3.5 Vortices in a rotating cylinder

Suppose we start with solid helium in a can – obtained by putting it under lots of pressure near 0° K. Next we rotate the can. Then release the pressure to melt the helium.

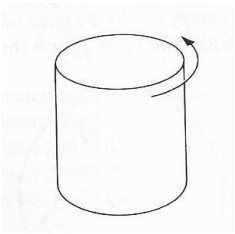


Figure 5: Rotating can of helium

Since the helium starts with a non-zero angular momentum, after melting it should have the same angular momentum. The temperature is close to 0, so (following the rules of statistical mechanics) the system will settle into a

state with the lowest energy that preserves the initial angular momentum. We will allow the system to have vortex lines and as we'll see, those vortex lines will be essential parts of that lowest-energy state.

Start the analysis by computing the initial kinetic energy of the just-melted helium. The rotation is rigid, with each particle rotating around the central axis with angular velocity ω. Assume the can has a radius R and a length L. Furthermore, assume the initial particle density is uniformly ρ_0 and that the mass density is $m\rho_0$. The kinetic energy E_R^K of rigid motion is

$$\begin{aligned} E_R^K &= 2\pi L \int_0^R \left(m\rho_0 \frac{(r\omega)^2}{2} \right) r dr \\ &= \pi L m \rho_0 \omega^2 \frac{R^4}{4}. \end{aligned} \tag{34}$$

Also, the angular momentum is

$$\begin{aligned} J &= 2\pi L \int_0^R \left(m\rho_0 r^2 \omega \right) r dr \\ &= \pi L m \rho_0 \omega \frac{R^4}{2}. \end{aligned} \tag{35}$$

Notice that I've been inconsistent in notation, and am now using the variable r to measure the distance from the axis – whereas earlier I used the variable d. Sorry ...

Let's compare this initial state with a single cylindrical vortex state with winding number l. We've computed in Eq. (33) that its energy, outside of a critical radius r_c is

$$E_{>r_c} = \frac{\pi L \rho_0 l^2}{m} \left(\log(R/r_c) + \mathcal{O}(\frac{1}{\tilde{r}_c^2}) \right).$$

We can also compute the angular momentum J_v of this vortex state.

$$\begin{aligned} J_v &= 2\pi L \int_0^R \left(m\rho_0 r \frac{l}{rm} \right) r dr \\ &= \pi L \rho_0 l R^2, \end{aligned} \tag{36}$$

where we've used the relationship for cylindrical vortices, that $v = l/rm$. Since angular momentum is conserved, if the solid helium were eventually to become a single cylindrical vortex, then we'd have to set $J_v = J$. This leads to an equation for the winding number,

$$l = m\omega \frac{R^2}{2}. \tag{37}$$

In case you're wondering about units, there is an implicit factor of $\hbar$ which we've set to 1 (so reinserting $\hbar$ we get $l = m\omega\frac{R^2}{2\hbar}$). Also, since l needs to be an integer, the equation is only approximate – we set l to be the closest integer to the RHS.

Then, with this value of the winding number, we can evaluate $E_{>r_c}$ using Eq. (33).

$$\begin{aligned} E_{>r_c} &= \frac{\pi L \rho_0 l^2}{m}\left(\log(R/r_c) + \mathcal{O}(\frac{1}{\tilde{r}_c^2})\right) \\ &= \frac{\pi L m\omega^2 \rho_0 R^4}{4}\left(\log(R/r_c) + \mathcal{O}(\frac{1}{\tilde{r}_c^2})\right) \\ &= E_R^K\left(\log(R/r_c) + \mathcal{O}(\frac{1}{\tilde{r}_c^2})\right), \end{aligned} \tag{38}$$

where the last line is obtained using Eq. (34).

What this shows us, is that the single cylindrical vortex has a higher energy than the kinetic energy of the original solid helium. However, the solid helium also has a large (negative) potential energy – required for keeping the helium atoms stuck rigidly together – which we haven't evaluated. So this single vortex state isn't thermodynamically favored over the original rigid configuration.

However, if we can find a vortex configuration whose total energy is less than the original kinetic energy of solid helium, then that vortex configuration could be thermodynamically favored. We hypothesize that this can be accomplished with a vortex array – a collection of evenly spaced vortex lines as in Fig. 6.

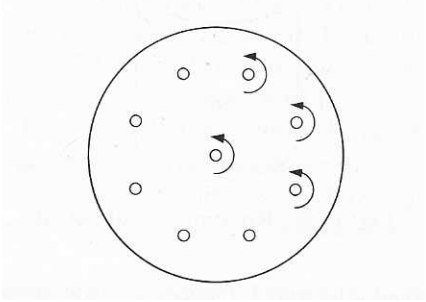


Figure 6: A circular portion of the helium container, seen from above

Assume that the lines are closely spaced and of uniform density throughout

the container.[5] Furthermore, assume that around **each** vortex line, the phase winding number is $l = 1$. Then if we draw a circular contour of radius r around the central axis of the helium container, it will enclose a number $l(r)$ of vortex lines. See, for example, Fig. 6). The winding number around that contour will be $l(r)$. Then with the same argument used previously to obtain the velocity as a function of r, we find

$$v(r) = \frac{l(r)}{rm}. \tag{39}$$

Setting the total number of lines to be $l(R)$, it is easy to see that

$$l(r) = \frac{r^2}{R^2} l(R), \tag{40}$$

and therefore

$$v(r) = \left(\frac{l(R)}{mR^2}\right) r. \tag{41}$$

Notice that this equation shows that the (average) fluid velocity is that of a rigid solid rotating around the center axis with an angular velocity $\omega' = \frac{l(R)}{mR^2}$.

We can calculate the total angular momentum J_A of the vortex array similarly to what we did before in Eq. (36). It's initially a bit puzzling that the fluid seems to behave rigidly, despite the fact that one would expect at the microscopic level, to see mini-"tornadoes" encircling each vortex line. However, looking at Fig. 6, we notice that in between two vortex lines, the respective vortex velocity fields are in opposite directions and thus in the computation of total angular momentum, they tend to cancel (since the velocity appears linearly in that calculation). This has the effect of leading to the total angular momentum of a rigid body.

$$\begin{aligned} J_A &= 2\pi L \int_0^R \left(m\rho_0 r \frac{l(r)}{rm}\right) r dr \\ &= 2\pi L \int_0^R \left(m\rho_0 r \frac{r^2 l(R)}{R^2 rm}\right) r dr \\ &= \frac{\pi L \rho_0 l(R) R^2}{2}. \end{aligned} \tag{42}$$

Notice that the minimum winding number for a line is $l = 1$. That is, the vorticity is quantized. To find the total number $l(R)$, recall that angular

[5]Further argumentation is needed in order to convince ourselves that the lowest-energy configuration is one where the lines are of constant density. I haven't constructed that argument although it seems reasonable.

momentum is conserved and therefore $J_A = J$. Using Eq. (35) we get

$$l(R) = m\omega R^2, \tag{43}$$

and the line density is $\tilde{l} = \frac{l(R)}{\pi R^2} = \frac{m\omega}{\pi\hbar}$, where I've re-inserted $\hbar$. Notice that $\omega = \omega'$. That is, if the original solid helium rotates with angular velocity ω, then the superfluid flow also rotates uniformly with the same angular velocity ($\omega' = \omega$) but unlike the solid helium, the superfluid is traversed by an array of vortices.

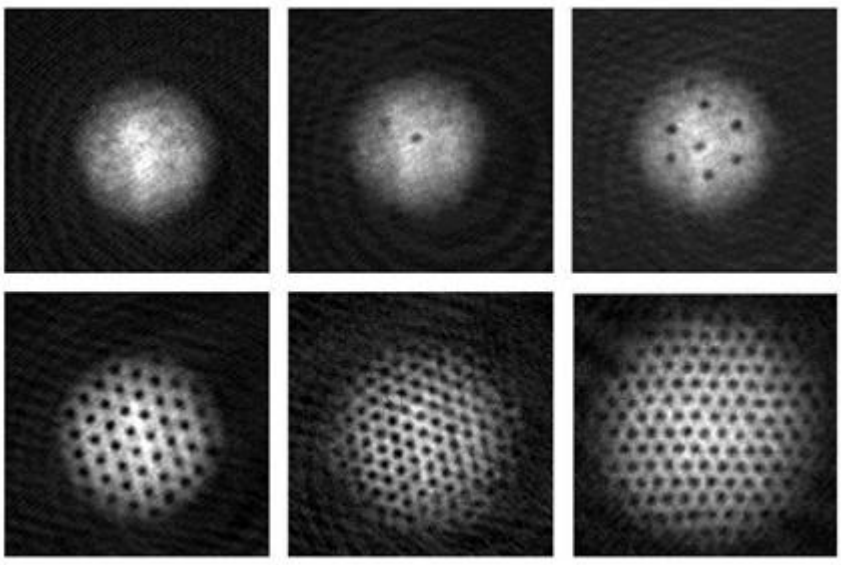


Figure 7: Vortices in a Bose Einstein Condensate. The dark spots are the cores of the vortices.

Feynman (p. 339 of Statistical Mechanics) computes (by taking the square-root of $\tilde{l}$) that if the rotation speed is $\omega = 1$ radian/sec, then the cores are about 2 mm apart.

Finally, we are ready to compute the energy of the vortex array. Unlike the computation of angular momentum, the kinetic energy involves the square of the velocity and that quantity doesn't cancel between vortex lines. On the other hand, the squared velocity drops off quickly between vortex cores, and the energy calculation can be done by assuming each vortex has a radius of $b = 2$mm and a cutoff $r_c = 4.0$Å. Then, since each vortex has a winding

number $l = 1$, we obtain for the energy E_A of the array,

$$\begin{aligned} E_A &= l(R) \times \text{Energy of a vortex of radius b} \\ &= (m\omega R^2)\frac{\pi L \rho_0}{m}\log(b/r_c) \\ &= \omega L \pi R^2 \rho_0 \log(b/r_c) \\ &= 14\rho_0 \omega \pi L R^2, \end{aligned} \tag{44}$$

where the second line comes from Eq. (33) but substituting, in the expression for the vortex energy, b for R and setting l=1. Note that even if we were to change the core-width from $4.0\mathring{A}$ to $0.4\mathring{A}$, that would only change the result by about 20%.

We are now ready to compare the energy of the vortex array to the kinetic energy of the rigidly rotating helium, as given by Eq. (34).

$$\begin{aligned} \frac{E_A}{E_R^K} &= \frac{14\rho_0\omega\pi L R^2}{\pi L \frac{m}{\hbar}\rho_0 \omega^2 \frac{R^4}{4}} \\ &= 4\hbar\frac{14}{m\omega R^2}. \end{aligned} \tag{45}$$

If we take R = 1 cm and $\omega = 1$ rad/sec, then the above energy ratio is about 10^{-2}. The vortex array is thus thermodynamically preferred at low temperature to the rigid configuration[6], and is what we'd expect to see at low temperatures.

3.6 Ring vortices

Recall that a vortex core either is a line whose endpoints are terminated on a boundary, or is closed loop. The most symmetric[7] loop is a circular ring such as the smoke ring shown in Fig. 8

[6]Since the rigid configuration has a negative potential energy, it seems possible to me that its total energy ends up less than that of the vortex array. I haven't seen an argument against that, although maybe it would be unreasonable to assume that the potential energy has roughly the same magnitude as the kinetic energy.

[7]A circular ring is presumably the lowest-energy loop of a given perimeter. I haven't thought too much about this, but it's a typical kind of result.

Figure 8: A smoke ring. The core is the center and all around it, there is a vortex flow.

We will proceed to show that a ring vortex moves with a characteristic velocity perpendicular to the ring, and we will compute both its energy and momentum. The dispersion curve (energy versus momentum) of these rings can be regarded as the dispersion curve of elementary excitations, from which we can compute a Landau critical velocity. If the fluid velocity is greater than the ring Landau critical velocity, then ring formations will be thermodynamically favored and the fluid velocity will decrease. In other words, above that critical velocity, the fluid is not a superfluid.

The reason this is important, is that for typical configurations, the ring critical velocity is much smaller than the critical velocity computed from normal modes (the Bogoliubov theory). This explains why experimentally, we find much smaller critical velocities than what is predicted by the Bogoliubov theory.

The ring-vortex dispersion relation is derived in Landau-Lifshitz Vol. 9 (Statistical Physics) on pp. 115 and 116 (Problem 1 and its solution). The derivation relies on a generalization of Eq. (31), which in turn was derived from the circulation around a straight-line vortex core. For an arbitrary-shape vortex filament, this generalized equation for a winding number l, is

$$\mathbf{v} = \frac{l}{m} \int \frac{d\mathbf{l} \times \mathbf{r}}{r^3} \tag{46}$$

where the integration is along the filament, and $\mathbf{r}$ is the radius vector from $d\mathbf{l}$ to the point where the velocity is observed. This equation is derived in the same way that the Biot-Savart law is derived for magnetostatics, since the relation between vorticity and velocity is the same as the relationship between current and magnetic field.

PROBLEM 1. Find the velocity and momentum of a circular vortex ring.

SOLUTION. Each element of the ring moves with the velocity $\mathbf{v}_s$ at a given point, and from the symmetry of a circular ring this velocity is the same at every point of it. It is therefore sufficient to determine the velocity $\mathbf{v}_s$ at any one point P of the ring due to the rest of the ring. The elements $d\mathbf{l}$ of the ring and the radius vectors $\mathbf{R}$ from $d\mathbf{l}$ to the point P are in the plane of the ring; hence the velocity at the point P, given by (29.4), is perpendicular to the plane of the ring, as a result of which the ring moves without change of shape or size.

Let us define the position of the element $d\mathbf{l}$ by the angle ϑ (Fig. 3). Then

$$dl = R_0\, d\vartheta, \quad R = 2R_0 \sin \tfrac{1}{2}\vartheta, \quad |d\mathbf{l}\times\mathbf{R}| = R \sin \tfrac{1}{2}\vartheta . dl,$$

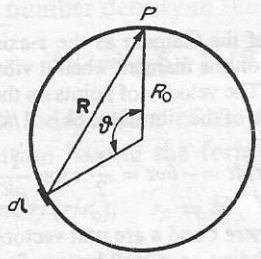


FIG. 3.

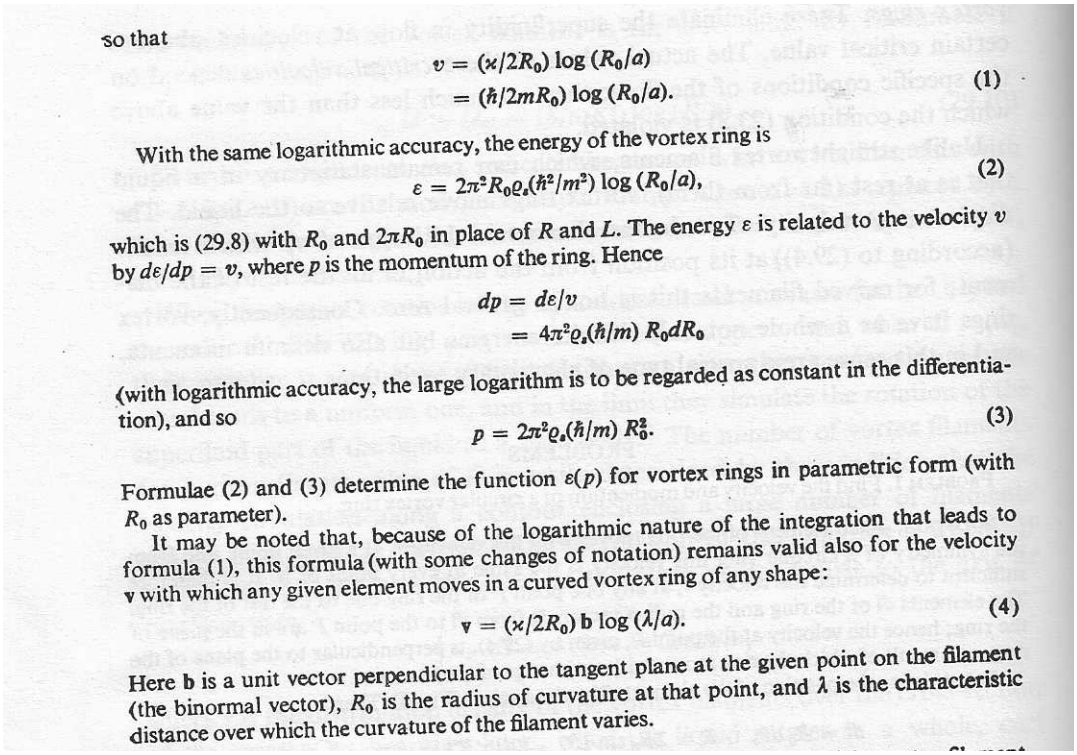
where R_0 is the radius of the ring, and we find from (29.4) for the ring velocity v

$$v = \frac{\varkappa}{8R_0} . 2 \int_0^{\pi} \frac{d\vartheta}{\sin \frac{1}{2}\vartheta} .$$

This integral, however, is logarithmically divergent at the lower limit, and must be cut off at a value $\vartheta \sim a/R_0$ corresponding to atomic distances ($\sim a$) of the element $d\mathbf{l}$ from the point P. The integral is determined, with logarithmic accuracy, by the range of values $a/R_0 \ll \vartheta \ll \pi$, and is

$$\int_{-a/R_0}^{-1} \frac{2\, d\vartheta}{\vartheta} = 2 \log \frac{R_0}{a} ,$$

Figure 9: Derivation of vortex ring dispersion relation. P. 115 of Landau and Lifshitz Vol. 9.



so that

$$v = (\varkappa/2R_0)\log(R_0/a) = (\hbar/2mR_0)\log(R_0/a). \quad (1)$$

With the same logarithmic accuracy, the energy of the vortex ring is

$$\varepsilon = 2\pi^2 R_0 \varrho_s(\hbar^2/m^2)\log(R_0/a), \quad (2)$$

which is (29.8) with R_0 and $2\pi R_0$ in place of R and L. The energy ε is related to the velocity v by $d\varepsilon/dp = v$, where p is the momentum of the ring. Hence

$$dp = d\varepsilon/v = 4\pi^2\varrho_s(\hbar/m)\,R_0 dR_0$$

(with logarithmic accuracy, the large logarithm is to be regarded as constant in the differentiation), and so

$$p = 2\pi^2\varrho_s(\hbar/m)\,R_0^2. \quad (3)$$

Formulae (2) and (3) determine the function $\varepsilon(p)$ for vortex rings in parametric form (with R_0 as parameter).

It may be noted that, because of the logarithmic nature of the integration that leads to formula (1), this formula (with some changes of notation) remains valid also for the velocity $\mathbf{v}$ with which any given element moves in a curved vortex ring of any shape:

$$\mathbf{v} = (\varkappa/2R_0)\,\mathbf{b}\log(\lambda/a). \quad (4)$$

Here $\mathbf{b}$ is a unit vector perpendicular to the tangent plane at the given point on the filament (the binormal vector), R_0 is the radius of curvature at that point, and λ is the characteristic distance over which the curvature of the filament varies.

Figure 10: Derivation of vortex ring dispersion relation. P. 116 of Landau and Lifshitz Vol. 9.

In Fig. 10, the authors use ρ for mass-density, whereas we have been using ρ for number-density. I'll continue to use our convention (but it should be said that our convention is non-standard – usually the number-density is written n.)

The energy, $E_{\rm ring}$ of a vortex ring of radius R_r is

$$E_{\rm ring} = 2\pi^2 R_r \rho_0 \frac{\hbar^2}{m}\log(\frac{R_r}{r_c}), \quad (47)$$

and the momentum, $p_{\rm ring}$ of that ring is

$$p_{\rm ring} = 2\pi^2\rho_0\hbar R_r^2. \quad (48)$$

The energy equation is, as usual, logarithmically dependent on the vortex-core cutoff, so for large values of R_r the result is fairly insensitive to r_c. Also, for clarity I've re-inserted the correct factors of $\hbar$.

From these equations, we have the dispersion relation

$$E_{\text{ring}} = \sqrt{\left(2\pi\frac{\hbar}{m}\right)\left(\pi\rho_0\frac{\hbar^2}{m}\right)}\sqrt{p_{\text{ring}}}\log\left(\frac{p_{\text{ring}}}{\left(2\pi^2\rho_0\frac{\hbar}{m}\right)mr_c^2}\right). \tag{49}$$

This has the shape of Fig. 11.

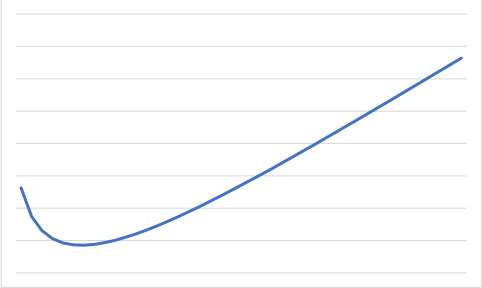


Figure 11: Vortex ring dispersion curve

Recall that for a vortex of radius R_r the equation for the Landau critical velocity is

$$v_c = \min_{p_{\text{ring}}} \frac{E_{\text{ring}}}{p_{\text{ring}}}. \tag{50}$$

Since there is exactly one value of p_{ring} for each ring-radius R_r, there is no need to take a minimum so from Eqs. (47) and (48) we get

$$v_c = \frac{\hbar}{m}\frac{1}{R_r}\log(\frac{R_r}{r_c}). \tag{51}$$

We see that the critical velocity decreases as the ring-radius increases. However, the ring-radius is approximately capped by the width D of the enclosure, so we'll take the superfluid vortex-induced critical velocity to be

$$v_c^V = \frac{\hbar}{m}\frac{1}{D}\log(\frac{D}{r_c}). \tag{52}$$

As an example, take $D = 10^{-5}$cms and $r_c = 4\mathring{A}$, then the critical velocity is about 80 cm/sec. The measured critical velocity is about 20 cm/sec (according to Feynman) so is in the right ballpark.