Conductivity – The Drude Theory of Metals

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January 21, 2025

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Introduction

- Goal: Prerequisites for Lancaster Chptr 44 on superconductors
- Summarize some key concepts of solid state physics
- Reference material:
  - James Annett Superconductivity, Superfluids and Condensates
  - Charles Kittell Introduction to Solid State Physics
  - Neil Ashcroft and N. David Merman Solid State Physics
- My versions of those texts are available at `https://www.dropbox.com/scl/fo/r34zxultk39xm75lespd9/ADgag5MONbaB3t5Vl5-0rDU?rlkey=gquuntsf650aid2gumn5eo9qf&st=8pbuxhqt&dl=0`.

Outline

- Follows Ashcroft and Mirman
- Drude theory developed around 1905
- Is basis for the modern understanding of conductivity
- Sections:
  - Assumptions and some definitions
  - DC Conductivity
  - Hall Effect
  - AC Conductivity

Assumptions I

- Metal has N fixed ions + electron gas (**<u>conduction electrons</u>**).
  - The electron gas consists of n_v valence electrons per nucleus.
  - The ions consist of the nucleus plus bound electrons.
  - Ionic charge $= Z - n_v e$
  - Total charge of electron gas $= n_v N e$.
  - We'll generally set $n_v = 1$.

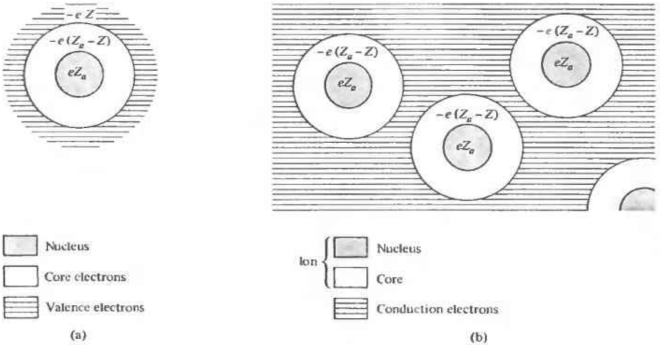


Figure 1.1
(a) Schematic picture of an isolated atom (not to scale). (b) In a metal the nucleus and ion core retain their configuration in the free atom, but the valence electrons leave the atom to form the electron gas.

Assumptions II

- **Independent electron approximation**

  Neglect $e^- - e^-$ interactions.

  A suprisingly good approximation for many metallic phenomena
- **Free electron approximation**

  Neglect ionic field on e^-.

  A poor approximation for many metallic phenomena
- Electron-ion collisions result in rapid velocity changes
- Between collisions, electrons obey $\mathbf{F} = m\mathbf{a}$ for external $\mathbf{F} = m\mathbf{a}$
- Electron probability of collision in time dt is $\frac{dt}{\tau}$.
  - τ is called the *relaxation time*
  - Electron will on average travel for τ before next collision.
  - Electron on average traveled for τ after last collision.
- τ is independent of electron position or velocity.
- Electrons maintain thermal equilibrium only thru collisions
  - Outgoing velocity is independent of ingoing velocity
  - Outgoing velocity is tied to local temperature
  - E.g. outgoing velocities are higher where local temp is higher

DC Conductivity I

- We will derive Ohm's Law $V = IR$ from the Drude Theory.
  - V, the voltage, is the work to move an electron across a wire.
  - I, the current, is the charge flowing per unit-time in the wire.
  - R, the resistance (of the wire), is independent of V and I.
- Assume for now, that
  - There is a straight metal wire of length L
  - The metal has constant density
  - The metal has x-sectional area A with unit normal $\hat{\mathbf{n}}$. $\mathbf{A} \equiv A\hat{\mathbf{n}}$.
  - There is a uniform electric field $\mathbf{E} \perp \mathbf{A}$

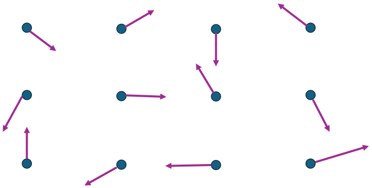
DC Conductivity II

- V_d is the work per charge, $\frac{W}{q}$, to move a charge q a distance d.
  - Electrons have charge $q = -e$.
  - $W_e = \mathbf{F} \cdot \mathbf{d} = -e\mathbf{E} \cdot \mathbf{d}$ so $\frac{W_e}{-e} = \mathbf{E} \cdot \mathbf{d}$.
  - Simplify by having $\mathbf{d} \perp \mathbf{A}$ so $\frac{W}{-e} = Ed$
  - Then $V = EL$ and $V_d = V\frac{d}{L}$
- Define the current density $\mathbf{j} = j\hat{\mathbf{n}}$.
  - j is the charge flowing per unit time per unit area.
  - So $I = jA$.
  - Define the resistivity, ρ, by $\mathbf{E} = \rho\mathbf{j}$.
    - Since $V = EL$ then $V = \rho jL = I\frac{\rho L}{A}$.
    - Since $R = \frac{V}{I}$, we see $R = \frac{\rho L}{A}$.

DC Conductivity III

To derive Ohm's Law, we must show ρ is independent of I and V.

- Recall thermal assumption



Collision outgoing velocities are independent of ingoing velocities

- Each electron accelerates according to $\mathbf{F} = m\mathbf{a}$ for a time Δt.
- So $\mathbf{v} = \mathbf{v}_o + \mathbf{a}\Delta t = \mathbf{v}_o + \frac{\mathbf{F}}{m}\Delta t$
- Each e^- travels for an average time $\overline{\Delta t} = \tau$ before next collision.
- Outgoing velocities $\mathbf{v}_o$ average to $\bar{\mathbf{v}}_o = 0$.
- Since $\mathbf{F} = -e\mathbf{E}$, $\mathbf{v} = \mathbf{v}_o - \frac{e}{m}\Delta t\mathbf{E}$ and $\bar{\mathbf{v}} = -\frac{e}{m}\tau\mathbf{E}$.

DC Conductivity IV

- Assume an electron density of n.
- In time dt electrons advance $\bar{v}dt$.
- So $n\bar{v}dt$ electrons cross area A in time $dt \implies$
  - $\mathbf{j} = -ne\bar{\mathbf{v}} = \frac{ne^2}{m}\tau\mathbf{E}$
- Recall that resistivity is defined by $\mathbf{j} = \frac{1}{\rho}\mathbf{E}$, so
  - $\rho = \frac{m}{ne^2\tau}$
  - **This shows that ρ is independent of voltage and current.**

- Measure ρ for metals using $\rho = \frac{AR}{L}$.
- Compute τ for metals using $\tau = \frac{m}{ne^2\rho}$.
- Mean free path $MFP = \bar{v}\tau$.
  - Classical kinetic theory $\implies \frac{1}{2}m\bar{v}^2 = \frac{3}{2}k_BT$
  - So[1] $MFP = \frac{\sqrt{3k_BmT}}{ne^2\rho}$
  - Substitute measured values of ρ and set T to room temp
- Gives typical values of MFP for metals as $1\AA - 10\AA$
  - $\approx$ interatomic spacing, consistent with Drude model
  - **But for free electrons, classical kinetic theory is WRONG**

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[1] It's common, but wrong, to say that $\bar{v} = \sqrt{\bar{v^2}}$, but for highly-peaked dists., the approximation is valid.

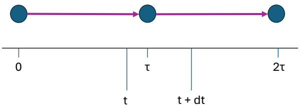
Generalized Force Law I

- We derived $\bar{\mathbf{v}} = -\frac{e}{m}\tau\mathbf{E}$ for a DC electric field.
- Next we will generalize this to

$$\frac{d\bar{\mathbf{p}}(t)}{dt} = -\frac{\bar{\mathbf{p}}}{\tau} + \mathbf{F}(t)$$

- Note that in steady state, the LHS $= 0$ so $\bar{\mathbf{p}} = \tau\mathbf{F}(t)$.
  - When $\mathbf{F}(t) = -e\mathbf{E}$ we get $\bar{\mathbf{v}} = -\frac{e}{m}\tau\mathbf{E}$.

- Consider an electron traveling between random time t and $t + dt$



The probability of a collision between t and $t + dt$ is $\frac{dt}{\tau}$.

- The probability of no collision between t and $t + dt$ is $1 - \frac{dt}{\tau}$

Generalized Force Law II

- First assume an electron has no collision.
  - Its momentum changes from $\mathbf{p}(t)$ to $\mathbf{p}(t) + \mathbf{F}(t)dt + \mathcal{O}(dt^2)$
  - Factor in the probability $1 - \frac{dt}{\tau}$
  - So the average momentum at $t + dt$ is

$$\begin{aligned}\bar{\mathbf{p}}(t+dt) &= \left(1 - \frac{dt}{\tau}\right)\left(\bar{\mathbf{p}}(t) + \mathbf{F}(t)dt + \mathcal{O}(dt^2)\right) \\ &= \bar{\mathbf{p}}(t) - \bar{\mathbf{p}}(t)\frac{dt}{\tau} + \mathbf{F}(t)dt + \mathcal{O}(dt^2)\end{aligned}$$

- Then add the electrons with a collision in interval dt.
  - Probability $\frac{dt}{\tau}$ multiplied by $\mathbf{F}(t)dt'$ with $dt' \leq dt$.
  - Contribution is $\mathcal{O}(dt^2)$.
- Only the no-collision situation matters.

$$\frac{\bar{\mathbf{p}}(t+dt) - \bar{\mathbf{p}}(t)}{dt} = -\frac{\bar{\mathbf{p}}(t)}{\tau} + \mathbf{F}(t) + \mathcal{O}(dt)$$

- Result is $\frac{d\bar{\mathbf{p}}(t)}{dt} = -\frac{\bar{\mathbf{p}}(t)}{\tau} + \mathbf{F}(t)$. Q.E.D.

Hall Effect I

In the late 1800's, Hall measured the effect of magnetism on currents. Drude theory explains many results.

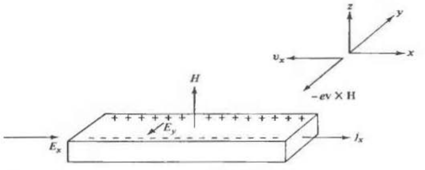


Figure 1.3
Schematic view of Hall's experiment.

- The setup
  - Apply electric field (aka voltage) to conductor in x direction
  - Apply magnetic field $\perp$ current in z direction
- What happens?
  - Current flows $\implies$ electrons have velocity $\mathbf{v}$ in x direction
  - Magnetic force is $-\frac{e}{c}\mathbf{v} \times \mathbf{H}$ in y direction
  - That pushes electrons to the edge as shown
  - ...which creates an electric field pushing them back
  - Motion reaches equilibrium; current moves in x direction

Hall Effect II

2 interesting questions: (1) **New resistivity?** (2) $\mathbf{E}_y$?

- The electron force is $\mathbf{F} = -e\left(\mathbf{E} + \mathbf{v} \times \frac{\mathbf{H}}{c}\right)$
- Recall the generalized force law $\frac{d\bar{\mathbf{p}}(t)}{dt} = -\frac{\bar{\mathbf{p}}(t)}{\tau} + \mathbf{F}(t)$.
- So $\frac{d\bar{\mathbf{p}}(t)}{dt} = -e\left(\mathbf{E} + \frac{\bar{\mathbf{p}}(t)}{mc} \times \mathbf{H}\right) - \frac{\bar{\mathbf{p}}(t)}{\tau}$
- In steady state, $\mathbf{p}$ is time-independent so

$$0 = -eE_x - \omega_c \bar{p}_y - \frac{\bar{p}_x}{\tau}$$
$$0 = -eE_y + \omega_c \bar{p}_x - \frac{\bar{p}_y}{\tau}$$

  where $\omega_c = \frac{eH}{mc}$

- Recall $\mathbf{j} = -ne\bar{\mathbf{v}} = -\frac{ne}{m}\bar{\mathbf{p}}$. Then multiply above by $-\frac{ne\tau}{m}$.

$$\frac{1}{\rho}E_x = \omega_c \tau \bar{j}_y + \bar{j}_x$$
$$\frac{1}{\rho}E_y = -\omega_c \tau \bar{j}_x - \bar{j}_y$$

  where ρ is the Drude result $\rho = \frac{m}{ne^2\tau}$ for resistivity.

Hall Effect III

- Recall that in equilibrium steady state, $\bar{j}_y = 0$.
- So $E_x = \rho \bar{j}_x$. **Magnetism doesn't effect resistivity!**
  - This was observed experimentally by Hall.
- Substitute $\bar{j}_y = 0$ into the second equation.

$$E_y = -\omega_c \rho \tau j_x = R_H H j_x$$

  where the Hall coefficient $R_H = -\frac{1}{nec}$.
- R_H **is independent of τ or other properties of the metal.**

We previously noted that the measured value of τ usually disagrees with the expected collision distance. Since R_H is τ-independent, we might expect the Drude prediction of R_H to be in good agreement with experiment. Mostly it is for alkali metals, not so much for the others.

AC Conductivity I

When is a metal wire transparent to light?

The method will be to check if Maxwell's equations permit radiation to flow along the wire.

- A trick to get cos and sin solutions is to make everything complex.
- A radiative solution has the form[2] $\mathbf{E}(t) = \mathbf{E}(\omega)e^{-i\omega t}$
- Recall the generalized force law $\frac{d\bar{\mathbf{p}}(t)}{dt} = -\frac{\bar{\mathbf{p}}(t)}{\tau} + \mathbf{F}(t) \implies$
  - $\frac{d\bar{\mathbf{p}}(t)}{dt} = -\frac{\bar{\mathbf{p}}(t)}{\tau} - e\mathbf{E}(\omega)e^{-i\omega t}$
- Seek solution of the form $\bar{\mathbf{p}}(t) = \bar{\mathbf{p}}(\omega)e^{-i\omega t}$ so
  - $-i\omega\bar{\mathbf{p}}(\omega) = -\frac{\bar{\mathbf{p}}(\omega)}{\tau} - e\mathbf{E}(\omega) \implies \bar{\mathbf{p}}(\omega) = -\frac{e\mathbf{E}(\omega)}{\frac{1}{\tau} - i\omega}$
- Since $\mathbf{j}(t) = -\frac{ne\bar{\mathbf{p}}(t)}{m}$ we have $\mathbf{j} = \mathbf{j}(\omega)e^{-i\omega t}$ where $\mathbf{j}(\omega) = \sigma(\omega)\mathbf{E}(\omega)$
  - $\sigma(\omega) = \frac{ne^2}{m\left(\frac{1}{\tau} - i\omega\right)} = \frac{1}{\rho}\frac{1}{1 - i\omega\tau}$. *$\sigma(\omega)$ is called "AC conductivity"*

[2] Really should be $\tilde{\mathbf{E}}(t)$ but we simplify notation. Also, superpositions are allowed.

AC Conductivity II

- We've made some (valid) assumptions.
  - Moving charges produce a magnetic field.
    - Induced force is $-e\bar{\mathbf{v}}/c \times \mathbf{H}$ where $\bar{\mathbf{v}}$ is small.
  - Derivation of generalized force law assumes uniform force.
    - But only between collisions
    - If wavelength is much larger than mean free path, we're OK
    - **But include position in j and E**, so $\mathbf{j}(\omega, \mathbf{x}) = \sigma(\omega)\mathbf{E}(\omega, \mathbf{x})$
- Now apply Maxwell's equations (assume no induced charge)

$$\nabla \cdot \mathbf{E} = 0; \;\; \nabla \cdot \mathbf{H} = 0; \;\; \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{dt}$$

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial E}{dt}$$

AC Conductivity III

- Substitute $\mathbf{j}(\omega, \mathbf{x}) = \sigma(\omega)\mathbf{E}(\omega, \mathbf{x})$ and get $-\nabla^2\mathbf{E} = \frac{\omega^2}{c^2}\epsilon(\omega)\mathbf{E}$.
  - $\epsilon(\omega) = 1 + \frac{4\pi i\sigma}{\omega}$ is the "complex dielectric constant".
- $\frac{4\pi i\sigma}{\omega} = \frac{4\pi i\tau ne^2}{m(1-i\omega\tau)}$.
- Assume very large frequency so $\omega\tau >> 1$.
- Then $\epsilon(\omega) \approx 1 - \frac{\omega_P^2}{\omega^2}$ where $\omega_P = \frac{4\pi ne^2}{m}$.
  - ω_P is the "plasma frequency"
  - By relationship of τ to resistivity, $\omega_P\tau >> 1$ for most metals
- When $\omega < \omega_P$ then $\epsilon(\omega) < 0$ and $\mathbf{E}$ dies exponentially
  - e.g. when the field is only x-dependent $\mathbf{E}(x) \propto e^{-\frac{\omega}{c}x}$
- When $\omega > \omega_P$, $\mathbf{E}$ oscillates
  - e.g. when the field is only x-dependent $\mathbf{E}(x) \propto e^{-i\frac{\omega}{c}x}$
  - This is radiation through the wire – i.e. **transparency**

AC Conductivity IV

ω_P can be computed based on electron density. For alkali metals, this is ultraviolet. Results are good.

OBSERVED AND THEORETICAL WAVELENGTHS BELOW WHICH THE ALKALI METALS BECOME TRANSPARENT

ELEMENT	THEORETICAL[a] λ (10^3 Å)	OBSERVED λ (10^3 Å)
Li	1.5	2.0
Na	2.0	2.1
K	2.8	3.1
Rb	3.1	3.6
Cs	3.5	4.4

[a] From Eq. (1.41).

Source: M. Born and E. Wolf. *Principles of Optics*, Pergamon, New York, 1964.