Superconductivity Part 1 Version 2

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Outline of superconducting notes

- Basic superconductivity assuming a Bose fluid of Cooper pairs
- BCS theory of pair-creation

Basic Superconductivity

Section 1

Introduction

Introduction – Remarks

- The semiclassical theory of conductivity:
  - Assumes current is carried by fermions (electrons)
  - Assumes a statistical distribution at room temperature
    - Fermi-Dirac distribution
  - Assumes chemical potential $\mu >> k_B T$
  - Assumes the electron size is much less than collision distance
- The theory of superconductivity = theory of charged superfluid
  - Assumes current is carried by bosons (electron-pairs)
  - Assumes a statistical distribution near $T = 0$
    - Bose-Einstein distribution
  - Assumes most pairs are in the ground state (condensate)
  - Assumes the pair-size is larger than collision distance

Introduction – References

- Physics Club Notes – Superfluids Part I
  https://billcelmaster.com/wp-content/uploads/2024/05/Superfluids-Part-I.pdf
- Physics Club Notes – Superfluids Part II
  https://billcelmaster.com/wp-content/uploads/2024/06/Superfluids-Part-II.pdf
- Feynman Lectures on Physics (Vol. 3 section 21)
  https://joepucc.io/static_assets/projects/feynman-lectures-on-physics/vol3.pdf
- Superfluid States of Matter (Chptr. 5) Svistunov et al.
  https://people.umass.edu/bvs/Book.pdf
- Superconductivity, Superfluids and Condensates James Annett
  https://www.dropbox.com/scl/fi/qyt2wlb5n78r6eprypdkn/Annett-Superconductivity.pdf?rlkey=vi7joo4ys5vvmpgeo75gsn5vh&st=jmk464j4&dl=0
- Quantum Field Theory for the Gifted Amateur (Chptr. 44) Lancaster & Blundell

Section 2

Schrodinger's equation

Schrodinger's equation – For coherent states

- The Schrodinger equation for a (nonrelativistic) charged particle is

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi = \frac{1}{2m}\left(-i\hbar\mathbf{\nabla} - q\mathbf{A}\right)\cdot\frac{1}{2m}\left(-i\hbar\mathbf{\nabla} - q\mathbf{A}\right)\psi + q\phi\psi$$

  where ϕ is the electric potential, $\mathbf{A}$ is the EM vector potential, and q is the charge. (Derived as NR limit of QED in coherent state.)

- At $T \approx 0$, electrons favor Cooper-pair configurations
- Treat each pair as a particle with charge $q = -2e$ and $m = 2m_e$.
- Near $T = 0$, almost all pairs (bosons) are in the same state. Then $\psi^*\psi$ is the *pair density*; not just *wavefunction probability density*.
- Let $P(\mathbf{r}, t) = [\psi^*\psi](\mathbf{r}, t)$ and $\mathbf{J}_p = \frac{1}{2m}\left[\psi^*\left(-i\hbar\mathbf{\nabla} - q\mathbf{A}\right)\psi + \psi\left(i\hbar\mathbf{\nabla} - q\mathbf{A}\right)\psi^*\right]$
  - From Schrodinger's equation, we can prove

$$\frac{\partial[\psi^*\psi]}{\partial t} = -\mathbf{\nabla}\cdot\frac{1}{2m}\left[\psi^*\left(-i\hbar\mathbf{\nabla} - q\mathbf{A}\right)\psi + \psi\left(i\hbar\mathbf{\nabla} - q\mathbf{A}\right)\psi^*\right]$$

  - So $\frac{\partial P}{\partial t} = -\mathbf{\nabla}\cdot\mathbf{J}_p \implies \mathbf{J}_p$ is the pair current density.

Schrodinger's equation – The current and the phase

The following approach is known as the *Ginsburg-Landau theory*

In Physics Club Notes (Superfluids Part II) we encountered the phase θ.

- The charge density is $\rho_{\text{ch}} = qP(\mathbf{r})$
- Write $\psi(\mathbf{r}) = \sqrt{\frac{\rho_{\text{ch}}(\mathbf{r})}{q}} e^{i\theta(\mathbf{r})}$
- The charge current density $\mathbf{J}$ is $\mathbf{J} = q\mathbf{J}_p$.
- Then $\mathbf{J} = \frac{\hbar}{m}\left(\boldsymbol{\nabla}\theta - \frac{q}{\hbar}\mathbf{A}\right)\rho_{\text{ch}}$
- Since $\mathbf{J} = \rho_{\text{ch}}\mathbf{v}$, we have $m\mathbf{v} = \hbar\left(\boldsymbol{\nabla}\theta - \frac{q}{\hbar}\mathbf{A}\right)$

NOTE: The current is independent of ϕ, the electric potential!

Schrodinger's equation – 0-resistivity

- Constitutive equation: $\mathbf{E} = \rho_{\text{res}} \mathbf{J}$ where ρ_{res} is the **resistivity**.
  - If $\mathbf{E}$ is independent of $\mathbf{J}$ then $\rho_{\text{res}} = 0$!
  - SYSTEM HAS NO RESISTANCE. i.e. SUPERCONDUCTOR

- As T increases, there is a sudden shift to the *normal* electron gas.

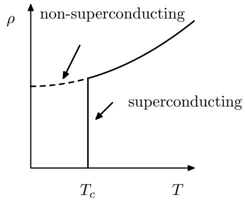

Fig. 1.1 Resistivity of a typical metal as a function of temperature.

Figure: Resistivity curve – from Annett

Schrodinger's equation – Meissner effect

- From constitutive equation, $\mathbf{E} = 0$.
  - Maxwell's equation (Faraday's Law): $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial \mathbf{t}}$
  - If $T < T_c$ and $\mathbf{B}$ initially 0 everywhere, it stays 0 in the superconductor even if it is turned on outside
    - $\implies$ current induced in the SC; magnetic flux cancels $\mathbf{B}$
- By thermal equilibrium, this doesn't depend on starting conditions.

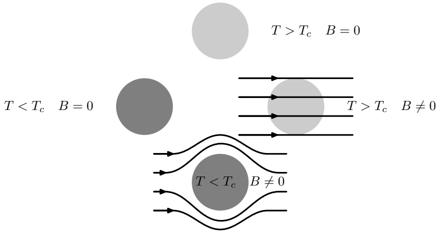

Fig. 1.5 The Meissner-Ochsenfeld effect in superconductors. If a sample initially at high temperature and in zero magnetic field (top) is first cooled (left) and then placed in a magnetic field (bottom), then the magnetic field cannot enter the material (bottom). This is a consequence of zero resistivity. On the other hand a normal sample (top) can be first placed in a magnetic field (right) and **then** cooled (bottom). In the case the magnetic field is **expelled** from the system.

Figure: Expulsion of magnetic field – from Annett

Section 3

London equation and magnetic penetration

London equation – not Feynman's version

- Since the ionic charge is uniform, it only neutralizes electron charge if $\rho(\mathbf{r})$ = constant.
- Recall $\mathbf{J} = \frac{\hbar}{m}\left(\boldsymbol{\nabla}\theta - \frac{q}{\hbar}\mathbf{A}\right)\rho_{\text{ch}}$.
  - Since the curl of a gradient is 0 and the curl of ρ_{ch} is 0

$$\boldsymbol{\nabla} \times \mathbf{J} = -\rho_{\text{ch}}\frac{q}{m}\mathbf{B}$$

.
  - **This is known as the <u>London equation</u>**.
- In a simply connected region (no holes) state this in terms of $\mathbf{A}$
  - $\boldsymbol{\nabla} \times \mathbf{J} = -\rho_{\text{ch}}\frac{q}{m}\boldsymbol{\nabla} \times \mathbf{A} \implies \boldsymbol{\nabla} \times \left(\mathbf{J} - \rho_{\text{ch}}\frac{q}{m}\mathbf{A}\right) = 0$
  - $\boldsymbol{\nabla} \times \left(\mathbf{J} - \rho_{\text{ch}}\frac{q}{m}\mathbf{A}\right) = 0 \implies \left(\mathbf{J} - \rho_{\text{ch}}\frac{q}{m}\mathbf{A}\right) = \boldsymbol{\nabla}\xi$ for some ξ.
  - So $\mathbf{J} = \rho_{\text{ch}}\frac{q}{m}\mathbf{A} + \boldsymbol{\nabla}\xi$
  - Change gauge: $\mathbf{A} \to \mathbf{A}'$ where $\mathbf{A}' = \mathbf{A} - \frac{\boldsymbol{\nabla}\xi}{\rho_{\text{ch}}\frac{q}{m}}$.
- Then $\mathbf{J} = \rho_{\text{ch}}\frac{q}{m}\mathbf{A}'$
- Recall $\frac{\partial P}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{J}_p$ so in steady-state, $\boldsymbol{\nabla} \cdot \mathbf{J}_p = 0 \implies \boldsymbol{\nabla} \cdot \mathbf{J} = 0$.
- Thus $\boldsymbol{\nabla} \cdot \mathbf{A}' = 0$. *Known as the London gauge.*

London equation – Magnetic penetration (equations)

- From Maxwell's equations in steady-state

$$\nabla \times \mathbf{B} = \frac{1}{\epsilon_0 c^2}\mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

- Then using $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{J} = \epsilon_0 c^2 \nabla \times \nabla \times \mathbf{B} = -\epsilon_0 c^2 \nabla^2 \mathbf{B}$
- Then from the London equation

$$-\rho_{\text{ch}}\frac{q}{m}\mathbf{B} = -\epsilon_0 c^2 \nabla^2 \mathbf{B}$$

  so $\nabla^2 \mathbf{B} = \lambda^2 \mathbf{B}$ where $\lambda^2 = \rho_{\text{ch}} \frac{q}{\epsilon_0 m c^2}$.
- Consider an ansatz $\mathbf{B}(\mathbf{r}) = (0, 0, B_z(x, 0, 0))$.
  - Then the most general solution for B_z would be

$$B_z(x,0,0) = \alpha e^{\lambda x} + \beta e^{-\lambda x}$$

  - SC boundaries: $x = 0, X$. $B_z(0,0,0) = B_z(X,0,0) = \mathcal{B}$
    - So $B_z(x,0,0) = 2\mathcal{B}\frac{\sinh\frac{\lambda X}{2}\cosh\lambda\left(\frac{X}{2}-x\right)}{\sinh\lambda X}$
    - Near $x = 0$, $B_z(x,0,0) \approx \mathcal{B}e^{-\lambda x}$
    - So penetration depth is $\frac{1}{\lambda} \approx \mathcal{O}\left(10^{-5}\right)$ cms
  - This is another proof of the Meissner effect (with penetration)

London equation – Magnetic penetration (figures)

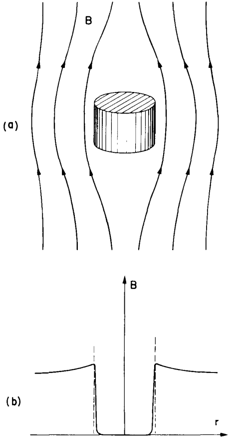

Fig 21–3 (a) A superconducting cylinder is a magnetic field; (b) the magnetic field B as a function of r.

Figure: Penetration and expulsion of magnetic field – from Feynman

Section 4

Flux quantization

Flux quantization – Trapped flux lines

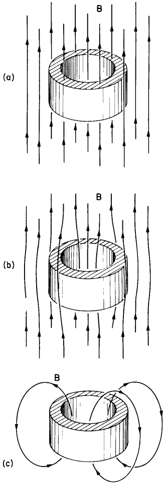

Fig. 21–4. A ring in a magnetic field: (a) in the normal state; (b) in the superconducting state; (c) after the external field is removed.

Figure: Trapping of flux lines – from Feynman

Flux quantization – Trapped flux lines cont'd

Why does flux remain when external field is removed?

- Consider a surface S bounding a circular path inside the SC
- The flux, Φ, of $\mathbf{B}$ through S is $\Phi = \int_S \mathbf{B} \cdot d\boldsymbol{\sigma} = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{s}$
- Use Faraday's law: $\oint_{\partial S} \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\boldsymbol{\sigma}$
- $\mathbf{E} = 0$ inside the SC so LHS $= 0$.
- Thus RHS doesn't change so flux remains on inside of ring.

Flux quantization – Role of the phase

Take a superconducting ring whose average distance from the origin is R.

The contour ∂S inside the ring and defined by $r = R$, encircles the region S covering the hole.

- The contour is well beyond the **B** penetration depth
- So on the contour, $\mathbf{B} = \mathbf{E} = 0$ so
  - (*Maxwell*) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \implies \mathbf{J} = 0$
- (Recall) $\mathbf{J} = \frac{\hbar}{m}\left(\nabla\theta - \frac{q}{\hbar}\mathbf{A}\right)\rho_{\text{ch}} \implies \hbar\nabla\theta = q\mathbf{A}$
- Note that ψ is continuous and single-valued $\implies$
  - In polar coordinates
    $e^{i\theta(r,\phi+2\pi)} = e^{i\theta(r,\phi)} \implies \theta(r,\phi+2\pi) = \theta(r,\phi) + 2n\pi$
  - So $\oint_{\partial S} \nabla\theta \cdot d\mathbf{s} = \int_0^{2\pi} d\phi \frac{d\theta}{d\phi}(r,\phi) = \theta(r,\phi+2\pi) - \theta(r,\phi) = 2n\pi$
- So $2\pi n\hbar = \hbar \oint_{\partial S} \nabla\theta \cdot d\mathbf{s} = q \oint_{\partial S} \mathbf{A} \cdot d\mathbf{s} = q \int_S \mathbf{B} \cdot d\boldsymbol{\sigma} \equiv q\Phi$
  - $\implies$ on ∂S, $\mathbf{A} \neq 0$ but $\mathbf{J} = 0 \implies$ no London gauge
- **FLUX QUANTIZATION:** $\Phi = n\frac{2\pi\hbar}{q}$.
  - Measured in 1961 with $q = 2q_e$. See Feynman.

Section 5

Alternative derivations of London equation

Alternatives – Feynman derivation

- Recall $\frac{\partial P}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{J}_p$ so in steady-state, $\boldsymbol{\nabla} \cdot \mathbf{J}_p = 0$
- Since the ionic charge is uniform, it only neutralizes electron charge if $\rho(\mathbf{r}) =$ constant.
- Pick the London gauge $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$.
- Recall $\mathbf{J} = \frac{\hbar}{m} \left(\boldsymbol{\nabla}\theta - \frac{q}{\hbar}\mathbf{A} \right) \rho_{\text{ch}}$. Then, using the above,

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{J} &= \frac{\hbar}{m} \left[\left(\boldsymbol{\nabla}^2\theta - \frac{q}{\hbar}\boldsymbol{\nabla} \cdot \mathbf{A} \right) \rho_{\text{ch}} + \left(\boldsymbol{\nabla}\theta - \frac{q}{\hbar}\mathbf{A} \right) \cdot \boldsymbol{\nabla}\rho_{\text{ch}} \right] \\ &= \frac{\hbar}{m} \left(\boldsymbol{\nabla}^2\theta \right) \rho_{\text{ch}} \end{aligned}$$

- Feynman says LHS $= 0$, so $\boldsymbol{\nabla}^2\theta = 0 \Rightarrow \theta =$ constant $\Rightarrow \boldsymbol{\nabla}\theta = 0$.
  - This appears to be a bogus argument.
- Substitute in the equation for **J** leading to the London equation

$$\mathbf{J} = -\rho_{\text{ch}} \frac{q}{m} \mathbf{A}$$

Alternatives – Even simpler than Feynman

- Since the ionic charge is uniform, it only neutralizes electron charge if $\rho(\mathbf{r}) =$ constant.
- Recall physics is gauge invariant: $\mathbf{A} \to \mathbf{A}'$ where $\mathbf{A}' = \mathbf{A} + \nabla\xi$
  - Choose $\xi = -\frac{\hbar}{q}\theta$
    - Probably unjustified when $\theta(\phi = 2\pi) \neq \theta(\phi = 0)$
    - Maybe also unjustified since θ depends on A
  - Then $\mathbf{A'} = \mathbf{A} - \frac{\hbar}{q}\nabla\theta$
- Recall $\mathbf{J} = \frac{\hbar}{m}\left(\nabla\theta - \frac{q}{\hbar}\mathbf{A}\right)\rho_{\mathrm{ch}}$.
  - Then

$$\mathbf{J} = -\rho_{\mathrm{ch}}\frac{q}{m}\mathbf{A}'$$

  .
  - **This is the London equation**.
- Since $\nabla\rho_{\mathrm{ch}} = 0$, then $\nabla\mathbf{J} = -\rho_{\mathrm{ch}}\frac{q}{m}\nabla\mathbf{A}'$.
- Recall $\frac{\partial P}{\partial t} = -\nabla \cdot \mathbf{J}_p$ so in steady-state, $\nabla \cdot \mathbf{J}_p = 0 \implies \nabla \cdot \mathbf{J} = 0$.
- Hence $\nabla \cdot \mathbf{A}' = 0$. *Known as the London gauge.*