Rotating Black Holes

The first step toward first principles: Boyer/Lindquist coordinates

November 5, 2025

Boyer/Lindquist (BL) Coordinates: outline

- History
- Description
- Visualization
- Equations for visualization
- Equations for structural elements (in BL coordinates)
- Equations of radial distance

Next time: derivation of the equations.

Boyer/Lindquist (BL) Coordinates: history

- Introduced by Robert H. Boyer and Richard W. Lindquist (1967).
- Alternative to Kerr/Schild (KS) coordinates, as used in Kerr's 1963 paper. (Roy Kerr and Alfred Schild introduced these formally in 1965.)
- BL became the preferred coordinates for text books and analysis, Because many aspects of Kerr geometry are manifest or easily exposed:
  - Stationarity - geometry is time-independent
  - Rotation going to zero at spatial infinity
  - Frame dragging
  - Stationary limits
  - Event horizons
  - Causal structure - which events can influence other events
  - Separability of the equations of motion

Boyer/Lindquist (BL) Coordinates: description

BL coordinates are intentionally designed to be:

- Well-adapted to the Kerr geometry.
- As close to Schwarzschild coordinates (t, r, θ, ϕ) as possible.

Schwarzschild (S) coordinates are quasi-spherical, where:

- t is the coordinate time as measured by clocks at rest at spatial infinity.
- (r, θ, ϕ) are spatial coordinates resembling spherical polar (SP) coordinates.
- The angular coordinates $(\theta, \phi)_{\mathrm{Sch}}$ are everywhere equivalent to $(\theta, \phi)_{\mathrm{SP}}$.
- Distance along radial coordinate r differs from that in flat space, i.e.
  - Between nearby points r and $r + \delta r$, the *proper distance* $s(\delta r) > \delta r$.
  - As $r \to \infty$, $\delta r \to s(\delta r)$, so $(r, \theta, \phi)_{\mathrm{Sch}} \to (r, \theta, \phi)_{\mathrm{SP}}$.

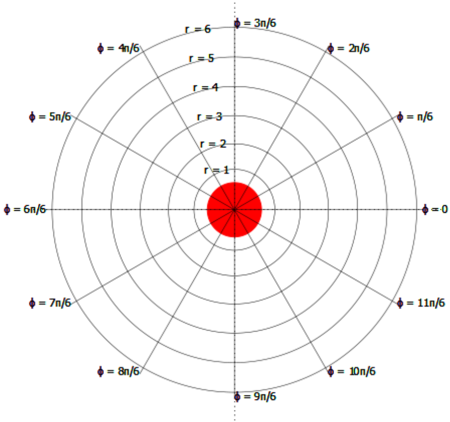
BL coordinates are also quasi-spherical, and also denoted by (t, r, θ, ϕ), where:

- $(t, r, \theta, \phi)_{\mathrm{BL}} \to (t, r, \theta, \phi)_{\mathrm{Sch}}$ as $r \to \infty$.
- At small r, (t, r, θ) are quite different from $(t, r, \theta)_{\mathrm{Sch}}$.

BL Spatial Coordinates: visualization

BL coordinates can be visualized by exhibiting cross-sections of surfaces of constant r and constant θ in Cartesian coordinates.

Note: In the figures below, r is in multiples of r_g, $a = 0.9r_g$; spatial relationships are correct, but distances are not accurate.



Equatorial cross-section

- Just like S (and SP) coordinates:
  - Curves of constant $r > 0$ are concentric circles.
  - Curves of constant ϕ are radial lines.
- Unlike S (and SP) coordinates:
  - $r = 0$ is not a point at the origin.
  - Instead, $r = 0$ everywhere on the red disk!
  - Red disk has radius a.

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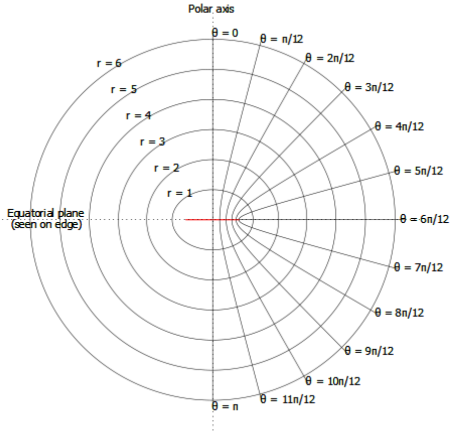

Polar cross-section

- Curves of constant r are concentric ellipses.
- The nearer the origin, the greater the eccentricity.
- With the exception of $\theta = 0$ and $\theta = \pi$, curves of constant θ do not emanate from the origin, but rather from some other point on the red disk.
- As θ goes from 0 to $\pi/2$, the point of emanation moves outward the center of the disk at $\theta = 0$ to the edge of the disk at $\theta = \pi/2$.
- Thus, points on the disk are not distinguished by r and ϕ, but by θ and ϕ.

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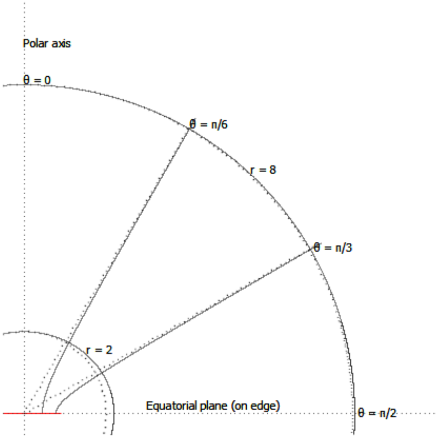

Asymptotic behavior: BL vs. S

- Solid curves are BL coordinates
- Dotted curves are S coordinates
- Distinguishable at small r.
- Nearly indistinguishable by $r = 8$.

BL Coordinates: equations for visualization

Recall that the geometry of a Kerr black hole is completely determined by two quantities:

- Its mass, denoted by M.
- Its angular momentum, denoted by J.

From these, define the *rotation parameter* $a = J/(Mc)$, a length.

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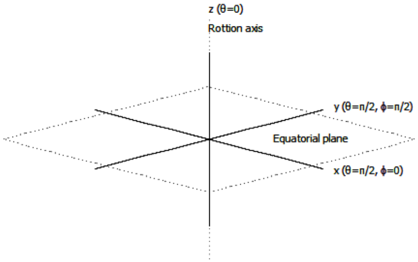
Visualization is achieved by mapping BL to Cartesian coordinates (in $\mathbb{E}^3$).



x and y axes lie in the equatorial plane.
x and y axes align with $\phi = 0$ and $\phi = \pi/2$.
z-axis aligns with the rotation axis, $\theta = 0$.

The transformation:

$$\begin{aligned} x &= \sqrt{r^2 + a^2} \sin\theta \cos\phi \\ y &= \sqrt{r^2 + a^2} \sin\theta \sin\phi \\ z &= r\cos\theta \end{aligned}$$

At $r = 0$:

$$\begin{aligned} x &= a\sin\theta\cos\phi \\ y &= a\sin\theta\sin\phi \\ z &= 0 \\ \Longrightarrow x^2 + y^2 &= a^2 \sin^2\theta \\ \Longrightarrow x^2 + y^2 &\leq a^2, \ \theta \leq \pi/2 \end{aligned}$$

Thus, $r = 0$ everywhere on the disk with edge radius a.

Equations for Structural Elements

From the mass M, define the *gravitational radius* $r_g = GM/c^2$, a length (like a).

In terms of BL coordinates and the parameters a and r_g, the equations of the structural elements are very simple.

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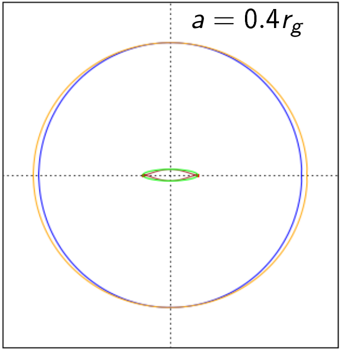

Equations:

- Outer stationary limit: $r = r_g + \sqrt{r_g^2 - a^2 \cos^2 \theta}$
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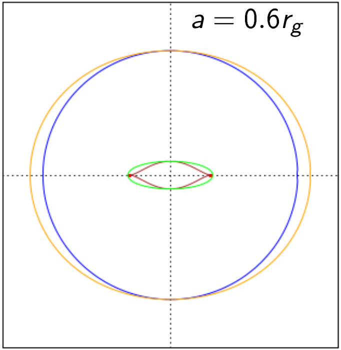

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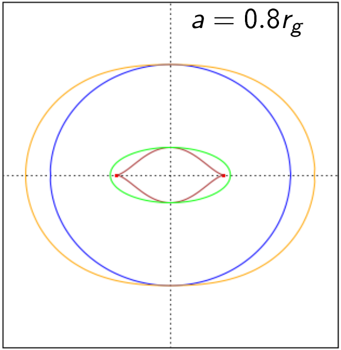

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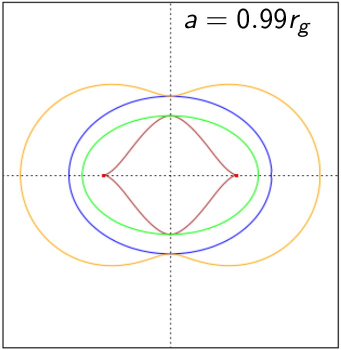

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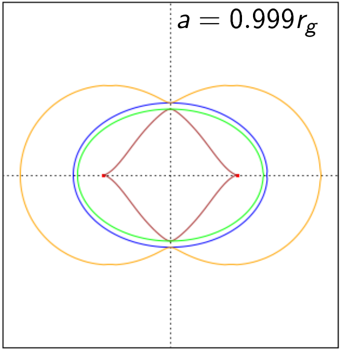

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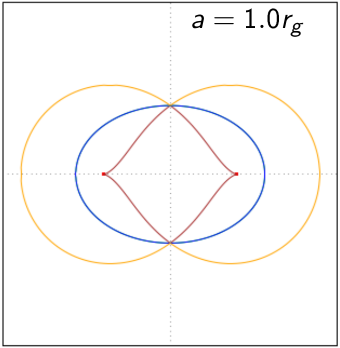

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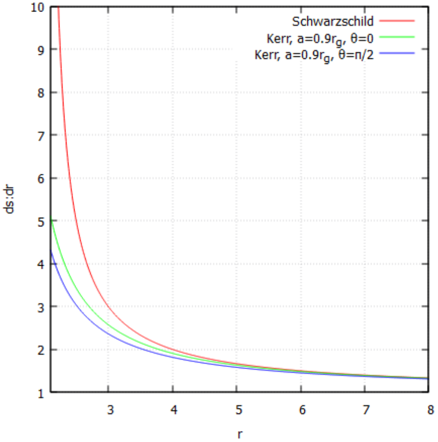
Notes:

- It is customary to denote the outer and inner horizons by r_+ and r_-, respectively.
- Horizon equations can then be written compactly as $r_\pm = r_g \pm \sqrt{r_g^2 - a^2}$.
- Evidently, we must have $a \leq r_g$; horizons the same at $r = a$ (extremal black hole).

Proper Distance Along Radii



Equations:

- Schwarzschild

$$\frac{ds}{dr} = \frac{1}{\sqrt{1 - 2r_g/r}} \tag{1}$$

- Kerr (BL)

$$\frac{ds}{dr} = \sqrt{\frac{r^2 + a^2 \cos^2\theta}{r^2 - 2r_g r + a^2}} \tag{2}$$

It is easy to show that, at $a = 0$, Eq. (2) reduces to Eq. (1).

Note: although technically r_{Sch}, r_{BL} and r_{SP} converge at spatial infinity, as a practical matter, r_{Sch} and r_{BL} are essentially indistinguishable beyond $r = 8r_g$.

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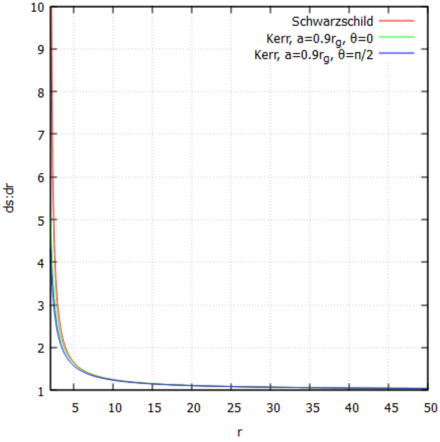

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It is easy to show that, at $a = 0$, Eq. (2) reduces to Eq. (1).

Note: all three, i.e. r_{Sch}, r_{BL} and r_{SP}, are essentially indistinguishable beyond $r = 50r_g$.