Rotating Black Holes

The second step toward first principles: The Kerr metric in Boyer/Lindquist coordinates

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The Kerr Metric: outline

- Review of metrics.
- Review of metrics in GR
- The Kerr metric in Boyer/Lindquist coordinates.
- Applying the Kerr metric:
  - Visualization of BL coordinates
  - Proper radial distance
  - Stationary limit surfaces
  - Event horizons
  - Singularities

Review of metrics

On a n-dimensional manifold $\mathbb{M}^n$, a metric is a quadratic form

$$ds^2 = g_{u\nu} dq^\mu dq^\nu = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} g_{u\nu} dq^\mu dq^\nu \tag{1}$$

where:

- ds is the differential distance.
- q^μ, $\mu = 1, 2, ..., n$ are arbitrary coordinates.
- dq^μ $\mu = 1, 2, ..., n$ are components of a differential displacement vector.
- $g_{u\nu}$ $\mu, \nu = 1, 2, ..., n$ are components of the metric tensor.
- For a Riemannian metric, the metric tensor is symmetric, i.e. $g_{u\nu} = g_{\nu\mu}$, and positive-definite,
  which implies $ds^2 \geq 0$, where $ds^2 = 0$ iff $dq^\mu = 0$.
- For a pseudo-Riemannian metric, the metric tensor is not positive definite, which implies ds^2 can be zero even if the $dq^\mu \neq 0$.

Review of metrics in GR

In general relativity,

- $n = 4$ and by convention
  - Coordinate indices run from 0 to 3 (rather than 1 to 4).
  - Index zero is associated with the time coordinate.
- Metric is pseudo-R with Lorentz signature $(-,+,+,+)$ or $(+,-,-,-)$.

We will be using the latter convention which implies that

- $ds^2 > 0$ on time-like curves.
- $ds^2 = 0$ on light-like (null) curves.
- $ds^2 < 0$ on space-like curves.

The Kerr Metric in BL Coordinates

Form of the Kerr Metric in BL Coordinates

In Boyer/Lindquist coordinates, the Kerr metric takes the form

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2$$

where:

- (t, r, θ, ϕ) are the Boyer/Lindquist coordinates.
- $g_{tt}, g_{rr}, g_{\theta\theta}, g_{\phi\phi}$ and $g_{t\phi} = g_{\phi t}$ are
  - The only non-zero metric coefficients.
  - Functions of r and θ, only.

The Kerr Metric in BL Coordinates

The Metric Coefficients

$$
\begin{aligned}
g_{tt}(r,\theta) &= c^2\left(1 - 2r_g\frac{r}{\rho^2(r,\theta)}\right) \\
g_{rr}(r,\theta) &= -\frac{\rho^2(r,\theta)}{\Delta(r)} \\
g_{\theta\theta}(r,\theta) &= -\rho^2(r,\theta) \\
g_{\phi\phi}(r,\theta) &= -\left(r^2 + a^2 + 2r_g a^2\frac{r\sin^2\theta}{\rho^2(r,\theta)}\right)\sin^2\theta \\
g_{t\phi}(r,\theta) = g_{\phi t}(r,\theta) &= 2cr_g a\frac{r\sin^2\theta}{\rho^2(r,\theta)}
\end{aligned}
$$

where:

$$
\begin{aligned}
\rho^2(r,\theta) &= r^2 + a^2\cos^2\theta \\
\Delta(r) &= r^2 - 2r_g r + a^2
\end{aligned}
$$

Reminder: $r_g = GM/c^2$ and $a = J/(mc)$ are the gravitational radius and rotation parameter, respectively.

Visualization of BL Coordinates

Recall that Visualization is achieved by mapping BL to Cartesian coordinates (in $\mathbb{E}^3$), i.e. when $M \to 0$, which implies $r_g \to 0$.
If $r_g \to 0$:

$$
\begin{aligned}
\rho^2(r,\theta) &= r^2 + a^2\cos^2\theta && \textit{unchanged} \\
\Delta(r) &= r^2 - 2r_g r + a^2 && \to \quad r^2 + a^2 \\
g_{tt}(r,\theta) &= c^2\left(1 - 2r_g\frac{r}{\rho^2(r,\theta)}\right) && \to \quad c^2 \\
g_{rr}(r,\theta) &= -\frac{\rho^2(r,\theta)}{\Delta(r)} && \to \quad -\frac{\rho^2(r,\theta)}{r^2+a^2} \\
g_{\theta\theta}(r,\theta) &= -\rho^2(r,\theta) && \textit{unchanged} \\
g_{\phi\phi}(r,\theta) &= -\left(r^2 + a^2 + 2r_g a^2\frac{r\sin^2\theta}{\rho^2(r,\theta)}\right)\sin^2\theta && \to \quad -\left(r^2+a^2\right)\sin^2\theta \\
g_{t\phi}(r,\theta) &= 2cr_g a\frac{r\sin^2\theta}{\rho^2(r,\theta)} && \to \quad 0
\end{aligned}
$$

Thus, the metric can be rewritten as

$$
ds^2 = c^2dt^2 - \frac{\rho^2(r,\theta)}{r^2+a^2}dr^2 - \rho^2(r,\theta)d\theta^2 - \left(r^2+a^2\right)\sin^2\theta\, d\phi^2
$$

This is just the metric for Minkowski space expressed in oblate-spheroidal coordinates instead of Cartesian coordinates.

Visualization of BL Coordinates (continued)

It can be shown that the oblate-spheroidal metric

$$ds^2 = c^2 dt^2 - \frac{\rho^2(r,\theta)}{r^2 + a^2} dr^2 - \rho^2(r,\theta) d\theta^2 - \left(r^2 + a^2\right) \sin^2\theta d\phi^2 = c^2 dt^2 - dl_{\mathrm{O}}^2 \quad (2)$$

is related to the usual the Minkowski metric

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 dt^2 - dl_{\mathrm{C}}^2 \quad (3)$$

by a coordinate transformation. The 1st term already matches, so we focus on dl^2.

Visualization of BL Coordinates (continued)

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$$ds^2 = c^2 dt^2 - \frac{\rho^2(r,\theta)}{r^2 + a^2} dr^2 - \rho^2(r,\theta) d\theta^2 - \left(r^2 + a^2\right) \sin^2\theta d\phi^2 = c^2 dt^2 - dl_{\mathrm{O}}^2 \quad (2)$$

is related to the usual the Minkowski metric

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 dt^2 - dl_{\mathrm{C}}^2 \quad (3)$$

by a coordinate transformation. The 1st term already matches, so we focus on dl^2.

We have already been introduced to the required, spatial coordinate transformation, i.e.

$$\begin{aligned} x &= \sqrt{r^2 + a^2} \sin\theta \cos\phi \\ y &= \sqrt{r^2 + a^2} \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned} \quad (4)$$

Visualization of BL Coordinates (continued)

In general, under a coordinate transformation $p^a = f^a(q^b)$, a vector transforms as $v^a = \mathcal{J}^a_b v^b$, where $\mathcal{J}^a_b = \partial p^a / \partial q^b$ is the Jacobian of f^a.
Thus, if $(p_1, p_2, p_3) = (x, y, z)$ and $(q_1, q_2, q_3) = (r, \theta, \phi)$, then $v^a = \mathcal{J}^a_b v^b$ becomes

$$
\begin{aligned}
dx &= \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi \\
dy &= \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi \\
dz &= \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial \phi} d\phi
\end{aligned} \tag{5}
$$

Then,

1. Taking all partials of (4),
2. Substituting them into (5)
3. Squaring the differentials dx, dy, dz
4. Summing the squares and simplifying

we obtain dl^2_{O}.

Proper Radial Distance

Previously, we saw that proper radial distance in Kerr spacetime is given by

$$dl = \sqrt{\frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2r_g r + a^2}}\, dr \tag{6}$$

Where does this come from?

If we are interested in measuring proper distance only along the radial direction, then we let $d\theta = d\phi = 0$ in the Kerr metric, i.e.

$$\begin{aligned} dl^2 &= g_{rr} dr^2 \\ &= \frac{\rho^2(r,\theta)}{\Delta(r)} dr^2 \\ &= \frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2r_g r + a^2} dr^2 \end{aligned}$$

Thus, dl is given by (6).

Stationary Limit Surfaces

Previously, we saw that the stationary limit surfaces are in Kerr spacetime are given by

$$r_{s\pm} = r_g \pm \sqrt{r_g^2 - a^2 \cos^2 \theta}$$

where do these come from?
Consider a photon emitted in the $\pm\phi$-direction, i.e. $dr = d\theta = 0$. Then,

$$ds^2 = g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 = 0$$

Dividing by dt^2, we obtain

$$g_{tt} + 2g_{t\phi} \frac{d\phi}{dt} + g_{\phi\phi} \left(\frac{d\phi}{dt} \right)^2 = 0$$

and solving for $d\phi/dt$ (quadratic formula), we obtain

$$\frac{d\phi}{dt} = \frac{-2g_{t\phi} \pm \sqrt{4g_{t\phi}^2 - 4g_{\phi\phi} g_{tt}}}{2g_{\phi\phi}}$$

Stationary Limits Surfaces (continued)

Now, consider what happens on the surface $g_{tt}(r, \theta) = 0$. From (8), we obtain

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \frac{g_{t\phi}}{g_{\phi\phi}}$$

Thus,

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} > 0 \qquad \text{or} \qquad \frac{d\phi}{dt} = 0$$

corresponding to the photon moving with or against rotation.

Physical interpretation: stationary observers at spatial infinity see the counter rotating photon momentarily at rest.
Why momentarily? The photon immediately begins to move inward, where it begins to be dragged in the $+\phi$ direction.
If this is true for photons, it is certainly true for massive particles.

Stationary Limits Surfaces (continued)

Thus, the equation for the stationary limit surfaces is

$$\begin{aligned} g_{tt}(r,\theta) &= 0 \\ 1 - 2r_g \frac{r}{\rho^2} &= 0 \end{aligned}$$

Then, multiplying by ρ^2, we obtain

$$\begin{aligned} \rho^2 - 2r_g r &= 0 \\ r^2 + a^2 \cos^2\theta - 2r_g r &= 0 \end{aligned}$$

and solving for r (quadratic formula), we obtain

$$\begin{aligned} r &= \frac{2r_g \pm \sqrt{4r_g^2 - 4a^2\cos^2\theta}}{2} \\ &= r_g \pm \sqrt{r_g^2 - a^2\cos^2\theta} \end{aligned}$$

as previously given.

Event Horizons

Definition

Defining property of an event horizon: a *null hypersurface*.

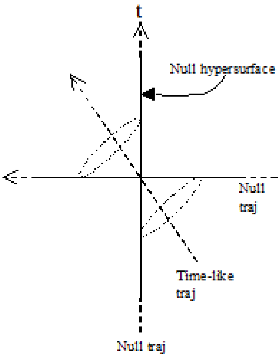

Illustration of a null hypersurface (at one event):

- Light cones are tangent to the surface.
- Implies one-way surface for real particles.
  - Time-like or null trajectories can pass through the event from right to left.
  - Null trajectories originating in the surface remain in the surface (photons are trapped).
  - Only space-like trajectories can pass through the event from left to right.
- All events on the hypersurface are the same.

Event Horizons in Kerr/BL Spacetime

Deriving the equation for a null hypersurface in BL coordinates

1. If a point $q^\mu,\ \mu = 1, 2, ..., n$ is restricted to lie in some hypersurface, then that point must satisfy one constraint equation, i.e.

$$f(q^1, q^2, ..., q^n) = 0$$

2. The normal *covector* to the hypersurface is the gradient

$$n_\mu = \partial_\mu f$$

3. Define a hypersurface of constant BL coordinate r, i.e. $r = const$

$$f(t, r, \theta, \phi) = r - const$$

4. In this specific case,

$$n_\mu = \partial_\mu f = \partial_\mu (r - const) = (0, 1, 0, 0)$$

5. Null hypersurface: surface on which the normal is a null vector. Formally

$$|n|^2 = g_{\mu\nu} n^\mu n^\nu = g^{\mu\nu} n_\mu n_\nu = 0$$

6. Substituting the specific covector $n_\mu = (0, 1, 0, 0)$ and the Kerr metric, yields

$$g^{rr} = 0$$

Event Horizons in Kerr/BL Spacetime

Deriving the equations for the event horizons

1. The meaning of g^{rr}: the contravariant component of the metric tensor corresponding to the covariant component g_{rr}.
2. The contravariant form of the metric tensor is the inverse of the covariant form, most easily determined using the matrix representation of the the latter, i.e.

$$\begin{bmatrix} g_{tt} & g_{t\phi} & 0 & 0 \\ g_{\phi t} & g_{\phi\phi} & 0 & 0 \\ 0 & 0 & g_{rr} & 0 \\ 0 & 0 & 0 & g_{\theta\theta} \end{bmatrix}$$

   Note: the matrix is organized to emphasize the block-diagonal nature of the metric tensor.
3. In taking the inverse of the matrix, it's obvious that

$$g^{rr} = \frac{1}{g_{rr}} = \frac{\Delta}{\rho^2}$$

4. Therefore,

$$g^{rr} = 0 \Longrightarrow \Delta = 0 \Longrightarrow r^2 - 2r_g r + a^2 = 0$$

5. Solving for r (quadratic formula) yields

$$\begin{aligned} r &= \frac{2r_g \pm \sqrt{4r_g^2 - 4a^2}}{2} \\ r = r_\pm &= r_g \pm \sqrt{r_g^2 - a^2} \end{aligned}$$

Singularities in the Kerr/BL Metric

The reasoning process

1. A metric is singular at points where any of its coefficients is singular.
2. The Kerr metric (in BL coordinates) is singular at $\rho^2 = 0$ and $\Delta = 0$.
3. We have already seen that $\Delta = 0$ at the horizons, so the metric is singular at $r_{\pm}$.
4. At $\rho^2 = r^2 + a^2 \cos^2\theta = 0$, the metric is singular on the locus of points $(r, \theta) = (0, \pi/2)$, which we have seen is the ring with radius a.
5. At $\Delta = 0$ the metric has a *coordinate singularity*: can be removed by a suitable coordinate transformation.
6. At $\rho^2 = 0$ the metric has a *curvature singularity*: cannot be removed and is thus a singularity of Kerr spacetime.

Singularities in the Kerr/BL Metric

Telling the difference between coordinate and curvature singularities

The key is the *Kretschmann scalar*,

$$K = R_{\mu\nu\sigma\xi} R^{\mu\nu\sigma\xi} \quad \text{(quadruple sum)}$$

where

- $R_{\mu\nu\sigma\xi}$ is the fully covariant form of the Riemann tensor.
- $R^{\mu\nu\sigma\xi}$ is the fully contravariant form of the Riemann tensor.

These are obtained from the standard (mixed) form $R^{\xi}_{\ \mu\sigma\nu}$ via

$$\begin{aligned} R_{\xi\mu\nu\sigma} &= g_{\xi a} R^{a}_{\ \mu\sigma\nu} \\ R^{\xi\mu\nu\sigma} &= g^{\mu a} g^{\nu b} g^{\sigma c} R^{\xi}_{\ abc} \end{aligned}$$

where

$$R^{\xi}_{\ \mu\sigma\nu} = \partial_{\sigma}\Gamma^{\xi}_{\mu\nu} - \partial_{\nu}\Gamma^{\xi}_{\sigma\mu} + \Gamma^{\xi}_{\sigma a}\Gamma^{a}_{\mu\nu} - \Gamma^{\xi}_{\nu a}\Gamma^{a}_{\sigma\mu}$$

and the $\Gamma^{\xi}_{\mu\nu}$ are the connection coefficients, given by

$$\Gamma^{\xi}_{\ \mu\nu} = \frac{1}{2} g^{\xi a} \left(\partial_a g_{\mu\nu} + \partial_\nu g_{\mu a} - \partial_\mu g_{\nu a}\right)$$

Singularities in the Kerr/BL Metric

The Kerr spacetime singularity

Substituting the Kerr metric into the above equations, the Kretschmann scalar turns out to be

$$K = \frac{48 r_g^2 \left(r^6 - 15 a^2 r^4 \cos^2\theta + 15 a^4 r^2 \cos^4\theta - a^6 \cos^6\theta\right)}{(\rho^2)^6}$$

We have seen that $\rho^2 = 0$ only on the ring, so:

- $K \to \infty$ only on the ring.
- K is finite at $r_\pm$.
- K is a scalar, so
  - K invariant under coordinate transformations.
  - If $K \to \infty$ at $\rho^2 = 0$ in BL coordinates, it does so in all coordinate systems.
  - $\rho^2 = 0$ is a true singularity in Kerr spacetime ($\Delta = 0$ is not).