Rotating Black Holes

First Principles: from Einstein to Kerr (Part I)

November 18, 2025

First Principles: outline (Part I)

- Einstein's field equation
- Properties of the field equation
- Types of constraints
- Applying geometric constraints

Einstein's Field Equation

Assuming negligible cosmological constant, the Einstein field equation is given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -kT_{\mu\nu}$$

where:

- $R_{\mu\nu}$ is the Ricci tensor.
- R is the Ricci scalar.
- $g_{\mu\nu}$ is the metric tensor.
- $T_{\mu\nu}$ is the stress-energy-momentum tensor.
- $k = 8\pi G/c^4$, where G is Newton's constant.
- $-$ sign on RHS required for consistency with convention $(+,-,-,-)$.

$R_{\mu\nu}$, R, $g_{\mu\nu}$ and $T_{\mu\nu}$ are smooth functions of the spacetime coordinates, i.e. they are tensor fields.

Properties of the Field Equation

$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -k T_{\mu\nu}$:

- Tensor equation $\Longrightarrow$ form invariant under coordinate transformations.
- Rank-2 equation (2 indices) $\Longrightarrow$16 component equations.
- Each term is symmetric, i.e. $\mathscr{T}_{\mu\nu} = \mathscr{T}_{\nu\mu} \Longrightarrow$ 10 unique equations.
- Component equations are PDEs
  - Second-order in the metric coefficients, $g_{\mu\nu}$.
  - Non-linear $\Longrightarrow$products of metric coefficients, their 1st derivatives, mixed products.
  - Coupled $\Longrightarrow$ each of the component equations involves several of the same metric coefficients and/or the same 1st and 2nd partial derivatives.

A general solution of Einstein's equation is hopeless.
Specific solutions emerge from physically motivated constraints on the metric.

Constraints on the Metric

In order to obtain a specific, exact solution, we impose constraints on the metric, which fall into four distinct categories:

- **Geometric constraints** - symmetries directly restrict on the functional form of $g_{\mu\nu}$. For Kerr:
  - Stationarity - metric is invariant under time translation.
  - Axial symmetry - metric is invariant under rotation about a line.
- **Physical constraints** - matter content of spacetime, i.e. choice of $T_{\mu\nu}$, indirectly constrains the metric via the field equation. For all black hole solutions, $T_{\mu\nu} = 0$.
- **Global constraints** - boundary conditions fix constants of integration. For all black hole solutions, $g_{\mu\nu} \to \eta_{\mu\nu}$ as $r \to \infty$, i.e. asymptotic flatness.
- **Consistency constraints** - conditions that force the metric to reduce smoothly to a simpler, already known solution, as one or more parameters go to zero. Of the three constraints below, only the first two are needed and the third will be used as a check.
  - Consistency with weak-field, small-velocity solution (Newtonian gravity) at large r.
  - Consistency with weak-field, slow-rotation solution (Lense-Thirring, 1918) at large r.
  - Consistency with spherically symmetric solution (Schwarzschild, 1916) as $J \to 0$.

Applying Geometric Constraints

Reducing in the number of variables

Recall:

- The general metric is given by

$$ds^2 = g_{\mu\nu} dq^\mu dq^\nu$$

  where:

  - q^μ, $\mu = 0, 1, 2, 3$ are arbitrary coordinates, as before.
  - Implied summation over 16 terms (10 unique).
  - The coefficient $g_{\mu\nu}$ of each term is a function of the 4 coordinates.
  - 10 unknown functions of 4 variables.

Introduce some specificity for the coordinates, with an eye toward BL coordinates:

- Let $q^0 = t$ be the temporal coordinate.
- Let $q^3 = \phi$ be the angular coordinate about the axis of rotation.
- Let $q^1 = x$ and $q^2 = y$ remain arbitrary space-like coordinates(for now).

Apply the symmetries:

- Stationarity $\Longrightarrow$ the coefficients $g_{\mu\nu}$ cannot be functions of t.
- Axial symmetry $\Longrightarrow$ the coefficients $g_{\mu\nu}$ cannot be functions of ϕ.
- $\Longrightarrow$ the coefficients are functions of x and y, only, (whatever they are).

The metric now has 10 functions of 2 variables.

Properites of the Coordinate Basis

Definitions:

- Let q^μ, $\mu = 0, 1, 2, 3$ be arbitrary spacetime coordinates.
- Let $\{\boldsymbol{b}_\mu,\ \mu = 0, 1, 2, 3\}$ be the corresponding set of coordinate basis vectors, defined by $\boldsymbol{b}_\mu := \partial/\partial q^\mu$, implying that $\boldsymbol{b}_\mu$ points along the direction of increasing q^μ, i.e.
  - Component of$\boldsymbol{b}_\mu$ along q^μ has coordinate length 1 (not proper length).
  - All other components are zero.
  - Note: the index μ in $\boldsymbol{b}_\mu$ distinguishes among basis vectors, not vector components.
- Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be arbitrary spacetime vectors. Then, the inner product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by
  - $(\boldsymbol{u}, \boldsymbol{v}) := g_p(\boldsymbol{u}, \boldsymbol{v})$, where g_p is a bi-linear map, $g_p : T_p\mathbb{M} \times T_p\mathbb{M} \to \mathbb{R}$ (coordinate-free notation).
  - $(\boldsymbol{u}, \boldsymbol{v}) := g_{\mu\nu} u^\mu v^\nu$ (in terms of contravariant components).

Consequently:

- $g_p(\boldsymbol{u}, \boldsymbol{v}) = g_{\mu\nu} u^\mu v^\nu$, in general, so
  $g_p(\boldsymbol{b}_\zeta, \boldsymbol{b}_\xi) = g_{\mu\nu} \left(\boldsymbol{b}_\zeta\right)^\mu \left(\boldsymbol{b}_\xi\right)^\nu$, in particular.
- From the definition $\boldsymbol{b}_\mu := \partial/\partial q^\mu$,

  the contravariant components of $\boldsymbol{b}_\mu$ are given by $\left(\boldsymbol{b}_\zeta\right)^\mu = \delta^\mu_\zeta$, so

  $g_p(\boldsymbol{b}_\zeta, \boldsymbol{b}_\xi) = g_{\mu\nu}\delta^\mu_\zeta\delta^\nu_\xi = g_{\zeta\xi}$.

So, the inner product of any pair of basis vectors equals the corresponding component of the metric tensor.

Applying Geometric Constraints

Reducing the number of (non-zero) coefficients

Recall:

- A coordinate basis vector $\boldsymbol{b}_\mu$ points in the direction of increasing q^μ.
- Inner product of two basis vectors given by $(\boldsymbol{b}_\mu, \boldsymbol{b}_\nu) = g_p(\boldsymbol{b}_\mu, \boldsymbol{b}_\nu) = g_{\mu\nu}$.
- The map g_p is bi-linear $\Longrightarrow$
  $g_p(-\boldsymbol{b}_\mu, \boldsymbol{b}_\nu) = -g_{\mu\nu}$, $(g_p(\boldsymbol{b}_\mu, -\boldsymbol{b}_\nu) = -g_{\mu\nu}$, $(g_p(-\boldsymbol{b}_\mu, -\boldsymbol{b}_\nu) = g_{\mu\nu}$

Apply the symmetries:

- Stationarity and axial symmetry together $\Longrightarrow$metric unchanged under

  reversal of the direction of t, i.e.$t \to -t$ and

  reversal of the direction of ϕ, i.e. $\phi \to -\phi$
  combined. (Let's call this $t\phi$-invariance.)
- $t \to -t$ and $\phi \to -\phi \Longrightarrow \boldsymbol{b}_t \to -\boldsymbol{b}_t$ and $\boldsymbol{b}_\phi \to -\boldsymbol{b}_\phi \Longrightarrow$

$(\boldsymbol{b}_t, \boldsymbol{b}_t) \to (-\boldsymbol{b}_t, -\boldsymbol{b}_t) \Longrightarrow g_{tt} \to g_{tt}$	$(\boldsymbol{b}_t, \boldsymbol{b}_x) \to (-\boldsymbol{b}_t, \boldsymbol{b}_x) = -g_{tx}$
$(\boldsymbol{b}_\phi, \boldsymbol{b}_\phi) \to (-\boldsymbol{b}_\phi, -\boldsymbol{b}_\phi) \Longrightarrow g_{\phi\phi} \to g_{\phi\phi}$	$(\boldsymbol{b}_t, \boldsymbol{b}_y) \to (-\boldsymbol{b}_t, \boldsymbol{b}_y) = -g_{ty}$
$(\boldsymbol{b}_t, \boldsymbol{b}_\phi) \to (-\boldsymbol{b}_t, -\boldsymbol{b}_\phi) \Longrightarrow g_{t\phi} \to g_{t\phi}$	$(\boldsymbol{b}_\phi, \boldsymbol{b}_x) \to (-\boldsymbol{b}_\phi, \boldsymbol{b}_x) = -g_{\phi x}$
	$(\boldsymbol{b}_\phi, \boldsymbol{b}_y) \to (-\boldsymbol{b}_\phi, \boldsymbol{b}_y) = -g_{\phi y}$

- Therefore, $t\phi$-invariance requires
  $g_{tx} = g_{ty} = g_{\phi x} = g_{\phi y} = 0$

The metric now has 6 unknown functions of 2 variables, i.e

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{xx}dx^2 + 2g_{xy}dxdy + g_{yy}dy^2$$

Applying Geometric Constraints

Further reduction in the number of (non-zero) coefficients

We have seen that combined symmetries $\Longrightarrow g_{tx} = g_{ty} = g_{\phi x} = g_{\phi y} = 0$.
$\Longrightarrow ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{xx}dx^2 + 2g_{xy}dxdy + g_{yy}dy^2$.

We now note also that:

- $g_{tx} = g_{ty} = g_{\phi x} = g_{\phi y} = 0 \Longrightarrow \boldsymbol{b}_t$ and $\boldsymbol{b}_\phi$ are each $\perp$ to $\boldsymbol{b}_x$ and $\boldsymbol{b}_y$.
  $\Longrightarrow$ 2d subspace spanned by $\{\boldsymbol{b}_t, \boldsymbol{b}_\phi\} \perp$ 2d subspace spanned by $\{\boldsymbol{b}_x, \boldsymbol{b}_y\}$.
- The latter is an intrinsically-defined, 2d submanifold with induced metric
  $d\sigma^2 = g_{xx}dx^2 + 2g_{xy}dxdy + g_{yy}dy^2$. (The former is not.)
- Every 2d manifold $\mathbb{M}^2$ is *conformally flat*,
  $\Longrightarrow \exists$ a smooth, scalar-valued function f such that

$$g_{\mu\nu}(p) = f(p)\delta_{\mu\nu}$$

  at every point p of $\mathbb{M}^2$.
  $\Longrightarrow \exists$ a transformation, $(x, y) \to (x', y')$, that diagonalizes the 2d metric, i.e.

$$d\sigma^2 = f(x', y')\left((dx')^2 + (dy')^2\right)$$

- Thus, $ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + f(x', y')\left((dx')^2 + (dy')^2\right)$,
  where $g_{x'x'} = g_{y'y'} = f(x', y')$ and g_{tt}, $g_{t\phi}$, $g_{\phi\phi}$ are now functions of (x', y').

The metric now has 4 unknown functions of 2 variables.

Applying Geometric Constraints

A slight relaxation of the constraints

We have seen that, under the geometric constraints, the metric has been reduced to 4 unknown functions of 2 variables

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + f(x', y')\left((dx')^2 + (dy')^2\right)$$

which is actually too restrictive for our purposes, because:

- We would now like to add specificity to the remaining space-like coordinates.
- With an eye toward BL coordinates, we would like these to be a radial coordinate r and a polar coordinate θ.
- We are free to make any further coordinate transformation on $\mathbb{M}^2$, so we choose $(x', y') \rightarrow (r, \theta)$.
- This implies that the 2d metric coefficients can no longer be equal, i.e. $f(x', y')\left((dx')^2 + (dy')^2\right) \rightarrow g_{rr}(r, \theta)dr^2 + g_{\theta\theta}(r, \theta)d\theta^2$ and g_{tt}, $g_{t\phi}$ and $g_{\phi\phi}$ are now functions of (r, θ), as well.

The full Kerr metric now has the form

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2$$

as presented in the previous talk.