Rotating Black Holes

First Principles: from Einstein to Kerr (Part II)

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First Principles: Outline (Part II)

- Applying physical constraint, $T_{\mu\nu} = 0$
- Vacuum solutions
- Constrained structure of the Ricci tensor
- Contracted Bianchi identities
- Isolating four Independent PDEs
- Form of the four independent PDEs
- Adding global constraints
- Solution methodology
- The family of black-hole solutions
- Applying the Newtonian Limit
- Applying the slow-rotation limit

Applying Physical constraint, $T_{\mu\nu} = 0$

Relationship between R and the trace of $T_{\mu\nu}$

Starting from $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -kT_{\mu\nu}$:

1. Raise an index (using inverse metric tensor)

$$g^{\sigma\mu}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -kT_{\mu\nu}\right)$$

2. Multiply through

$$\begin{aligned} g^{\sigma\mu}R_{\mu\nu} - \frac{1}{2}Rg^{\sigma\mu}g_{\mu\nu} &= -kg^{\sigma\mu}T_{\mu\nu} \\ R^{\sigma}_{\nu} - \frac{1}{2}R\delta^{\sigma}_{\nu} &= -kT^{\sigma}_{\nu} \end{aligned}$$

3. Contract

$$\begin{aligned} R^{\nu}_{\nu} - \frac{1}{2}R\delta^{\nu}_{\nu} &= kT^{\nu}_{\nu} \\ R - \frac{1}{2}R \times 4 &= -kT \\ R &= kT \end{aligned}$$

Applying Physical constraint, $T_{\mu\nu} = 0$

The trace-reversed field equation

Starting from $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -kT_{\mu\nu}$:

1. Move 2nd term on LHS to RHS

$$R_{\mu\nu} = \frac{1}{2}Rg_{\mu\nu} - kT_{\mu\nu}$$

2. Substitute $R = kT$ (from previous slide)

$$R_{\mu\nu} = \frac{1}{2}kTg_{\mu\nu} - kT_{\mu\nu}$$

3. Factor out k

$$R_{\mu\nu} = k\left(\frac{1}{2}Tg_{\mu\nu} - T_{\mu\nu}\right)$$

It is obvious the preceding equation that in a vacuum, i.e. $T_{\mu\nu} = 0$, we have

$$R_{\mu\nu} = 0$$

i.e. the *vacuum field equation*.

Vacuum Solutions

In all generality, the Ricci tensor is given by

$$R_{uv} = \partial_\nu \Gamma^a_{\ \mu a} - \partial_a \Gamma^a_{\ \mu\nu} + \Gamma^a_{\ \mu\nu}\Gamma^b_{\ ab} - \Gamma^a_{\ \mu b}\Gamma^b_{\ \nu a} \tag{1}$$

where

$$\Gamma^\xi_{\ \mu\nu} = \frac{1}{2} g^{\xi a} \left(\partial_a g_{\mu\nu} + \partial_\nu g_{\mu a} - \partial_\mu g_{\nu a}\right)$$

Trivial vacuum solution: $g_{\mu\nu} = \eta_{\mu\nu} \implies$ all $\partial_a g_{\mu\nu} = 0 \implies$ all $\Gamma^\xi_{\ \mu\nu} = 0$ $\implies R_{\mu\nu} = 0$.

Non-trivial vacuum solutions: the metric is such that:

- At least some components of the $\Gamma^\xi_{\ \mu\nu} \neq 0 \implies$ non-zero terms in (1).
- The sum of these non-zero terms is zero.

Constrained Structure of the Ricci Tensor

We have seen that, the coordinates (t, r, ϕ, θ), the stationary, axisymmetric metric has the form

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2$$

which implies that the metric tensor and the Ricci tensor are block diagonal:

$$\begin{bmatrix} g_{tt}(r,\theta) & g_{t\phi}(r,\theta) & 0 & 0 \\ g_{\phi t}(r,\theta) & g_{\phi\phi}(r,\theta) & 0 & 0 \\ 0 & 0 & g_{rr}(r,\theta) & 0 \\ 0 & 0 & 0 & g_{\theta\theta}(r,\theta) \end{bmatrix} \Longrightarrow \begin{bmatrix} R_{tt}(r,\theta) & R_{t\phi}(r,\theta) & 0 & 0 \\ R_{\phi t}(r,\theta) & R_{\phi\phi}(r,\theta) & 0 & 0 \\ 0 & 0 & R_{rr}(r,\theta) & R_{r\theta}(r,\theta) \\ 0 & 0 & R_{\theta r}(r,\theta) & R_{\theta\theta}(r,\theta) \end{bmatrix}$$

That is, the block diagonal structure of the metric tensor guarantees that:

- The Ricci tensor has the same block diagonal structure.
- The lower-right block of $R_{\mu\nu}$ is NOT, itself, diagonal.

There are thus 6 unique, non-trivial vacuum field equations, i.e.

Upper-Left Block	Lower-Right Block
$R_{tt}(r,\theta) = 0$	$R_{rr}(r,\theta) = 0$
$R_{\phi t}(r,\theta) = 0$	$R_{\bar{r}\theta}(r,\theta) = 0$
$R_{\phi\phi}(r,\theta) = 0$	$R_{\theta\theta}(r,\theta) = 0$

The Contracted Bianchi Identities

The Bianchi identities (rediscovered in 1902 by Italian mathematician Luigi Bianchi) relate covariant derivatives of the Riemann tensor, specifically

$$\nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0$$

Contraction of this equation leads to the contracted Bianchi identities (CBIs)

$$\nabla_a G^a_b = 0$$

where $G^a_b = g^{ac}\left(R_{bc} - \frac{1}{2} R g_{bc}\right)$ is the mixed Einstein tensor.
If $R_{\mu\nu} = 0$ (as in vacuum solutions), then the CBIs reduce to

$$\nabla_a R^a_b = 0$$

In general, the CBIs remove 4 out of 10 the degrees of freedom (DOF) from the field equations.

For stationary, axially-symmetric, vacuum spacetimes, 2 CBIs become trivial, i.e.

$$\nabla_a R^a_t = 0 \implies 0 = 0 \qquad \text{and} \qquad \nabla_a R^a_\phi = 0 \implies 0 = 0$$

Remaining (non-trivial) constraints are

$$\nabla_a R^a_r = 0 \qquad \text{and} \qquad \nabla_a R^a_\theta = 0$$

Thus, the CBIs remove 2 out of 6 DOF. However, in the current coordinates, the remaining 4 DOF are distributed among the 6 PDEs.

Isolating Four Independent PDEs

There is a system, wherein the 4 DOF are distributed among 4 independent PDEs, known as *Weyl-Papapetrou* (WP) coordinates, wherein the metric takes the form

$$ds^2 = f\,dt^2 - 2fh\,dtd\phi + \frac{1}{f}\left(f^2h^2 - \mathtt{r}^2\right)d\phi^2 - \frac{1}{f}e^{2p}\left(d\mathtt{r}^2 + dz^2\right)$$

where there are only three unknown functions f, h and p of coordinates $(\mathtt{r}, z)$, and where $(\mathtt{r}, z)$ are quasi-cylindrical coordinates. That is:

- $\mathtt{r}$, the radial coordinate of the cylinder, is given by

  $$\mathtt{r}^2 = \det\begin{bmatrix} g_{tt}(r,\theta) & g_{t\phi}(r,\theta) \\ g_{\phi t}(r,\theta) & g_{\phi\phi}(r,\theta) \end{bmatrix} = g_{tt}g_{\phi\phi} - g_{t\phi}^2$$

  Note: $\mathtt{r}(r,\theta)$ is a harmonic function $\Longrightarrow$ satisfies $\nabla^2\mathtt{r} = 0$, where

  $$\nabla^2 = \frac{1}{w}\left[\partial_r\left(wg^{rr}\partial_r\right) + \partial_\theta\left(wg^{\theta\theta}\partial_\theta\right)\right], \quad w = \sqrt{g_{rr}g_{\theta\theta}}$$

- z, the coordinate along the axis of the cylinder, is the harmonic conjugate of $\mathtt{r}$,

  i.e.
  z is constrained such that $\mathtt{r}$ and z satisfy the generalized Cauchy-Riemann equations, i.e.

  $$\begin{aligned} wg^{rr}\partial_r\mathtt{r} &= \pm\partial_\theta z \\ wg^{\theta\theta}\partial_\theta\mathtt{r} &= \mp\partial_r z \end{aligned}$$

Isolating 4 Independent PDEs (continued)

The resultant system of vacuum equations has the same block diagonal structure, i.e.

$$\begin{bmatrix} f & -fh & 0 & 0 \\ -fh & \frac{1}{f}\left(f^2h^2 - \mathtt{r}^2\right) & 0 & 0 \\ 0 & 0 & -\frac{1}{f}e^{2p} & 0 \\ 0 & 0 & 0 & -\frac{1}{f}e^{2p} \end{bmatrix} \implies \begin{bmatrix} R_{tt}(\mathtt{r},z) & R_{t\phi}(\mathtt{r},z) & 0 & 0 \\ R_{\phi t}(\mathtt{r},z) & R_{\phi\phi}(\mathtt{r},z) & 0 & 0 \\ 0 & 0 & R_{\mathtt{rr}}(\mathtt{r},z) & R_{\mathtt{r}z}(\mathtt{r},z) \\ 0 & 0 & R_{\theta\mathtt{r}}(\mathtt{r},z) & R_{zz}(\mathtt{r},z) \end{bmatrix}$$

But now, for the upper-left block, we have:

- Three unique vacuum field equations:
  $$R_{tt}(\mathtt{r},z) = 0,\ R_{t\phi}(\mathtt{r},z) = 0,\ R_{\phi\phi}(\mathtt{r},z) = 0.$$
- Each of these is 2nd-order in the functions f and h.
- The CBI $\nabla_a G^a_\phi = 0$ reduces this two 2 independent PDEs.
- Solving this systems determines f and h.

And for the lower-right block:

- Three unique vacuum field equations:
  $$R_{\mathtt{rr}}(\mathtt{r},z) = 0,\ R_{\mathtt{r}z}(\mathtt{r},z) = 0,\ R_{zz}(\mathtt{r},z) = 0.$$
- Each of these is 1st-order in the function p.
- The CBI $\nabla_a G^a_z = 0$ reduces this to 2 independent PDEs.
- Solving this system determines p.

Form of the Independent PDEs

The two, independent, 2nd-order PDEs are: $R_{tt}(\mathtt{r}, z) = 0$ and $R_{t\phi}(\mathtt{r}, z) = 0$, which (after some reorganization) have the following form:

$$\nabla^2 f = \frac{1}{f}\left[|\nabla f|^2 - \frac{f^4}{\mathtt{r}^2}|\nabla h|^2\right]$$

$$\nabla^2 h = -\frac{2}{f}\nabla f \cdot \nabla h$$

where $\nabla = (\partial_{\mathtt{r}}, \partial_{\mathtt{z}})$ and $\nabla^2 = \partial_{\mathtt{r}}^2 + \partial_{\mathtt{z}}^2$ are the flat-space gradient and Laplacian. Note: this system is elliptic $\Longrightarrow$ no waves, no discontinuities, just smooth behavior determined entirely by boundary conditions.

The two, independent 1st-order PDEs are: $R_{\mathtt{rr}}(\mathtt{r}, z) = 0$ and $R_{\mathtt{r}z}(\mathtt{r}, z) = 0$, which (after some reorganization) have the following form:

$$\partial_{\mathtt{r}} p = \frac{\mathtt{r}}{4f^2}\left[(\partial_{\mathtt{r}} f)^2 - (\partial_{\mathtt{z}} f)^2 + \frac{f^4}{\mathtt{r}^2}\left((\partial_z h)^2 - (\partial_{\mathtt{r}} h)^2\right)\right]$$

$$\partial_z p = \frac{\mathtt{r}}{2f^2}\left(\partial_{\mathtt{r}} f \partial_{\mathtt{z}} f - \frac{f^4}{\mathtt{r}^2}\partial_{\mathtt{r}} h \partial_{\mathtt{z}} h\right)$$

Adding Global Constraints

Current status:

$$\left.\begin{array}{c}\text{Stationarity}\\ \text{Axial symmetry}\\ T_{\mu\nu}=0\end{array}\right\} \Longrightarrow \text{WP metric} \Longrightarrow \left\{\begin{array}{c}\text{2D, 2nd order, system of PDEs}\\ \text{2D, 1st order system of PDEs}\end{array}\right.$$

Note: 2nd-order system is elliptic.

To this, we add Global constraints:

- Asymptotic flatness at spatial infinity.
- Even horizon must exist and be
  - Single piece $\Longrightarrow$ topologically compact, S^2
  - Smooth (C^∞) $\Longrightarrow$ no discontinuities, kinks, corners or cusps
  - Regular $\Longrightarrow$ No curvature singularities, analytically extendable
  - Non-degenerate $\Longrightarrow$ non-extremal
- Axis regularity

  $\Longrightarrow$ as $\mathtt{r} \rightarrow 0$, geometry looks like flat space in cylindrical coordinates.

Solution Methodolgy

1. Solve 2nd-order system:
   1.1 Compute the boundary data from the global constraints, i.e. values of f, h and p at spatial infinity, on axis, on horizon.
   1.2 Obtain expressions $f(\mathtt{r}, z)$ and $h(\mathtt{r}, z)$.
   Note 1: elliptic system $\Longrightarrow$ interior solution fully determined by boundary data.
   Note 2: asymptotic flatness introduces two free parameters (say $P_1$and P_2).
2. Solve 1st-order system:
   2.1 Differentiate to obtain expressions for $\partial_{\mathtt{r}} f$, $\partial_{\mathtt{z}} f$, $\partial_{\mathtt{r}} h$ and $\partial_{\mathtt{z}} h$.
   2.2 Substitute results of 1.2 and 2.1 into one 1st-order equation.
   Note: elliptic structure of 2nd-order system
   $\Longrightarrow$ the two integrals (of the 1st-order system) are identical, up to a constant.
   2.3 Integrate to obtain an expression for $p(\mathtt{r}, z)$.
   Note: asymptotic flatness $\Longrightarrow$ constant of integration is zero.
3. Map back to original coordinates, i.e. $(\mathtt{r}, z) \to (r, \theta)$.

The Family of Black-Hole Solutions

After mapping back to (r, θ), the metric functions take the form

$$g_{tt}(r,\theta) = 1 - 2P_1 \frac{r}{\rho^2(r,\theta)}$$

$$g_{t\phi}(r,\theta) = 2P_1 P_2 \frac{r\sin^2\theta}{\rho^2(r,\theta)}$$

$$g_{\phi\phi}(r,\theta) = -\left(r^2 + P_2^2 + 2P_1 P_2^2 \frac{r\sin^2\theta}{\rho^2(r,\theta)}\right)\sin^2\theta$$

$$g_{rr}(r,\theta) = \frac{\rho^2}{\Delta} \qquad g_{\phi\phi}(r,\theta) = -\rho^2$$

where

$$\rho^2(r,\theta) = r^2 + P_2^2\cos^2\theta$$

$$\Delta(r) = r^2 - 2P_1 r + P_2^2$$

Applying the Newtonian Limit

In the Newtonian limit, i.e.

- Weak gravitational field, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $|h_{\mu\nu}| \ll 1$.
- Completely static (or slowly changing) field
- Slow velocities, $v \ll c$.

With signature $(+ - --)$, the standard weak-field expansion of g_{tt} (in SI units) is

$$g_{tt} = 1 + 2\frac{\Phi}{c^2} + 2\frac{\Phi^2}{c^4} + \ldots$$

where Φ is the Newtonian gravitational potential, i.e.

$$\Phi = -\frac{GM}{r}$$

Substituting, we have to 1st order

$$g_{tt} = 1 - 2\frac{GM}{c^2}\frac{1}{r}$$

In the stationary, axially-symmetric, vacuum spacetime,
large $r \implies \rho^2(r,\theta) \to r^2 \implies$

$$g_{tt}(r,\theta) = 1 - 2P_1\frac{r}{\rho^2(r,\theta)} \to 1 - 2P_1\frac{1}{r}$$

Comparing the preceding two equations, it is obvious that $P_1 = \frac{GM}{c^2} = r_g$.

Applying the Slow-Rotation Limit

With signature $(+ - - -)$, the rotation component of the Lense-therring metric is

$$\begin{aligned} g_{t\phi} &= 2\frac{\mathtt{G}J}{c^3}\frac{1}{r}\sin^2\theta \\ &= 2\frac{\mathtt{G}M}{c^2}\frac{J}{cM}\frac{1}{r}\sin^2\theta \\ &= 2r_g a\frac{1}{r}\sin^2\theta \end{aligned}$$

In the stationary, axially-symmetric, vacuum spacetime, large $r \Longrightarrow \rho^2(r,\theta) \to r^2 \Longrightarrow$

$$\begin{aligned} g_{t\phi} &= 2P_1 P_2 \frac{r\sin^2\theta}{\rho^2(r,\theta)} \to 2P_1 P_2 \frac{1}{r}\sin^2\theta \\ &= 2r_g P_2 \frac{1}{r}\sin^2\theta \end{aligned}$$

Comparing, it is obvious that $P_2 = a$.