Geodesics

December 16, 2025

Geodesics: outline

- Geodesics on Riemannian manifolds
- Geodesics on spacetime manifolds
- The geodesic equation
- Three ways of obtaining the geodesic equation
  - Geodesic equation via the notion of parallel transport
  - Geodesic equation via generalization of Newton's first law
  - Geodesic equation via the Euler-Lagrange equations
- The geodesic equation: finding solutions in spacetime

Geodesics on Riemannian Manifolds

For any n-dimensional, Riemannian manifold:

- There are infinitely-many curves connecting any pair of points.
- Among these curves, those that minimize the distance between them (relative to nearby curves) are the geodesics.
- If the manifold is open and either flat or hyperbolic
  - There there is only one geodesic passing through any pair of points.
  - The points are connected by a single segment of that geodesic.
- If the manifold is closed
  - There exists at least one geodesic passing through any pair of points.
  - But there may be multiple (even infinitely many) geodesics or geodesic segments connecting them.

Examples:

- If the manifold is the Euclidean plane $\mathbb{E}^2$
  - The geodesics are straight lines.
  - A single geodesic (straight line) passes through any pair of points and the points are connected by a singe geodesic (line) segment.
- If the manifold is the 2-sphere $\mathbb{S}^2$
  - The geodesics are great circles.
  - A single geodesic passes through any pair of non-antipodal points. and the points are connected by two unequal geodesic segments.
  - Infinitely many geodesics pass through any pair of antipodal points and, on each geodesic, the points are connected by two equal geodesic segments.

Geodesics on the Spacetime Manifold

For any pseudo-Riemannian manifold, with metric

$$ds^2 = g_{u\nu} dq^\mu dq^\nu, \; \mu, \nu = 0, 1, 2, 3$$

having Lorentz signature, there exists three distinct types of geodesics. Using the signature convention $(+ - --)$, these are

- For $ds^2 > 0 \implies$ time-like:
  - There are infinitely-many *time-like curves* connecting any pair of time-like-separated events.
  - Among these curves, those that maximize the proper distance s between them (relative to nearby curves) are the *time-like geodesics*.
  - There may be one or more time-like geodesics or geodesic segments connecting the two events.
- For $ds^2 < 0 \implies$ space-like:
  - There are infinitely-many *space-like curves* connecting any pair of space-like-separated events.
  - Among these curves, those that minimize the proper distance s between them (relative to nearby curves) are the *space-like geodesics*.
  - There may be one or more space-like geodesics or geodesic segments connecting the two events.
- For $ds^2 = 0 \implies$ null: there is one and only one null curve connecting any pair of null-separated events and that curve is the *null geodesic*.

The Geodesic Equation

Regardless of whether we have a Riemannian or spacetime manifold, the geodesics are solutions of the geodesic equation

$$\frac{d^2 q^\mu}{d\lambda^2} + \Gamma^\mu_{ab} \frac{dq^a}{d\lambda} \frac{dq^b}{d\lambda} = 0$$

where:

- A solution is a set of n functions $q^\mu(\lambda)$, $\mu = 0, 1, ..., n$.
- Together, the n equations represent the geodesic in parametric form.
- The parameter λ is monotonically increasing along the curve.
- The RHS of the geodesic equation is zero only if λ is an affine parameter

  meaning distance s along the curve $q^\mu(\lambda)$ is linearly related to λ, i.e. $s = a\lambda + b$, where a and b are constants and $a \neq 0$.
- For null geodesics on spacetime manifolds, λ cannot be defined as above, because $s = 0$; otherwise, we would have $\lambda = -b/a$, contradicting the fact that λ is monotonically increasing. (Affine parameters do exist for null geodesics, but they are defined differently.)
- Henceforth, we will not be concerned with space-like geodesics.

Derivations of the Geodesic Equation

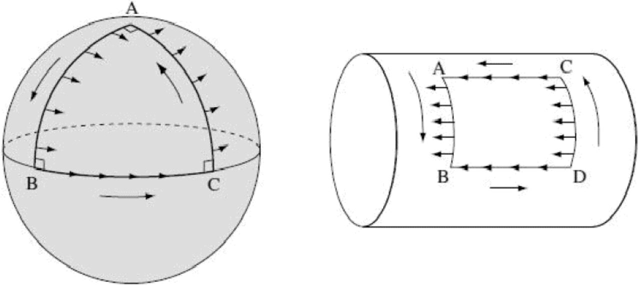
There are three ways to arrive at the geodesic equation:

- Via the notion of parallel transport
- Via generalization of Newton's first law
- Via the Euler-Lagrange equations

We will briefly discuss each of these in the next three slides

Geodesic Equation via Parallel Transport

Parallel transport of a vector along an arbitrary curve is defined by moving the vector in infinitesimal increments along the curve, while preserving its length and keeping it parallel to itself (to 1st order)



Parallel transport is one of the key concepts in Riemannian geometry, because:

- It provides a precise definition of intrinsic curvature, and leads to the curvature tensor.
- It provides a means to generate geodesics, and leads to the geodesic equation.

Geodesic Equation via Parallel Transport (continued)

Let:

- $q^{\mu}(\lambda)$ be an arbitrary curve in parametric form,
- $\{\boldsymbol{b}_{\mu}(\lambda)\}$ be the coordinate basis at λ,
- $\boldsymbol{v}(\lambda) = v^{\mu}\boldsymbol{b}_{\mu}(\lambda)$ be arbitrary vector at λ.

As $\lambda \to (\lambda + d\lambda)$, $\boldsymbol{v}(\lambda) \to \boldsymbol{v}(\lambda + d\lambda)$, $\boldsymbol{b}_{\mu}(\lambda) \to \boldsymbol{b}_{\mu}(\lambda + d\lambda)$
and in components:

1. $\boldsymbol{v}(\lambda + d\lambda) = v^{\mu} + dv^{\mu}$
   $\boldsymbol{b}_{\mu}(\lambda + d\lambda) = \boldsymbol{b}_{\mu} + d\boldsymbol{b}_{\mu}$.
2. Then, $\boldsymbol{v}(\lambda + d\lambda) = (v^{\mu} + dv^{\mu})\,(\boldsymbol{b}_{\mu} + d\boldsymbol{b}_{\mu})$
3. So to 1st order $\boldsymbol{v}(\lambda + d\lambda) \approx v^{\mu}\boldsymbol{b}_{\mu} + v^{\mu}d\boldsymbol{b}_{\mu} + \boldsymbol{b}_{\mu}dv^{\mu}$.
4. Then, the 1st-order change in $\boldsymbol{v}$ is
   $$\delta\boldsymbol{v} := \boldsymbol{v}(\lambda + d\lambda) - \boldsymbol{v}(\lambda) \approx \boldsymbol{b}_{\mu}dv^{\mu} + v^{\mu}d\boldsymbol{b}_{\mu}.$$
5. Keeping $\boldsymbol{v}$ the same length and parallel to itself to 1st order implies
   $\delta\boldsymbol{v} = 0$, so
   $$\boldsymbol{b}_{\mu}dv^{\mu} + v^{\mu}d\boldsymbol{b}_{\mu} = 0.$$

Geodesic Equation via Parallel Transport (continued)

We continue the derivation of the equation for parallel transport of a vector. In last step (5, previous slide), we arrived at $\boldsymbol{b}_\mu dv^\mu + v^\mu d\boldsymbol{b}_\mu = 0$.

6. The expression for the change in basis vectors $d\boldsymbol{b}_\mu$ is

$$
\begin{aligned}
d\boldsymbol{b}_\mu &= \frac{\partial}{\partial q^\mu}\left(\frac{\partial p^\sigma}{\partial q^a}\right)\frac{\partial q^a}{\partial p^\nu} dq^\nu \boldsymbol{b}_\sigma \\
&= \Gamma^\sigma_{\mu\nu} dq^\nu \boldsymbol{b}_\sigma
\end{aligned}
$$

   where $p^\sigma := q^\sigma + dq^\sigma$ and the red factor is literally the definition of $\Gamma^\sigma_{\mu\nu}$.

7. Substituting the result of step 6 into step 5, we arrive at the equation for parallel transport

$$
\begin{aligned}
\boldsymbol{b}_\mu dv^\mu + v^\mu \Gamma^\sigma_{\mu\nu} dq^\nu \boldsymbol{b}_\sigma &= 0 \\
\boldsymbol{b}_\sigma dv^\sigma + v^\mu \Gamma^\sigma_{\mu\nu} dq^\nu \boldsymbol{b}_\sigma &= 0 \\
\boldsymbol{b}_\sigma \left(dv^\sigma + \Gamma^\sigma_{\mu\nu} v^\mu dq^\nu\right) &= 0 \\
dv^\sigma + v^\mu \Gamma^\sigma_{\mu\nu} dq^\nu &= 0 \\
\frac{dv^\sigma}{d\lambda} + \Gamma^\sigma_{\mu\nu} v^\mu \frac{dq^\nu}{d\lambda} &= 0
\end{aligned}
$$

Geodesic Equation via Parallel Transport (continued)

In the last step (7, preceding slide), we arrived at

$$\frac{d}{d\lambda}(v^\sigma) + \Gamma^\sigma_{\mu\nu} v^\mu \frac{dq^\nu}{d\lambda} = 0$$

i.e. the equation for parallel transport of an arbitrary vector.

Then:

1. Recalling that $\frac{dq^\nu}{d\lambda}$ is the tangent vector to the curve $q^\nu(\lambda)$, we take the arbitrary vector v^μ to be a tangent vector, i.e.

$$v^\sigma = \frac{dq^\sigma}{d\lambda} \quad and \quad v^\sigma = \frac{dq^\mu}{d\lambda}$$

2. Substituting Step 1 into the equation for parallel transport (top), we arrive at

$$\begin{aligned} \frac{d}{d\lambda}\left(\frac{dq^\sigma}{d\lambda}\right) + \Gamma^\sigma_{\mu\nu}\frac{dq^\mu}{d\lambda}\frac{dq^\nu}{d\lambda} &= 0 \\ \frac{d^2 q^\sigma}{d\lambda^2} + \Gamma^\sigma_{\mu\nu}\frac{dq^\mu}{d\lambda}\frac{dq^\nu}{d\lambda} &= 0 \end{aligned}$$

   i.e. the geodesic equation.

Geodesic Equation via Newton's First Law

We may write Newton's first law in parametric form and Cartesian coordinates x^μ, $\mu = 1, 2, 3$ as

$$\frac{d^2 x^\mu(t)}{dt^2} = 0.$$

Generalization:

1. Recognize that t is an affine parameter because, if $\boldsymbol{v}$ is velocity

$$s(t) = \int_{t_o}^{t} |\boldsymbol{v}| dt' = |\boldsymbol{v}|(t - t_0) = |\boldsymbol{v}|t + const$$

   i.e. s is linearly related to t.

2. It is mathematically permissible to

   2.1 Replace t with any other affine parameter λ, where $\lambda = \alpha t + \beta$, and where α and β are constant and $\alpha \neq 0$.

   2.2 Let the number of dimensions be any integer $n > 2$.

   2.3 We then have

$$\frac{d^2 x^\mu(\lambda)}{d\lambda^2} = 0, \ \mu = 1, 2, ..., n \tag{1}$$

Geodesic Equation via Newton's First Law (continued)

We continue with generalization of Newton's first law, by making it valid for arbitrary coordinates, as follows:

3. Transform to from Cartesian coordinates x^μ to arbitrary coordinates q^μ, i.e.

$$x^\mu(\lambda) \to q^\mu(x^\mu(\lambda))$$

4. The total derivative of q^μ along the curve is given by

$$\frac{dq^\mu}{d\lambda} = \frac{\partial q^\mu}{\partial x^a}\frac{dx^a}{d\lambda} \tag{2}$$

5. Taking 2nd derivatives, we have

$$\frac{d^2 q^\mu(\lambda)}{d\lambda^2} = \frac{\partial q^\mu}{\partial x^a}\frac{d^2 x^a}{d\lambda^2} + \frac{d}{d\lambda}\left(\frac{\partial q^\mu}{\partial x^a}\right)\frac{dx^a}{d\lambda} \tag{3}$$

where the 1st factor of the 2nd term (red) is once again a total derivative, i.e.

$$\frac{d}{d\lambda}\left(\frac{\partial q^\mu}{\partial x^a}\right) = \frac{\partial}{\partial x^b}\left(\frac{\partial q^\mu}{\partial x^a}\right)\frac{dx^b}{d\lambda} \tag{4}$$

and substituting the result from (4) (magenta) into (3), yields

$$\frac{d^2 q^\mu(\lambda)}{d\lambda^2} = \frac{\partial q^\mu}{\partial x^a}\frac{d^2 x^a}{d\lambda^2} + \frac{\partial}{\partial x^b}\left(\frac{\partial q^\mu}{\partial x^a}\right)\frac{dx^a}{d\lambda}\frac{dx^a}{d\lambda}. \tag{5}$$

Geodesic Equation via Newton's First Law (continued)

In the last step (5, preceding slide) we arrived at

$$\frac{d^2 q^\mu(\lambda)}{d\lambda^2} = \frac{\partial q^\mu}{\partial x^a}\frac{d^2 x^a}{d\lambda^2} + \frac{\partial}{\partial x^b}\left(\frac{\partial q^\mu}{\partial x^a}\right)\frac{dx^a}{d\lambda}\frac{dx^a}{d\lambda}$$

We continue the transformation from Cartesian to arbitrary coordinates:

6. Since $\frac{d^2 q^\mu(\lambda)}{d\lambda^2} = 0$, we have

$$\frac{\partial q^\mu}{\partial x^a}\frac{d^2 x^a}{d\lambda^2} + \frac{\partial}{\partial x^b}\left(\frac{\partial q^\mu}{\partial x^a}\right)\frac{dx^b}{d\lambda}\frac{dx^a}{d\lambda} = 0 \qquad (6)$$

7. Now, if we multiply (6) by $\frac{\partial x^\nu}{\partial q^\mu}$, we arrive at the geodesic equation

$$\frac{\partial x^\nu}{\partial q^\mu}\left(\frac{\partial q^\mu}{\partial x^a}\frac{d^2 x^a}{d\lambda^2} + \frac{\partial}{\partial x^b}\left(\frac{\partial q^\mu}{\partial x^a}\right)\frac{dx^b}{d\lambda}\frac{dx^a}{d\lambda} = 0\right)$$

$$\begin{aligned}
\frac{\partial x^\nu}{\partial q^\mu}\frac{\partial q^\mu}{\partial x^a}\frac{d^2 x^a}{d\lambda^2} + \frac{\partial x^\nu}{\partial q^\mu}\frac{\partial}{\partial x^b}\left(\frac{\partial q^\mu}{\partial x^a}\right)\frac{dx^b}{d\lambda}\frac{dx^a}{d\lambda} &= 0 \\
\frac{\partial x^\nu}{\partial x^a}\frac{d^2 x^a}{d\lambda^2} + \frac{\partial}{\partial x^b}\left(\frac{\partial q^\mu}{\partial x^a}\right)\frac{\partial x^\nu}{\partial q^\mu}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} &= 0 \\
\delta^\nu_a\frac{d^2 x^a}{d\lambda^2} + \frac{\partial}{\partial x^b}\left(\frac{\partial q^\mu}{\partial x^a}\right)\frac{\partial x^\nu}{\partial q^\mu}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} &= 0 \\
\frac{d^2 x^\nu}{d\lambda^2} + \Gamma^\nu_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} &= 0
\end{aligned}$$

Geodesic Equation via Euler-Lagrange Equation

We first need to define a Lagrangian:

1. On a Riemannian manifold, the distance s along any parameterized curve $q^\mu(\lambda)$ is given by

$$s = \int ds = \int \frac{ds}{d\lambda} d\lambda \tag{7}$$

where ds is determined by the Riemannian metric

$$ds^2 = g_{\mu\nu} dq^\mu dq^\nu . \tag{8}$$

2. Substituting (8) into (7), we have

$$s = \int \frac{\sqrt{g_{\mu\nu} dq^\mu dq^\nu}}{d\lambda} d\lambda = \int \sqrt{g_{\mu\nu} \frac{dq^\mu}{d\lambda} \frac{dq^\nu}{d\lambda}} d\lambda. \tag{9}$$

3. The geodesic between two points is the curve that minimizes s. Thus, (9) is formally an action

$$s = \int \mathcal{L} d\lambda$$

and we see that the Lagrangian is

$$\mathcal{L} = \sqrt{g_{\mu\nu} \frac{dq^\mu}{d\lambda} \frac{dq^\nu}{d\lambda}}. \tag{10}$$

4. Although $\frac{dq^\mu}{d\lambda}$ is a tangent vector, we can think of it as a generalized velocity, i.e. $\dot{q}^\mu = dq^\mu / d\lambda$and thus rewrite the Lagrangian as

$$\mathcal{L} = \sqrt{g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu} \tag{11}$$

Geodesic Equation via Euler-Lagrange Equation (continued)

The Lagrangian $\mathcal{L} = \sqrt{g_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu}}$ has two drawbacks:

- Derivatives with respect to generalized velocities are messy.
- In spacetime, this Lagrangian does not work for null geodesics.

It turns out that the quadratic Lagrangian

$$\mathcal{L}_Q = \frac{1}{2} g_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu} \tag{12}$$

has the following general properties:

- Most importantly, it gives the same geodesics as $\mathcal{L}$.
- Calculations are much simpler (no square root).

In spacetime:

- For time-like geodesics, $\dot{q}^{\mu}$ is actually the 4-velocity u^{μ}.
- Thus, the RHS of (12):
  - Looks formally like Newtonian kinetic energy (per unit mass).
  - Can be interpreted as generalized kinetic energy.
  - Becomes Newtonian kinetic energy in the Newtonian limit.
- For freely-falling bodies, the potential energy is zero, so $\mathcal{L}_Q$ is physically the full Lagrangian.
- It works equally well for null geodesics.

Note: $\mathcal{L}_Q$ gives the same geodesics for any constant factor (not just 1/2);
The factor 1/2 is chosen so that:

- The generalized kinetic energy reduces to the Newtonian kinetic energy.
- The conjugate momentum does not have an extra factor.

Geodesic Equation via Euler-Lagrange Equation (continued)

Substitute the Lagrangian $\mathcal{L}_Q = \frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu$ into the Euler-Lagrange equation

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) - \frac{\partial \mathcal{L}}{\partial q^a} = 0 : \tag{13}$$

1. Obtain conjugate momentum, i.e. 1st term of (13)

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{q}^a} &= \frac{\partial}{\partial \dot{q}^a}\left(\frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu\right) = \frac{1}{2} g_{\mu\nu} \frac{\partial}{\partial \dot{q}^a}\left(\dot{q}^\mu \dot{q}^\nu\right) \\
&= \frac{1}{2} g_{\mu\nu}\left(\frac{\partial \dot{q}^\mu}{\partial \dot{q}^a}\dot{q}^\nu + \dot{q}^\mu \frac{\partial \dot{q}^\nu}{\partial \dot{q}^a}\right) = \frac{1}{2} g_{\mu\nu}\left(\delta^\mu_a \dot{q}^\nu + \dot{q}^\mu \delta^\nu_a\right) \\
&= \frac{1}{2}\left(g_{\mu\nu}\delta^\mu_a \dot{q}^\nu + g_{\mu\nu}\delta^\nu_a \dot{q}^\mu\right) = \frac{1}{2}\left(g_{a\nu}\dot{q}^\nu + g_{\mu a}\dot{q}^\mu\right) \\
&= \frac{1}{2}\left(g_{\mu a}\dot{q}^\mu + g_{\mu a}\dot{q}^\mu\right) = g_{\mu a}\dot{q}^\mu
\end{aligned} \tag{14}$$

2. Obtain derivative of conjugate momentum (14)

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) = \frac{d}{d\lambda}\left(g_{\mu a}\dot{q}^\mu\right) = \frac{d g_{\mu a}}{d\lambda}\dot{q}^\mu + g_{\mu a}\frac{d\dot{q}^\mu}{d\lambda} \tag{15}$$

where the magenta factor is $\ddot{q}^\mu$ and the red factor is a total derivative , i.e.

$$\frac{d g_{\mu a}}{d\lambda} = \frac{\partial g_{\mu a}}{\partial q^b}\frac{d q^b}{d\lambda} = \partial_b g_{\mu a}\dot{q}^b .$$

3. Substitute the previous two expressions into (15), yielding

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) = \partial_b g_{\mu a}\dot{q}^b \dot{q}^\mu + g_{\mu a}\ddot{q}^\mu \tag{16}$$

Geodesic Equation via Euler-Lagrange Equation (continued)

After substituting Lagrangian $\mathcal{L}_Q = \frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu$ into

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) - \frac{\partial \mathcal{L}}{\partial q^a} = 0 \quad (13) \textit{repeated}$$

we arrived at (step 3, previous slide)

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) = \partial_b g_{\mu a} \dot{q}^b \dot{q}^\mu + g_{\mu a} \ddot{q}^\mu \quad (16) \textit{ repeated.}$$

Now:

4. Obtain 2nd term of (13)

$$\frac{\partial \mathcal{L}}{\partial q^a} = \frac{\partial}{\partial q^a}\left(\frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu\right) = \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial q^a} \dot{q}^\mu \dot{q}^\nu = \frac{1}{2} \partial_a g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu \qquad (17)$$

5. Substituting (16) and (17) into (13) yields

$$g_{\mu a} \ddot{q}^\mu + \partial_b g_{\mu a} \dot{q}^b \dot{q}^\mu - \frac{1}{2} \partial_a g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu = 0 \qquad (18)$$

6. Multiply (18) by the inverse metric

$$g^{\sigma a} g_{\mu a} \ddot{q}^\mu + g^{\sigma a} \partial_b g_{\mu a} \dot{q}^b \dot{q}^\mu - \frac{1}{2} g^{\sigma a} \partial_a g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu = 0 \qquad (19)$$

Geodesic Equation via Euler-Lagrange Equation (continued)

After substituting the Lagrangian $\mathcal{L}_Q = \frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu$ into

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) - \frac{\partial \mathcal{L}}{\partial q^a} = 0 \quad (13) \textit{ repeated}$$

we arrived at (step 6, previous slide)

$$g^{\sigma a} g_{\mu a} \ddot{q}^\mu + g^{\sigma a} \partial_b g_{\mu a} \dot{q}^b \dot{q}^\mu - \frac{1}{2} g^{\sigma a} \partial_a g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu = 0 \quad (19) \textit{ repeated}$$

Now:

7. Because $\partial_b g_{\mu a}$ is contracted with $\dot{q}^b \dot{q}^\mu$, μ, b are dummy indices. This allows us to symmetrize $\partial_b g_{\mu a}$ on indices μ and b, i.e.

$$\partial_b g_{\mu a} = \frac{1}{2}\left(\partial_b g_{\mu a} + \partial_\mu g_{bs}\right) \quad (20)$$

8. Substitute (20) into (19)

$$\begin{aligned}
g^{\sigma a} g_{\mu a} \ddot{q}^\mu + \frac{1}{2} g^{\sigma a}\left(\partial_b g_{\mu a} + \partial_\mu g_{ab}\right) \dot{q}^b \dot{q}^\mu - \frac{1}{2} g^{\sigma a} \partial_a g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu &= 0 \\
\delta^\sigma_\mu \ddot{q}^\mu + \frac{1}{2} g^{\sigma a}\left(\partial_\nu g_{\mu a} + \partial_a g_{\mu\nu}\right) \dot{q}^\nu \dot{q}^\mu - \frac{1}{2} g^{\sigma a} \partial_a g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu &= 0 \\
\ddot{q}^\sigma + \frac{1}{2} g^{\sigma a}\left(\partial_\nu g_{\mu a} + \partial_a g_{\mu\nu} - \partial_a g_{\mu\nu}\right) \dot{q}^\mu \dot{q}^\nu &= 0 \\
\frac{d^2 q^\sigma}{d\lambda^2} + \Gamma^\sigma_{\mu\nu} \frac{dq^\mu}{d\lambda} \frac{dq^\nu}{d\lambda} &= 0 \quad (21)
\end{aligned}$$

The Geodesic Equation: Finding Solutions in Spacetime

The geodesic equation

$$\frac{d^2 q^\sigma}{d\lambda^2} + \Gamma^\sigma_{\mu\nu} \frac{dq^\mu}{d\lambda} \frac{dq^\nu}{d\lambda} = 0$$

is not easily solved, because:

- It is coupled system of 4, 2nd-order, nonlinear ODEs.
- The 2nd term is, in general, a summation over 16 terms.
- In each of those terms, the connection is given by

  $$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma a} \left(\partial_\nu g_{\mu a} + \partial_a g_{\mu\nu} - \partial_a g_{\mu\nu}\right)$$

  which is, itself, the sum of 4 terms, each involving 3 different partial derivatives of $g_{\mu\nu}$.

Thus, even if some components of the metric tensor are zero,
a direct attack is not advisable.

Instead, if one can find 4 quantities that remain constant along the geodesics, then the system can be reduced to 4 decoupled, 1st-order equations of the form

$$\frac{dq^\mu(\lambda)}{d\lambda} = f^\mu(q^\nu, \lambda), \quad \mu, \nu = 0, 1, 2, 3.$$

Thus

$$q^\mu(\lambda) = \int f^\mu(q^\nu, \lambda) d\lambda$$

so even if the f^μ cannot be integrated analytically,
the geodesics can be determined by simple numerical integration.