

Rotating Black Holes

Solving the Geodesic Equation

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Solving the Geodesic Equation: Outline

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Geodesic Equation: Review

Last time, we saw that the geodesic equation is

$$\frac{d^2 q^\mu}{d\lambda^2} + \Gamma_{ab}^\mu \frac{dq^a}{d\lambda} \frac{dq^b}{d\lambda} = 0,$$

where:

- ▶ The q^μ are arbitrary coordinates.
- ▶ The geodesic is a curve $q^\mu(\lambda)$, $\mu = 0, 1, \dots, n$ in parametric form.
- ▶ λ is an affine parameter along the curve $q^\mu(\lambda)$, which means the distance s along the curve $q^\mu(\lambda)$ is linearly related to λ , i.e.
 $s = a\lambda + b$, where a and b are constants and $a \neq 0$.
Note: not true for null geodesics in spacetime, because $s = 0$.
- ▶ The $dq^a/d\lambda$ are components of the tangent vector to the curve $q^\mu(\lambda)$.
- ▶ The Γ_{ab}^μ are connection coefficients.

Geodesic Equation: Further Considerations

Denoting the tangent vector by $u^a = dq^a/d\lambda$, the geodesic equation can be written as

$$\frac{du^\mu}{d\lambda} + \Gamma_{ab}^\mu u^a u^b = 0. \quad (1)$$

The LHS of (1) is actually the *total covariant derivative* $\nabla_{\vec{u}} u^a$ of \mathbf{u} along \mathbf{u} . To see what this means, note that:

- ▶ The covariant derivative of any contravariant vector u^μ along coordinate direction q^a is

$$\nabla_a u^\mu = \partial_a u^\mu + \Gamma_{ab}^\mu u^b \quad (2)$$

where the connection term corrects for curvature or non-Cartesian coordinates.

- ▶ If u^μ is a tangent to $q^a(\lambda)$, the total derivative of u^μ wrt λ is given by

$$\frac{du^\mu}{d\lambda} = \frac{dq^a}{d\lambda} \frac{\partial u^\mu}{dq^a} = u^a \partial_a u^\mu \quad (3)$$

- ▶ If we define $\nabla_{\vec{u}} u^\mu := u^a \nabla_a u^\mu$, and substitute (2) and then (3), we have

$$\nabla_{\vec{u}} u^\mu := u^a \nabla_a u^\mu = u^a \left(\partial_a u^\mu + \Gamma_{ab}^\mu u^b \right) = u^a \partial_a u^\mu + \Gamma_{ab}^\mu u^a u^b = \frac{du^\mu}{d\lambda} + \Gamma_{ab}^\mu u^a u^b$$

so the geodesic equation (1) can be expressed as $\nabla_{\vec{u}} u^a = 0$.

Geodesic Equation: Further Considerations (continued)

The preceding slide showed that the geodesic equation can be expressed as $\nabla_{\vec{u}} u^a = 0$, which literally means that a geodesic is a curve $q^\mu(\lambda)$, such that the tangent vector \mathbf{u} is constant along $q^\mu(\lambda)$.

It is possible to have the same geodesic, without requiring u^a to be constant, but only in a restricted sense. That is:

- The length $|\mathbf{u}|$ of \mathbf{u} can change, but not its direction.
- In the latter case, a different curve would be generated.

If we allow $|\mathbf{u}|$ to change, the geodesic equation

generalizes from $\nabla_{\vec{u}} u^a = 0$ to $\nabla_{\vec{u}} u^a = f(\lambda)u^a$, where f is any smooth, scalar-valued function of λ .

$|\mathbf{u}| \iff \lambda$ is an affine parameter. Thus:

- ▶ $\nabla_{\vec{u}} u^a = 0$ is more precisely called the affinely-parameterized geodesic equation.
- ▶ $\nabla_{\vec{u}} u^a = f(\lambda)u^a$ is the generally-parameterized geodesic equation.
- ▶ The latter reduces to the former when $f(\lambda) = 0$.

Note: in the derivation of $\nabla_{\vec{u}} u^a = 0$ via parallel transport, a question arose.

Where did the assumption of affine λ enter into the derivation?

Answer: it implicitly enters when we insist that $|\mathbf{u}| = \text{const}$.

Geodesic Equation \iff Equations of Motion

Previously, it was noted that the affinely-parameterized geodesic equation $\nabla_{\vec{u}} u^a = 0$ is a system of 4, coupled, nonlinear, 2nd-order ODEs.

If we interpret the tangent vector $u^\mu = dq^\mu/d\lambda$ as generalized velocity \dot{q}^μ , the geodesic equation $\nabla_{\vec{u}} u^a = 0$ can be written in the form

$$\ddot{q}^\mu + \Gamma_{ab}^\mu \dot{q}^a \dot{q}^b = 0 \quad (4)$$

which suggests that this system actually provides *equations of motion*.

This is **physically** true in GR, where

(4) gives the equations of motion for freely falling test particles.

Previously, it was noted that the system (4) is extremely difficult to solve.

However, if one can find 4 quantities that remain constant along the geodesic, then the system can be reduced to 4 decoupled, 1st-order equations of the form

$$\frac{dq^\mu(\lambda)}{d\lambda} = f^\mu(q^\nu, \lambda) \implies q^\mu(\lambda) = \int f^\mu(q^\nu, \lambda) d\lambda.$$

The 4 quantities are the *constants of motion*., which we denote by C_i , $i = 1, 2, 3, 4$. We have already seen the first of these, i.e. $C_1 = |\mathbf{u}|$. True for any manifold.

C_2 and C_3 will be obtained via the Euler-Lagrange equation for Kerr spacetime.

C_4 will be obtained via the separation of the Hamilton-Jacobi equation for Kerr spacetime.

Euler-Lagrange Equations: Review

We will find two additional constants of motion via the Euler-Lagrange equations

$$\frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} \right) - \frac{\partial \mathcal{L}}{\partial q^\mu} = 0. \quad (5)$$

In the previous talk, we saw that:

- ▶ The Lagrangian for GR (in the absence of non-gravitational forces) is

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu \quad (6)$$

- ▶ Then from (6)

$$\frac{\partial \mathcal{L}}{\partial q^\mu} = \frac{1}{2} \partial_\mu g_{ab} \dot{q}^a \dot{q}^b \quad (7)$$

and the conjugate momentum is

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} = g_{\mu a} \dot{q}^a \quad (8)$$

The 2nd Constant of Motion

For the Boyer-Lindquist coordinate t :

- ▶ The Euler-Lagrange equation (5) is

$$\frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial \dot{t}} \right) - \frac{\partial \mathcal{L}}{\partial t} = 0 \quad (9)$$

- ▶ Recalling that for the Kerr Metric the g_{ab} are functions of r and θ , only, the 2nd term on the LHS (9) is

$$\frac{\partial \mathcal{L}}{\partial t} = \frac{1}{2} \partial_t g_{ab}(r, \theta) \dot{q}^a \dot{q}^b = 0 \quad \text{timetranslation symmetry.} \quad (10)$$

and the conjugate momentum is

$$p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} \quad \text{corresponding conserved quantity} \quad (11)$$

Substituting (10) and (11) into (9), we have

$$\frac{d}{d\lambda} (p_t) - 0 = 0 \implies \frac{d}{d\lambda} (p_t) = 0 \implies p_t = C_2 \quad (12)$$

The 3rd Constant of Motion

For the Boyer-Lindquist coordinate ϕ :

- ▶ The Euler-Lagrange equation (5) is

$$\frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad (13)$$

- ▶ Recalling that for the Kerr Metric the g_{ab} are functions of r and θ , only, the 2nd term on the LHS (15) is

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{1}{2} \partial_{\phi} g_{ab}(r, \theta) \dot{q}^a \dot{q}^b = 0 \quad \text{axial symmetry.} \quad (14)$$

and the conjugate momentum is

$$p_{\phi} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad \text{corresponding conserved quantity} \quad (15)$$

Substituting (14) and (15) into (13), we have

$$\frac{d}{d\lambda} (p_{\phi}) - 0 = 0 \implies \frac{d}{d\lambda} (p_{\phi}) = 0 \implies p_{\phi} = C_3 \quad (16)$$

The 4th Constant of Motion

The revelation of a 4th constant of motion is considerably more complicated than the previous three.

Some history:

- ▶ Initially, it was not known whether a 4th constant of motion existed.
- ▶ In 1968, Brandon Carter deduced the existence of the 4th constant, now called *Carter's constant*.
- ▶ Carter did this by explicitly demonstrating the separability of the Hamilton-Jacobi equation.

Note: this is the same Carter who, in 1968, demonstrated the global structure of Kerr spacetime, i.e. the maximally-extended Kerr solution.

The methodology requires that we discuss:

1. *Hamilton's principle function*
2. *The Hamilton-Jacobi equation*
3. The Hamilton-Jacobi equation in Kerr spacetime.
4. Separation of the Hamilton-Jacobi equation.

Hamilton's Principle Function in Spacetime

Definition: Hamilton's principle function $S(q^\mu, \lambda)$ is the action evaluated along a geodesic $q^\mu(\lambda)$, from a fixed initial point (q_0^μ, λ_0) to some endpoint (q^μ, λ) . Thus,

$$S(q^\mu, \lambda) = \int_{\lambda_0}^{\lambda} \mathcal{L}(q^\mu(\lambda'), \dot{q}^\mu(\lambda'), \lambda') d\lambda'.$$

Properties:

- ▶ S is a smooth, scalar-valued function from an augmented, i.e. 5-dimensional, space to the real numbers. Specifically,

$$S : \mathbb{M}^4 \times \mathbb{I} \longrightarrow \mathbb{R}$$

where \mathbb{M}^4 is the spacetime manifold and $\mathbb{I} \subset \mathbb{R} = [\lambda_0, \lambda]$.

- ▶ The derivative of S wrt the coordinates q^μ is the conjugate momentum, i.e.

$$\frac{\partial S}{\partial q^\mu} = \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} = p_\mu. \quad (17)$$

- ▶ S satisfies the Hamilton-Jacobi equation (to be discussed next)

The Hamilton-Jacobi Equation in Spacetime

In all generality, the Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial \lambda} + \mathcal{H} = 0 \quad (18)$$

where \mathcal{H} is the Hamiltonian. In a spacetime with no (non-gravitational) forces acting

$$\mathcal{H} = \mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu. \quad (19)$$

However, it will be convenient to write \mathcal{H} in terms of the conjugate momentum. From (7), we recalled that

$$\begin{aligned} g_{\mu a} \dot{q}^a &= p_\mu \\ (g^{\sigma\mu} g_{\mu a}) \dot{q}^a &= g^{\sigma\mu} p_\mu \\ (\delta_a^\sigma) \dot{q}^a &= g^{\sigma\mu} p_\mu \\ \dot{q}^\sigma &= g^{\sigma a} p_a \end{aligned} \quad (20)$$

Then, substituting (20) into (19), we have

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu \\ &= \frac{1}{2} g_{\mu\nu} (g^{\mu a} p_a) (g^{\nu b} p_b) \\ &= \frac{1}{2} \delta_\nu^a p_a (g^{\nu a} p_a) = \frac{1}{2} p_\nu g^{\nu a} p_a = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \end{aligned} \quad (21)$$

The Hamilton-Jacobi Equation in Spacetime (continued)

In the previous slide, we saw that the Hamilton-Jacobi equation is given, in general, by (18), i.e.

$$\frac{\partial S}{\partial \lambda} + \mathcal{H} = 0$$

and that in spacetime, with no (non-gravitational) forces acting, the Hamiltonian \mathcal{H} can be written as shown in (21), i.e.

$$\mathcal{H} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu.$$

Now, recalling from (17) that

$$p_\mu = \frac{\partial S}{\partial q^\mu}$$

and substituting into (21), the Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial q^\mu} \frac{\partial S}{\partial q^\nu} \quad (22)$$

and substituting (22) into (18), we arrive at the desired form of the Hamilton-Jacobi equation in spacetime, i.e.

$$\frac{\partial S}{\partial \lambda} + \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial q^\mu} \frac{\partial S}{\partial q^\nu} = 0. \quad (23)$$

The Hamilton-Jacobi Equation in Kerr Spacetime

We have just seen that, in a general spacetime, the HJE can be written as

$$\frac{\partial S}{\partial \lambda} + \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial q^\mu} \frac{\partial S}{\partial q^\nu} = 0 \quad \implies \quad -2 \frac{\partial S}{\partial \lambda} = g^{\mu\nu} \frac{\partial S}{\partial q^\mu} \frac{\partial S}{\partial q^\nu}. \quad (24)$$

Expanding the RHS of the latter or the Kerr metric (in BL coordinates), yields

$$-2 \frac{\partial S}{\partial \lambda} = g^{tt} \left(\frac{\partial S}{\partial t} \right)^2 + 2g^{t\phi} \frac{\partial S}{\partial t} \frac{\partial S}{\partial \phi} + g^{\phi\phi} \left(\frac{\partial S}{\partial \phi} \right)^2 + g^{rr} \left(\frac{\partial S}{\partial r} \right)^2 + g^{\theta\theta} \left(\frac{\partial S}{\partial \theta} \right)^2. \quad (25)$$

Now, 3 of the derivatives in (25) can be written in terms the constants of motion:

- ▶ From the initial form (18) of HJE and the Hamiltonian (19), we have

$$\frac{\partial S}{\partial \lambda} = -\mathcal{H} = -\frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu = -\frac{1}{2} g_{\mu\nu} u^\mu u^\nu = -\frac{1}{2} C_1. \quad (26)$$

- ▶ From (17) and (12), we have

$$\frac{\partial S}{\partial t} = p_t = C_2 \quad (27)$$

- ▶ From (17) and (16), we have

$$\frac{\partial S}{\partial \phi} = p_\phi = C_3. \quad (28)$$

Then, substituting (26), (27) and (28) into (25), we have

$$C_1 = C_2^2 g^{tt} + 2C_2 C_3 g^{t\phi} + C_3^2 g^{\phi\phi} + g^{rr} \left(\frac{\partial S}{\partial r} \right)^2 + g^{\theta\theta} \left(\frac{\partial S}{\partial \theta} \right)^2 \quad (29)$$

Separation of Variables and the 4th Constant of Motion

On the previous slide we obtained the form (29)

$$C_1 = C_2^2 g^{tt} + 2C_2 C_3 g^{t\phi} + C_3^2 g^{\phi\phi} + g^{rr} \left(\frac{\partial S}{\partial r} \right)^2 + g^{\theta\theta} \left(\frac{\partial S}{\partial \theta} \right)^2$$

of the HJE for Kerr spacetime (in BL coordinates), where:

- ▶ The $g^{\mu\nu}$ are the components of the inverse metric tensor.
- ▶ The C_i , $i = 1, 2, 3$ are the constants of motion (so far).
- ▶ The derivatives $\frac{\partial S}{\partial r}$ and $\frac{\partial S}{\partial \theta}$ are unknown.

Recall that, due to symmetries, the $g_{\mu\nu}$ and thus $g^{\mu\nu}$ are functions of r and θ , only. It turns out that we can:

1. Separate the r -dependencies from the θ -dependencies.
2. Reveal the 4th constant of motion.
3. Obtain expressions for the unknown derivatives.

Of course, to begin this process, we will require explicit expressions for the $g^{\mu\nu}$.

Components of the Metric Tensor: Review

In previous talks, the expressions for the Kerr metric coefficients (or metric tensor components) were given in terms of BL coordinates and SI units. These are repeated here in matrix form:

$$\begin{bmatrix} g_{tt} & g_{t\phi} & 0 & 0 \\ g_{\phi t} & g_{\phi\phi} & 0 & 0 \\ 0 & 0 & g_{rr} & 0 \\ 0 & 0 & 0 & g_{\theta\theta} \end{bmatrix} = \begin{bmatrix} c^2 \left(1 - \frac{2r_g r}{\rho^2}\right) & 2c \frac{r_g a r \sin^2 \theta}{\rho^2} & 0 & 0 \\ 2c \frac{r_g a r \sin^2 \theta}{\rho^2} & -\frac{\Sigma^2 \sin^2 \theta}{\rho^2} & 0 & 0 \\ 0 & 0 & -\frac{\rho^2}{\Delta} & 0 \\ 0 & 0 & 0 & -\rho^2 \end{bmatrix} \quad (30)$$

where:

- ▶ c is the speed of light.
- ▶ r_g is the gravitational radius, i.e. GM/c^2 .
- ▶ a is the rotation parameter, i.e. $a = J/(Mc)$.
- ▶ $\rho^2(r, \theta) = r^2 + a^2 \cos^2 \theta$.
- ▶ $\Delta = r^2 - 2r_g r + a^2$.
- ▶ $\Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta$, allowing a more compact form for $g_{\phi\phi}$.

Note: previously, we had $g_{\phi\phi} = -\left(r^2 + a^2 + 2r_g a^2 \frac{r \sin^2 \theta}{\rho^2(r, \theta)}\right) \sin^2 \theta$;
it can be shown (via a page of algebra) that the two are equivalent.

Components of the Inverse Metric Tensor

We obtain the inverse metric components by taking the matrix inverse of (30).
The block-diagonal makes this simple, i.e.

$$\begin{bmatrix} g_{tt} & g_{t\phi} & 0 & 0 \\ g_{\phi t} & g_{\phi\phi} & 0 & 0 \\ 0 & 0 & g_{rr} & 0 \\ 0 & 0 & 0 & g_{\theta\theta} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{g_{\phi\phi}}{g_{tt}g_{\phi\phi} - g_{t\phi}g_{\phi t}} & -\frac{g_{t\phi}}{g_{tt}g_{\phi\phi} - g_{t\phi}g_{\phi t}} & 0 & 0 \\ -\frac{g_{\phi t}}{g_{tt}g_{\phi\phi} - g_{t\phi}g_{\phi t}} & \frac{g_{tt}}{g_{tt}g_{\phi\phi} - g_{t\phi}g_{\phi t}} & 0 & 0 \\ 0 & 0 & \frac{1}{g_{rr}} & 0 \\ 0 & 0 & 0 & \frac{1}{g_{\theta\theta}} \end{bmatrix}$$

Then, substituting the metric components on the RHS of (30), we have

$$\begin{bmatrix} g^{tt} & g^{t\phi} & 0 & 0 \\ g^{\phi t} & g^{\phi\phi} & 0 & 0 \\ 0 & 0 & g^{rr} & 0 \\ 0 & 0 & 0 & g^{\theta\theta} \end{bmatrix} = \begin{bmatrix} \frac{\Sigma^2}{c^2 \rho^2 \Delta} & \frac{2r_g ar}{c \rho^2 \Delta} & 0 & 0 \\ \frac{2r_g ar}{c \rho^2 \Delta} & \frac{a^2 \sin^2 \theta - \Delta}{\rho^2 \Delta \sin^2 \theta} & 0 & 0 \\ 0 & 0 & -\frac{\Delta}{\rho^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{\rho^2} \end{bmatrix} \quad (31)$$

Separation of Variables (continued)

From the form (29) of the Hamilton-Jacobi equation, i.e.

$$C_1 = C_2^2 g^{tt} + 2C_2 C_3 g^{t\phi} + C_3^2 g^{\phi\phi} + g^{rr} \left(\frac{\partial S}{\partial r} \right)^2 + g^{\theta\theta} \left(\frac{\partial S}{\partial \theta} \right)^2$$

separation of variables r and θ can be achieved via the following steps:

1. Substitute the inverse metric components from (31), yielding

$$C_1 = C_2^2 \frac{\Sigma^2}{c^2 \rho^2 \Delta} + 2C_2 C_3 \frac{2r_g a r}{c \rho^2 \Delta} + C_3^2 \frac{a^2 \sin^2 \theta - \Delta}{\rho^2 \Delta \sin^2 \theta} - \frac{\Delta}{\rho^2} \left(\frac{\partial S}{\partial r} \right)^2 - \frac{1}{\rho^2} \left(\frac{\partial S}{\partial \theta} \right)^2. \quad (32)$$

2. Multiply (32) by ρ^2 , yielding

$$C_1 \rho^2 = C_2^2 \frac{\Sigma^2}{c^2 \Delta} + 2C_2 C_3 \frac{2r_g a r}{c \Delta} + C_3^2 \frac{a^2 \sin^2 \theta - \Delta}{\Delta \sin^2 \theta} - \Delta \left(\frac{\partial S}{\partial r} \right)^2 - \left(\frac{\partial S}{\partial \theta} \right)^2. \quad (33)$$

3. Substituting for ρ^2 and Σ^2 (but not Δ), and expand mixed terms

$$C_1 \rho^2 = C_1 (r^2 + a^2 \cos^2 \theta) = C_1 r^2 + C_1 a^2 \cos^2 \theta \quad (34)$$

$$C_2^2 \frac{\Sigma^2}{c^2 \Delta} = C_2^2 \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{c^2 \Delta} = C_2^2 \frac{(r^2 + a^2)^2}{c^2 \Delta} - C_2^2 \frac{a^2 \sin^2 \theta}{c^2} \quad (35)$$

$$C_3^2 \frac{a^2 \sin^2 \theta - \Delta}{\Delta \sin^2 \theta} = C_3^2 \frac{a^2}{\Delta} - C_3^2 \frac{1}{\sin^2 \theta} = C_3^2 \frac{a^2}{\Delta} - C_3^2 \csc^2 \theta \quad (36)$$

Separation of Variables (continued)

Continuing with the separation of r and θ , the next steps are:

4. Substitute (34), (35) and (36) into (33), move r - and θ -dependent terms to opposite sides and simplify, yielding

$$\begin{aligned} & a^2 C_1^2 \cos^2 \theta + \left(\frac{a C_2 \sin \theta}{c} - C_3 \csc \theta \right)^2 + \left(\frac{\partial S}{\partial \theta} \right)^2 \\ &= \frac{1}{\Delta} \left[\left(\frac{(r^2 + a^2) C_2}{c} + a C_3 \right)^2 - \Delta r^2 C_1^2 - \Delta^2 \left(\frac{\partial S}{\partial r} \right)^2 \right]. \end{aligned} \quad (37)$$

5. Now, we want (37) to be an identity (valid for all r and θ), not just an equation, so each side of (37) must be equal to the same constant, i.e.

$$a^2 C_1^2 \cos^2 \theta + \left(\frac{a C_2 \sin \theta}{c} - C_3 \csc \theta \right)^2 + \left(\frac{\partial S}{\partial \theta} \right)^2 = C_4 \quad (38)$$

$$\frac{1}{\Delta} \left[\left(\frac{(r^2 + a^2) C_2}{c} + a C_3 \right)^2 - \Delta r^2 C_1^2 - \Delta^2 \left(\frac{\partial S}{\partial r} \right)^2 \right] = C_4 \quad (39)$$

Expressions for the Previously Unknown Derivatives

From (38) and (39), expressions for the previously unknown derivatives $\frac{\partial S}{\partial r}$ and $\frac{\partial S}{\partial \theta}$ can be obtained (by a few simple algebraic manipulations):

$$\frac{\partial S}{\partial r} = \pm \frac{1}{\Delta} \sqrt{\left(\frac{(r^2 + a^2) C_2}{c} + a C_3 \right)^2 - \Delta (r^2 C_1^2 + C_4)}. \quad (40)$$

$$\frac{\partial S}{\partial \theta} = \pm \sqrt{C_4 - a^2 C_1^2 \cos^2 \theta - \left(\frac{a C_2 \sin \theta}{c} - C_3 \csc \theta \right)^2}. \quad (41)$$

It is customary to represent the expressions under the square roots in (40) and (41) by $\mathcal{R}(r)$ and $\Theta(\theta)$, respectively, and then write (40) and (41) more compactly:

$$\frac{\partial S}{\partial r} = \pm \frac{1}{\Delta} \sqrt{\mathcal{R}(r)}. \quad (42)$$

$$\frac{\partial S}{\partial \theta} = \pm \sqrt{\Theta(\theta)}. \quad (43)$$

Separated, 1st-Order Equations of Motion

Recall that $\dot{q}^\mu = g^{\mu a} p_a$ (20), so:

- ▶ $\dot{t} = g^{ta} p_a = g^{tt} p_t + g^{t\phi} p_\phi + g^{tr} p_r + g^{t\theta} p_\theta$
 - ▶ In Kerr spacetime, $\dot{t} = g^{tt} p_t + g^{t\phi} p_\phi$
 - ▶ Substituting $p_t = C_2$ (27) and $p_\phi = C_3$ (28), $\dot{t} = g^{tt} C_2 + g^{t\phi} C_3$
 - ▶ Substituting the inverse metric components

$$\dot{t} = C_2 \frac{\Sigma^2}{c^2 \rho^2 \Delta} + C_3 \frac{2r_g a r}{c \rho^2 \Delta} = \frac{1}{\rho^2 \Delta} \left[C_2 \frac{\Sigma^2}{c^2} + C_3 \frac{2r_g a r}{c} \right] \quad (44)$$

- ▶ $\dot{\phi} = g^{\phi a} p_a = g^{\phi t} p_t + g^{\phi\phi} p_\phi + g^{\phi r} p_r + g^{\phi\theta} p_\theta$
 - ▶ In Kerr spacetime, $\dot{\phi} = g^{\phi t} p_t + g^{\phi\phi} p_\phi$
 - ▶ Substituting $p_t = C_2$ (27) and $p_\phi = C_3$ (28), $\dot{\phi} = g^{\phi t} C_2 + g^{\phi\phi} C_3$
 - ▶ Substituting the inverse metric components

$$\dot{\phi} = C_2 \frac{2r_g a r}{c \rho^2 \Delta} + C_3 \frac{a^2 \sin^2 \theta - \Delta}{\rho^2 \Delta \sin^2 \theta} = \frac{1}{\rho^2 \Delta} \left[C_2 \frac{2r_g a r}{c} + C_3 \frac{a^2 \sin^2 \theta - \Delta}{\sin^2 \theta} \right] \quad (45)$$

Separated, 1st-Order Equations of Motion (continued)

Recall, once again that $\dot{q}^\mu = g^{\mu a} p_a$ (20), so:

▶ $\dot{r} = g^{ra} p_a = g^{rt} p_t + g^{r\phi} p_\phi + g^{rr} p_r + g^{r\theta} p_\theta$

▶ In Kerr spacetime, $\dot{r} = g^{rr} p_r$

▶ Substituting $p_r = \frac{\partial S}{\partial r}$ (17), $\dot{r} = g^{rr} \frac{\partial S}{\partial r}$

▶ Substituting $\frac{\partial S}{\partial r} = \pm \frac{1}{\Delta} \sqrt{\mathcal{R}(r)}$ (42), $\dot{r} = \pm g^{rr} \frac{1}{\Delta} \sqrt{\mathcal{R}(r)}$

▶ Substituting the inverse metric component and expanding $\mathcal{R}(r)$

$$\dot{r} = \pm \frac{\Delta}{\rho^2} \frac{1}{\Delta} \sqrt{\mathcal{R}(r)} = \pm \frac{1}{\rho^2} \sqrt{\mathcal{R}(r)} = \pm \frac{1}{\rho^2} \sqrt{\left(\frac{(r^2 + a^2) C_2}{c} + a C_3 \right)^2 - \Delta (r^2 C_1^2 + C_4)}$$

(46)

▶ $\dot{\theta} = g^{\theta a} p_a = g^{\theta t} p_t + g^{\theta\phi} p_\phi + g^{\theta r} p_r + g^{\theta\theta} p_\theta$

▶ In Kerr spacetime, $\dot{\theta} = g^{\theta\theta} p_\theta$

▶ Substituting $p_\theta = \frac{\partial S}{\partial \theta}$ (24), $\dot{\theta} = g^{\theta\theta} \frac{\partial S}{\partial \theta}$

▶ Substituting $\frac{\partial S}{\partial \theta} = \pm \sqrt{\Theta(\theta)}$ (43), $\dot{\theta} = \pm g^{\theta\theta} \sqrt{\Theta(\theta)}$

▶ Substituting the inverse metric component and expanding $\Theta(\theta)$

$$\dot{\theta} = \pm \frac{1}{\rho^2} \sqrt{\Theta(\theta)} = \pm \frac{1}{\rho^2} \sqrt{C_4 - a^2 C_1^2 \cos^2 \theta - \left(\frac{a C_2 \sin \theta}{c} - C_3 \csc \theta \right)^2}$$

(47)

Physical Interpretation of the Constants of Motion

Physical Interpretation of C_1

For affinely-parameterized geodesics $q^\mu(\lambda)$, the length of the tangent vector is constant by definition. Thus, we defined:

- ▶ $C_1 = |\mathbf{u}|$, where $\mathbf{u} = \frac{dq^\mu}{d\lambda}$ to is tangent to $q^\mu(\lambda)$,
- ▶ where $|\mathbf{u}|^2 = (\mathbf{u}, \mathbf{u}) = g_{\mu\nu} u^\mu u^\nu = g_{\mu\nu} \frac{dq^\mu}{d\lambda} \frac{dq^\nu}{d\lambda}$,
- ▶ so $|\mathbf{u}| = \sqrt{g_{\mu\nu} \frac{dq^\mu}{d\lambda} \frac{dq^\nu}{d\lambda}} = \sqrt{\frac{g_{\mu\nu} dq^\mu dq^\nu}{d\lambda^2}} = \sqrt{\frac{ds^2}{d\lambda^2}} = \frac{ds}{d\lambda}$.
- ▶ Then, if $q^\mu(\lambda)$ is a time-like geodesics, let $\lambda = \tau$, where τ is the proper time.
 - ▶ Since τ is an affine parameter, let $s = c\tau + b \implies \frac{ds}{d\tau} = c$.
 - ▶ So $C_1 = |\mathbf{u}| = \frac{ds}{d\lambda} = \frac{ds}{d\tau} = c$.
 - ▶ If c is the speed of light, \mathbf{u} is the 4-velocity of the a test particle on $q^\mu(\lambda)$.
- ▶ And for null geodesics, $\lambda \neq \tau$ and $C_1 = |\mathbf{u}| = \frac{ds}{d\lambda} = \frac{0}{d\lambda} = 0$.

In summary, $C_1 = |\mathbf{u}| = \begin{cases} c, & \text{timelike geodesics} \\ 0, & \text{null geodesics} \end{cases}$

Physical Interpretation of the Constants of Motion

Physical interpretation of C_2 and C_3

For affinely-parameterized geodesics $q^\mu(\lambda)$ in Kerr spacetime, with BL coordinates t and ϕ , we defined via (11) and (12):

- ▶ $C_2 := p_t$ as the conserved quantity associated with time-translation symmetry, \implies the conserved quantity is the *energy* E of a test particle traveling on $q^\mu(\lambda)$.
- ▶ $C_3 := p_\phi$ as the conserved quantity associated with axial symmetry, \implies the conserved quantity is the *z-component of angular momentum* L_z of a test particle traveling on $q^\mu(\lambda)$.

General sign conventions, as seen by asymptotic observers:

- ▶ $E > 0$ for a particle in geodesic motion, with $r_0 >$ the outer stationary limit.
- ▶ $L_z > 0$ for a particle in *prograde*, geodesic motion.

Effect of Lorentz signature convention:

- ▶ It can be shown that, for signature $(+, -, -, -)$,
 - ▶ $p_t > 0$, consistent with the general sign convention. Thus, $C_2 := E$.
 - ▶ $p_\phi < 0$, opposite to the general sign convention. Thus, $C_3 := -L_z$.
- ▶ For signature $(-, +, +, +)$, the signs are reversed.

Physical Interpretation of the Constants of Motion

Physical Interpretation of C_4

We have seen that the first three constants are:

- ▶ properties of the geodesics.
- ▶ But also properties of test particles traveling on those geodesics, actually measurable by asymptotic observers.

However, C_4 the constant of separation (of the Hamilton Jacobi equation):

- ▶ Cannot be interpreted as a physical property of the test particle.
- ▶ In fact, in its initial form, no physical interpretation has been found.
- ▶ But, in 1968 Carter also defined the constant

$$Q = C_4 - (L_z - aE)^2,$$

which provides a physical interpretation for entire geodesics, i.e.

- ▶ $Q = 0 \implies$ geodesics **may be** confined to the equatorial plane, or just touch it.
- ▶ $Q > 0 \implies$ oblique geodesics that cross the equatorial plane.
- ▶ $Q < 0 \implies$ oblique geodesics confined to one hemisphere.