Rotating Black Holes

Classification of Geodesics (Orbits)

February 3, 2026

Classification of Geodesics: Outline

- A couple of corrections
- The 1st-order equations of motion: review
- Orbit classification scheme
- Radial Classifications
- Polar configurations
- Combining radial and polar classifications
- An example

The 1st-Order Equations of Motion: Review

In the preceding talk (talk 7), we derived the separated, 1st order equations of motion:

- Derivative of t wrt λ

$$\dot{t} = \frac{1}{\rho^2 \Delta}\left[E\frac{\Sigma^2}{c^2} + L_z\frac{2r_g ar}{c}\right]$$

- Derivative of ϕ wrt λ

$$\dot{\phi} = \frac{1}{\rho^2 \Delta}\left[E\frac{2r_g ar}{c} + L_z\frac{a^2\sin^2\theta - \Delta}{\sin^2\theta}\right]$$

- Derivative of r wrt λ

$$\dot{r} = \pm\frac{1}{\rho^2}\sqrt{\mathcal{R}(r)} = \pm\frac{1}{\rho^2}\sqrt{\left(\frac{(r^2+a^2)\,E}{c} + aL_z\right)^2 - \Delta\left(r^2|\boldsymbol{u}|^2 + C_4\right))}$$

- Derivative of θ wrt λ

$$\dot{\theta} = \pm\frac{1}{\rho^2}\sqrt{\Theta(\theta)} = \pm\frac{1}{\rho^2}\sqrt{C_4 - a^2|\boldsymbol{u}|^2\cos^2\theta - \left(\frac{aE\sin\theta}{c} - L_z\csc\theta\right)^2}$$

where:

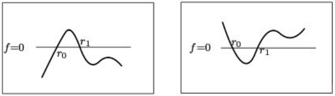
- c is the speed of light.
- r_g is the gravitational radius, i.e. GM/c^2.
- a is the rotation parameter, i.e. $a = J/(Mc)$.
- $\rho^2(r,\theta) = r^2 + a^2\cos^2\theta$.
- $\Delta = r^2 - 2r_g r + a^2$.
- $\Sigma^2 = \left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta$
- $|\boldsymbol{u}|$, E, L_z and $C_4 = Q + (L_z - aE)^2$ are the constants of motion.

Orbit Classification Scheme

Logical sequence:

1. Note that $\dot{r}$ and $\dot{\theta}$ depend on $\sqrt{\mathcal{R}(r)}$ and $\sqrt{\Theta(\theta)}$, respectively.
2. Thus, valid orbits can occur only where both $\mathcal{R} > 0$ and $\Theta > 0$.
3. A pair of curves $\mathcal{R}(r)$ and $\Theta(\theta)$ are uniquely determined by some combination of
   - 3.1 Black hole parameters r_g and a,
   - 3.2 Constants of motion $|\boldsymbol{u}|$, E, L_z and Q .
4. Define a *configuration* as the unique curve, $\mathcal{R}(r)$ or $\Theta(\theta)$, resulting from such a combination.
5. Define a *configuration class* as the set of all configurations with
   - 5.1 The same number of real roots and
   - 5.2 The same orientation, denoted by either N or P,

     for left-end Negative and left-end Positive, respectively.



6. Define a configuration space as an abstract space of n dimensions, where n is the number of parameters and each point is a unique configuration.
7. Without loss of generality, reduce n from 6 to 5, by fixing $r_g = 1$ and $a \in [0, 1]$.

Real Roots of the Radial Function $\mathcal{R}$

Rewrite the equation for $\mathcal{R}$, after having set $r_g = 1$ and , $c = 1$ (natural units)

$$\mathcal{R}(r) = \left(\left(r^2 + a^2\right) E + aL_z\right)^2 - \Delta \left(r^2 |\boldsymbol{u}|^2 + C_4\right) \tag{1}$$

where $\Delta = r^2 - 2r + a^2$. It can be shown (a page of algebra) that (1) is equivalent to

$$\mathcal{R}(r) = \alpha_4 r^4 + \alpha_3 r^3 + \alpha_2 r^2 + \alpha_1 r + \alpha_0,$$

where the coefficients α_i, $i = 0, 1, 2, 3, 4$ are given by:

- $\alpha_4 = E^2 - |\boldsymbol{u}|^2$
- $\alpha_3 = 2|\boldsymbol{u}|^2$
- $\alpha_2 = 2a^2 E^2 - 2aEL - a^2 |\boldsymbol{u}|^2 - C_4$
- $\alpha_1 = 2C_4$
- $\alpha_0 = a^4 E^2 - 2a^2 EL_z - a^2 L_z^2 - C_4$

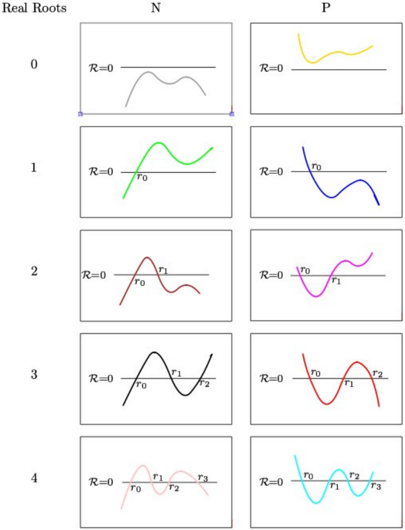
and where $C_4 = Q + (L_z - aE)^2$.
Note that:

- 4th-degree polynomial $\Longrightarrow$ # real roots = 0, 2 or 4.
- If $E = |\boldsymbol{u}|$, then $\alpha_4 = 0$ $\Longrightarrow$3rd-degree $\Longrightarrow$ # real roots = 1 or 3.
- If $E = |\boldsymbol{u}| = 0$, then $\alpha_4, \alpha_3 = 0$ $\Longrightarrow$ 2nd-degree $\Longrightarrow$ # real roots = 0 or 2.

Thus, the number of real roots may be 0, 1, 2, 3 or 4.

Configuration Classes for $\mathcal{R}$-Space

Representative sketches of all possible R-configuration classes are shown below:



A class can now be identified textually by its designation or visually by its color,

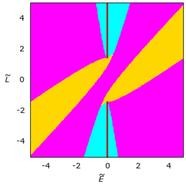
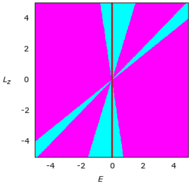
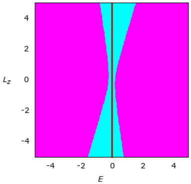
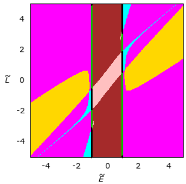
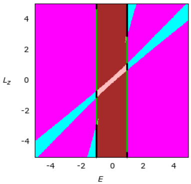
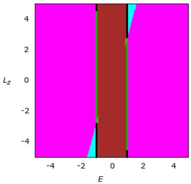
e.g. (0,P)

Sampling Algorithm for $\mathcal{R}$-Space

```
Select a ∈ [0,1], |u| ∈{0,1} and Q ∈ [Q_min, Q_max].

// Densely sample E, L_z-plane:
   For E = −5 to E = 5 in small increments
      For L_z = −5 to L_z = 5 in small increments
         Compute coefficients α_i, i = 0,1,2,3,4.

         // Compute roots of α_4 r^4 + α_3 r^3 + α_2 r^2 + α_1 r + α_0 = 0:
            If α_4 = α_3 = 0, quadratic formula.
            else if α_4 = 0, gsl_poly_complex_solve_cubic.
            else gsl_poly_complex_solve.

         Save set of real roots {r_i} and sort (ascending).

         // Determine orientation, where n is cardinality of {r_i}
            If n = 0,
               If R(5) > 0, class = (0,P)
               Else class = (0,N)
            Else
               If R(r_0 − 5) > 0, class = (n,P).
               Else class = (n,N).
```

Sampling Results for $\mathcal{R}$-Space

For $a = 0.8$:

$Q = -1$	$Q = 0$	$Q = 1$	$\|u\|$
			0
			1

Note that some configurations do not appear:
(0,N), (1,P) and (3,P)

Real Roots of the Polar Function Θ, $L_z \neq 0$

Rewrite the equation for Θ, after having set $r_g = 1$ and $c = 1$ (natural units)

$$\Theta(\theta) = C_4 - a^2 |\boldsymbol{u}|^2 \cos^2\theta - (aE\sin\theta - L_z/\sin\theta)^2 \,. \tag{2}$$

It can be shown that Θ can be written as

$$\Theta(\theta) = \frac{\alpha_4 \cos^4\theta + \alpha_2 \cos^2\theta + \alpha_0}{\sin^2\theta} \tag{3}$$

where the coefficients α_i, $i = 0, 2, 4$ are given by:

- $\alpha_4 = -a^2 \left(E^2 - |\boldsymbol{u}|^2\right)$
- $\alpha_2 = -\alpha_4 - b$
- $\alpha_0 = b - L_z^2$

and where $b = C_4 - a^2 E^2 + 2aEL_z$, with $C_4 = Q + (L_z - aE)^2$.
Note that:

- Biquadratic polynomial in $\cos\theta \implies$ # real roots = 0, 2 or 4.
- If $E = |\boldsymbol{u}| \implies \alpha_4 = 0 \implies$ 2nd-degree $\implies$ # real roots = 0 or 2.

Thus, the number of real roots remains 0, 2 or 4.

Note: the preceding analysis is valid for θ on the open interval $(0, \pi)$.

Real Roots of the Polar Function Θ, $L_z = 0$

We know that the original polar equation

$$\Theta(\theta) = C_4 - a^2|\boldsymbol{u}|^2 \cos^2\theta - (aE\sin\theta - L_z/\sin\theta)^2$$

is singular at $\theta = 0$ and $\theta = \pi$, i.e. at the poles.
However, the singularities are removed if $L_z = 0$.
It can be shown that, if $L_z = 0$, the preceding equation reduces to

$$\Theta(\theta) = \beta_2 \cos^2\theta + \beta_0, \tag{4}$$

where:

- $\beta_2 = a^2\left(E^2 - |\boldsymbol{u}|^2\right) = -\alpha_4$
- $\beta_0 = C_4 - a^2E^2$

and where $C_4 = Q + (L_z - aE)^2 = Q + a^2E^2 \implies \beta_0 = Q$.
Note that:

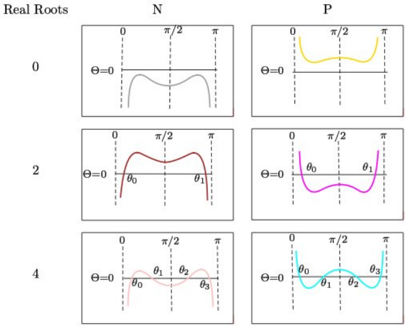
- 2nd-degree polynomial in $\cos\theta \implies$ # real roots = 0 or 2.
- If $E = |\boldsymbol{u}| \implies \beta_2 = 0 \implies \Theta(\theta) = \beta_0 = Q$, so:
    - If $Q \neq 0$, # real roots = 0.
    - If $Q = 0$, $\Theta(\theta) = 0 \implies \dot{\theta} = 0 \implies \theta = const$.

Note: the preceding analysis is valid for θ on the closed interval $[0, \pi]$.

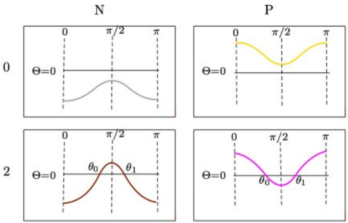
Configuration Classes for Θ-Space

Representative sketches of all possible Θ-configuration classes are shown below:

$L_z \neq 0$



$L_z = 0$

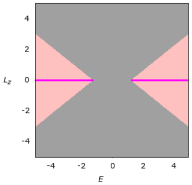
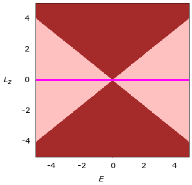
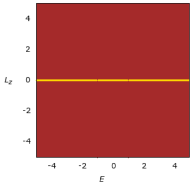
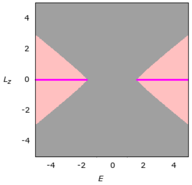
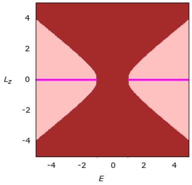
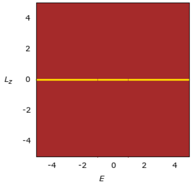


Sampling algorithm for Θ-Space

```
Select a ∈ [0,1], |u| ∈{0,1} and Q ∈ [Q_min, Q_max].

// Densely sample E, L_z-plane:
   For E = −5 to E = 5 in small increments
      For L_z = −5 to L_z = 5 in small increments

         If L_z = 0:
            Compute coefficients β_i, i = 0,2.
            If β_2 = 0: β_0 >,< 0 ⟹ class = (0,P),(0,N).
            Else compute roots of β_2 cos^2 θ + β_0 = 0.
         Else
            Compute coefficients α_i i = 0,2,4.
            If α_4 = 0, compute roots of α_2 cos^2 θ + α_0 = 0.
            Else compute roots of α_4 cos^4 θ + α_2 cos^2 θ + α_0 = 0.

         If L_z ≠ 0 & β_2 ≠ 0:
            Save set of real roots {θ_i} and sort (ascending).

            // Determine orientation, where n is cardinality of {θ_i}
               If n = 0,
                  If Θ(π/2) > 0, class = (0,P)
                  else class = (0,N)
               Else
                  If Θ(θ_0/2) > 0, class = (n,P).
                  Else class = (n, N).
```

Sampling Results for Θ-Space

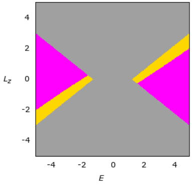
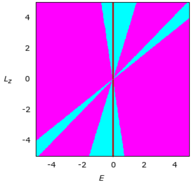
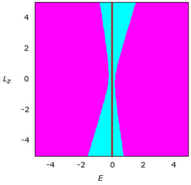
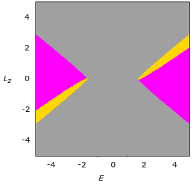
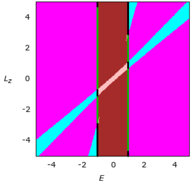
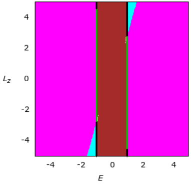
For $a = 0.8$:

$Q = -1$	$Q = 0$	$Q = 1$	$\lvert \boldsymbol{u} \rvert$
			0
			1

Note that one configuration does not appear: (4,P)
and for $Q = -1$, there is a large swath of (0,N)
$\Longrightarrow$ no real values for $\dot{\theta} \Longrightarrow$ no orbits.

Valid Points in $\mathcal{R}$-Space

For $a = 0.8$:

$Q = -1$	$Q = 0$	$Q = 1$	$\lvert \boldsymbol{u} \rvert$
			0
			1

Example: $a = 0.8,\ Q = -1,\ E = 4,\ L_z = 2$

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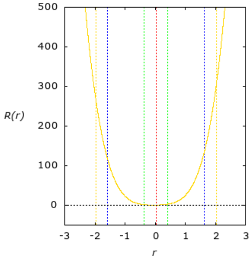
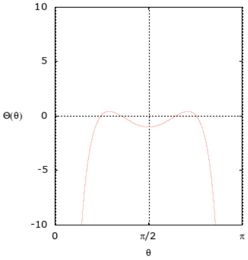
<tr><td>
</td><td>
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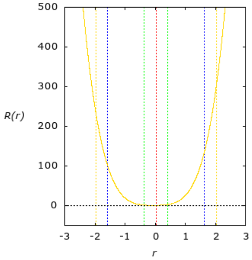
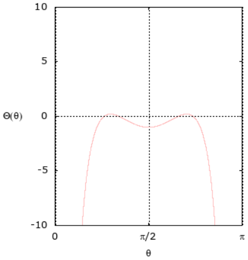
<tr><td>
</td><td>
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Interpretation:

- Plunging orbit (trajectory)
- Confined to one hemisphere
- $|\boldsymbol{u}| = 0, 1$: qualitatively identical.

Vertical, dotted lines $\mathcal{R}(r)$, $r = const$:

- Red - zero
- Green: $\pm$ Cauchy horizon
- Blue: $\pm$ at event horizon
- Yellow: $\pm$ outer stationary limit at $\theta = \pi/2$.