Rotating Black Holes

Orbit Generation

February 3, 2026

Orbit Generation: Outline

- Review: equations of motion in BL coordinates
- Equations of motion in co-moving coordinates
- Orbit generation algorithm
- Initial conditions
- Selected orbits

Review: Equations of Motion in BL Coordinates

The equations of motion in Boyer-Lindquist coordinates (t, r, θ, ϕ) are rewritten below, but with $r_g = c = 1$:

$$
\begin{aligned}
\dot{t} &= \frac{1}{\rho^2 \Delta}\left[E\Sigma^2 + 2L_z a r\right] \\
\dot{\phi} &= \frac{1}{\rho^2 \Delta}\left[2Ear + L_z \frac{a^2 \sin^2\theta - \Delta}{\sin^2\theta}\right] \\
\dot{r} &= \pm\frac{1}{\rho^2}\sqrt{\mathcal{R}(r)} = \pm\frac{1}{\rho^2}\sqrt{\left((r^2 + a^2)\,E + aL_z\right)^2 - \Delta\left(r^2|\boldsymbol{u}|^2 + C_4\right)} \\
\dot{\theta} &= \pm\frac{1}{\rho^2}\sqrt{\Theta(\theta)} = \pm\frac{1}{\rho^2}\sqrt{C_4 - a^2|\boldsymbol{u}|^2\cos^2\theta - \left(aE\sin\theta - L_z\csc\theta\right)^2}
\end{aligned}
$$

where:

- a is the rotation parameter
- $\rho^2(r, \theta) = r^2 + a^2\cos^2\theta$
- $\Delta = r^2 - 2r + a^2$
- $\Sigma^2 = \left(r^2 + a^2\right)^2 - a^2\Delta\sin^2\theta$
- $|\boldsymbol{u}|$, E, L_z and $C_4 = Q + (L_z - aE)^2$ are the constants of motion.

Equations of Motion in Co-Moving Coordinates

Why we need co-moving coordinates:

- The BL equations of motion generate orbits as seen by observers at rest as $r \to \infty$.
- Although these orbits are what is measured by the BL-observers, they represent a kind of illusion. That is, particles approach the event horizon, but never cross it.
- On the other hand, co-moving observers, see the particle cross the event horizon and continue on.
- Formally, the co-moving coordinates are Eddington-Finkelstein (EF) coordinates.
- In EF coordinates, the equations of motion for $\dot{r}$ and $\dot{\theta}$ are the same as for BL coordinates.
- The EF equations of motion for $\dot{t}$ and $\dot{\phi}$ are given in terms of the BL equations of motion by:

$$
\begin{aligned}
\dot{t}' &= \dot{t} + 2\frac{r}{\Delta}\dot{r} \\
\dot{\phi}' &= \dot{\phi} + \frac{a}{\Delta}\dot{r}
\end{aligned}
$$

Orbit Generation Algorithm: Outline

- Notation
- Iteration
- Initialization
- Turning points
- Termination

Orbit Generation Algorithm: Notation

Quantities of Interest:

- λ - affine parameter, initial value λ_{in}.
- $\delta\lambda$ - adaptive step size, initial value $(\delta\lambda)_{in}$.
- ϵ - a small number (fixed).
- t, r, θ, ϕ - Boyer/Lindquist (BL) coordinates, initial values $t_{in}, r_{in}, \theta_{in}, \phi_{in}$.
- σ^r, σ^θ - sign of $\dot{r}$ and $\dot{\theta}$, respectively, initial values $\sigma^r_{in}, \sigma^\theta_{in}$.
- r_i, θ_i - ith root of $\mathcal{R}(r) = 0$ and $\Theta(\theta) = 0$, respectively.
- In order to describe the algorithm compactly
  - q^μ, $\mu = 0, 1, 2, 3$ - proxies for BL coordinates, i.e. $(q^0, q^1, q^2, q^3) = (t, r, \theta, \phi)$.
  - $f^\mu\,(r, \theta)$, $\mu = 0, 1, 2, 3$ - proxies for $(f^0, f^1, f^2, f^3) = (\dot{t} \text{ or } \dot{t}', \sigma^r \dot{r}, \sigma^\theta \dot{\theta}, \dot{\phi} \text{ or } \dot{\phi}')$.
    Note: here we define $\dot{r} = \frac{1}{\rho^2}\sqrt{\mathcal{R}(r)}$ and $\dot{\theta} = \frac{1}{\rho^2}\sqrt{\Theta(\theta)}$, i.e. positive $\sqrt{}$ only.

Orbit Generation Algorithm: Iteration

Given the current values of q^μ, λ and $\delta\lambda$:
$q^\mu_{gvn} = q^\mu$, $\lambda_{gvn} = \lambda$.

stepDecreased = false.
converged = false.
Loop until converged = true:

// One full step
$\dot{q}^\mu = f^\mu\left(q^1, q^2\right), \mu = 0,1,2,3.$
$q^\mu_{full} = q^\mu + \dot{q}^\mu \delta\lambda, \mu = 0,1,2,3.$
$\lambda_{full} = \lambda_{in} + \delta\lambda.$

// Two half steps.
$q^\mu_{hlf1} = q^\mu + \dot{q}^\mu \delta\lambda/2, \mu = 0,1,2,3.$
$\lambda_{hlf1} = \lambda + \delta\lambda/2.$
$\dot{q}^\mu_{hlf1} = f^\mu\left(q^1_{hlf1}, q^2_{hlf1}\right), \mu = 0,1,2,3.$
$q^\mu_{hlf2} = q^\mu_{hlf1} + \dot{q}^\mu_{hlf1}\delta\lambda/2, \mu = 0,1,2,3.$

// Test for convergence.
If $\left|q^\mu_{full} - q^\mu_{hlf2}\right| < \epsilon,\ \mu = 0,1,2,3$:
$\left(q^\mu_{full}, \lambda_{full}\right) \longmapsto \left(q^\mu, \lambda\right).$
converged = true.
Else:
$\left(q^\mu_{hlf1}, \lambda_{haf1}\right) \longmapsto \left(q^\mu, \lambda\right),\ \delta\lambda \mapsto \delta\lambda/2.$
stepDecreased = true.

If stepDecreased = false:
// Status after 1 pass, at left:
// converged = true.
// $\dot{q}^\mu$corresponds to q^μ_{gvn}.
// q^μ corresponds to q^μ_{full}.
// λ corresponds to λ_{full}.
// $\delta\lambda$ is the original step.

Loop until converged = false:

// Do 2nd full step.
$\dot{q}^\mu_{full1} = f^\mu\left(q^1, q^2\right), \mu = 0,1,2,3.$
$q^\mu_{full2} = q^\mu + \dot{q}^\mu_{full1}\delta\lambda,\ \mu = 0,1,2,3.$
$q^\mu_{dbl} = q^\mu_{gvn} + 2\dot{q}^\mu\delta\lambda,\ \mu = 0,1,2,3.$
$\lambda_{dbl} = \lambda + 2\delta\lambda.$

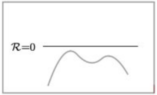
// Test for convergence:
If $\left|q^\mu_{full2} - q^\mu_{dbl}\right| < \epsilon,\ \mu = 0,1,2,3.$
$\left(q^\mu_{dbl}, \lambda_{dbl}\right) \longmapsto \left(q^\mu, \lambda\right),\ \delta\lambda \mapsto 2\delta\lambda.$
Else:
converged = false.

Orbit Generation Algorithm: Initialization

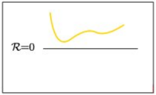
- Hard-wired initial values:
  - $\epsilon = 10^{-5}$
  - $q^0_{in} = t_{in} = 0$
  - $q^3_{in} = \phi_{in} = 0$
  - $\lambda_{in} = 0$
  - $\delta\lambda_{in} = 10^{-3}$
- User inputs: a, $|\boldsymbol{u}|$, E, $L_z$and Q
- Initial computations:
  - Roots of $\mathcal{R}(r) = 0$ and $\Theta(\theta) = 0$
  - Number of real roots n^r and n^θ
  - Orientation $\mathcal{O}^r$ and $\mathcal{O}^\theta$ of $\mathcal{R}$ and Θ, respectively.
  - Configuration classes for $\mathcal{R}$ and Θ, i.e. $(n^r,\mathcal{O}^r)$ and $(n^\theta,\mathcal{O}^\theta)$, respectively
  - From $(n^r,\mathcal{O}^r)$, determine r_{in} and σ^r_{in}
  - From $(n^\theta,\mathcal{O}^\theta)$, determine θ_{in} and σ^θ_{in}

Orbit Generation Algorithm: Initial r and σ^r

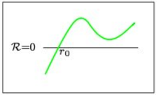
Focus on the most interesting part of orbits, no further out than $5r_g = 5$, i.e. 1 r_g above the outer stationary limit $r_{OSL} = r_g + \sqrt{r_g^2 - a^2\cos^2\theta}$ at its furthest point, at $\theta = \pi/2 \implies r_{in} = 3r_g = 3$. Move toward the black hole, i.e. $\sigma^r_{in} = -1$.



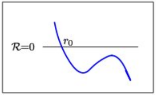
$\mathcal{R}$-config (0,N):
No orbits



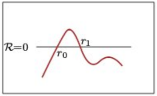
$\mathcal{R}$-config (0,P):
$r_{in} = 5$



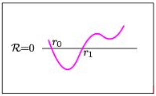
$\mathcal{R}$-config (1,N):
$r_{in} = \max(5r_0, 5)$



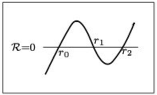
$\mathcal{R}$-config (1,P):
No orbits



$\mathcal{R}$-config (2,N):
If $r_1 > 0$, $r_{in} = r_1 - \epsilon$



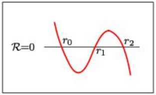
$\mathcal{R}$-config (2,P):
$r_{in} = \max(5r_1, 5)$



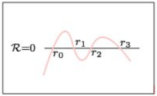
$\mathcal{R}$-config (3,N):

a. $r_{in} = \max(5r_2, 5)$

b. If $r_1 > 0$, $r_{in} = r_1 - \epsilon$



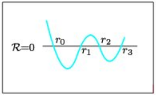
$\mathcal{R}$-config (3,P):
No orbits



$\mathcal{R}$-config (4,N):

a. $r_{in} = r_3 - \epsilon$

b. If $r_1 > 0$, $r_{in} = r_1 - \epsilon$



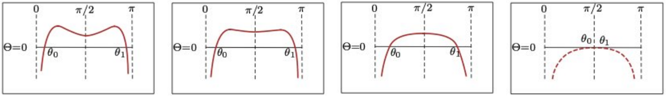
$\mathcal{R}$-config (4,P):

a. $r_{in} = \max(5r_3, 5)$

b. If $r_2 > 0$, $r_{in} = r_2 - \epsilon$

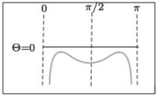
Orbit Generation Algorithm: Initial θ and σ^θ

There is typically a choice of hemispheres, unless θ is fixed a $\pi/2$.
This happens in polar class (2,N), as indicated below:

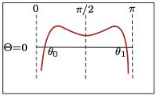


As the curve moves downward, it begins to flatten out in the center and the center becomes a maximum at $\theta = \pi/2$. Ultimately, the borderline case is a double root at $\theta = \pi/2$.

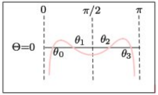
$L_z \neq 0$:



Θ-config (0,N):
No orbits

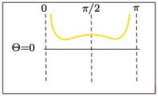


Θ-config (2,N):
$\theta_{in} = \theta_0 + \epsilon$, $\sigma^\theta = +1$

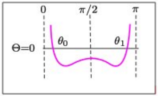


Θ-config (4,N):
nh. $\theta_{in} = \theta_0 + \epsilon$, $\sigma^\theta = +1$
sh. $\theta_{in} = \theta_3 - \epsilon$, $\sigma^\theta = -1$

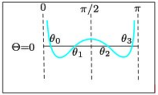
$L_z \neq 0$:



Θ-config (0,P):
$\theta_{in} = \epsilon$, $\sigma^\theta = +1$



Θ-config (2,P):
nh. $\theta_{in} = \epsilon$, $\sigma^\theta = +1$
sh. $\theta_{in} = \pi - \epsilon$, $\sigma^\theta = -1$



Θ-config (4,P):
No orbits

Orbit Generation Algorithm: Turning Points

Basic algorithm:

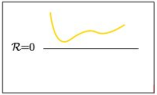
```
If ṙ is zero:
    If σ^r is negative,
        Set σ^r positive.
    Else
        Set σ^r negative.
If θ̇ is zero:
    If σ^θ is negative,
        Set σ^θ positive.
    Else
        Set σ^θ negative.
```

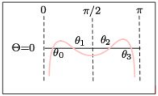
Selected Orbits for $a = 0.8$: Example 1

$(Q = -1,\ |\boldsymbol{u}| = 1,\ E = 4,\ L_z = 2) \Longrightarrow$

$\mathcal{R}$-class = (0,P) Θ-class = (4,N)

Roots: none $\pi/2 \pm 0.495, \pm 0.747$





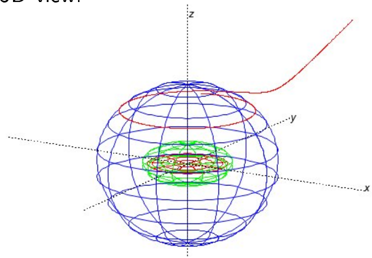
Choice of two plunging orbits:

- Northern hemisphere
- Southern hemisphere

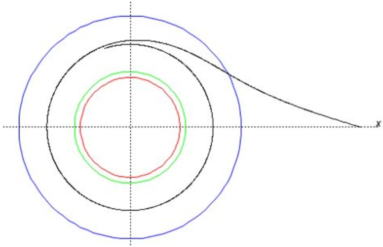
Example 1 (continued)

Northern hemisphere, plunging orbit as seen by remote observers:

3D view:



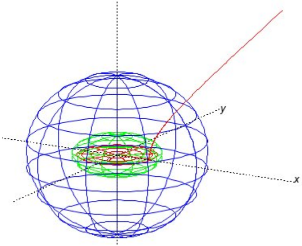
Projection onto equatorial plane:



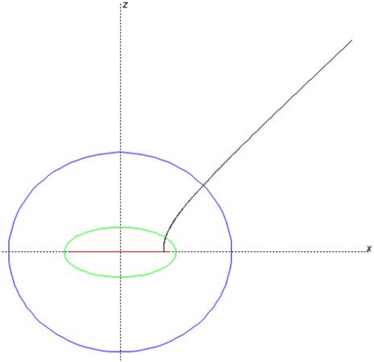
Example 1 (continued)

Northern hemisphere, plunging orbit as seen by co-moving observers:

3D view:



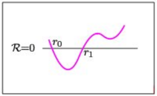
Projection onto xz-plane:



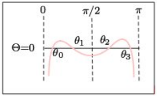
Selected Orbits for $a = 0.8$: Example 2

$(Q = 0, \; |\boldsymbol{u}| = 1, \; E = 10, \; L_z = 5) \Longrightarrow$

$\mathcal{R}$-class = (2,P) Roots: $-0.358, 0$



Θ-class = (4,N) $\pi/2 \pm 0, \pm 0.892$



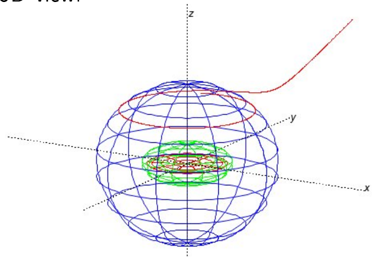
Choice of four (nominally) unbound orbits:

- $r \geq 0$
  - Northern hemisphere
  - Southern hemisphere
- $r < 0$ (the same choices)

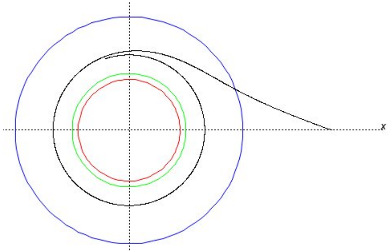
Example 2 (continued)

Northern hemisphere, (nominally) unbound orbit as seen by remote observers:

3D view:



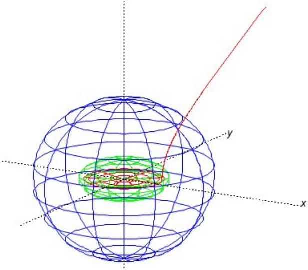
Projection onto equatorial plane:



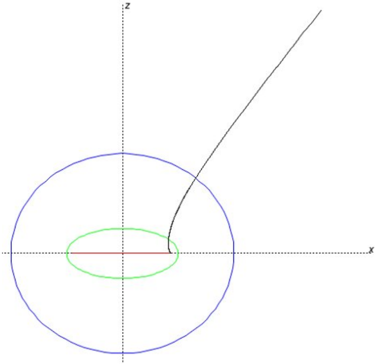
Example 2 (continued)

Northern hemisphere, (nominally) unbound orbit as seen by co-moving observers:

3D view:



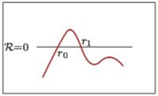
Projection onto xz-plane:



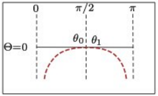
Selected Orbits for $a = 0.8$: Example 3

$$(Q = 0,\ |\boldsymbol{u}| = 1,\ E = 0.5,\ L_z = 2) \Longrightarrow$$

$\mathcal{R}$-class = (2,N)
Roots: 0,1.6

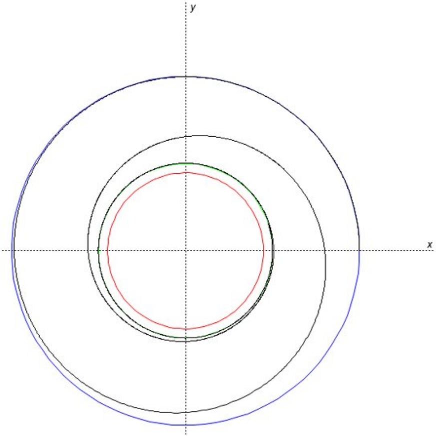


Θ-class = (2,N)
$\pi/2 \pm 0$



One orbit: (nominally) oscillating, equatorial

Equatorial orbit as seen by co-moving observers:
Equatorial plane:

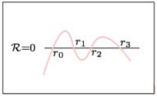


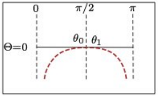
Selected Orbits for $a = 0.8$: Example 4

$$(Q = 0,\ |\boldsymbol{u}| = 1,\ E = 0.5,\ L_z = 0.5) \Longrightarrow$$

$\mathcal{R}$-class = (4,N) Θ-class = (2,N)

Roots: 0,1,0.03,0.4,2.24 $\pi/2 \pm 0$





Choice of two (nominally) oscillating, equatorial orbits:

- Inner $r_0 \leq r \leq r_1$
- Outer $r_2 \leq r \leq r_3$

Outer equatorial orbit as seen by co-moving observers: