Quantum Computing – 2 Qbits

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June 6, 2026

Overview

- 2-Qbit product states
- 2-Qbit general states
- 2 x 2 Quantum Gates (pictures, words), unitary matrices and quantum circuits
- A simple quantum circuit and algorithm for IBM's computer
  - Introduction to IBM's environment
  - Some Python scripts (for non-Python people)
  - Translating those scripts to pictures and gates
  - Predicting the quantum behavior of the quantum circuit
  - Execution of the algorithm on actual hardware

2-Qbit **product** states

- In vector notation $s_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}$
- Write $|s_1\rangle|s_2\rangle$ or $|s_1 s_2\rangle$.
- 4 basis states are $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$
- $|s_1 s_2\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$
  - Vector notation: $s_1 \otimes s_2 = \begin{pmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{pmatrix}$
- Let $\langle s_3 s_4 | s_1 s_2 \rangle \equiv \alpha_3^*\alpha_4^*\alpha_1\alpha_2 + \alpha_3^*\beta_4^*\alpha_1\beta_2 + \beta_3^*\alpha_4^*\beta_1\alpha_2 + \beta_3^*\beta_4^*\beta_1\beta_2$
  - Vector notation: $(s_3 \otimes s_4, s_1 \otimes s_2)$
- Prob. of measuring $|s_3 s_4\rangle$ if system is in state $|s_1 s_2\rangle$, is $|\langle s_3 s_4 | s_1 s_2\rangle|^2$

2-Qbit **general** states

- (Normalized) 2-Qbit general states superpose the 4 basis states.
- $|\psi\rangle = \alpha_1|00\rangle + \alpha_2|01\rangle + \alpha_3|10\rangle + \alpha_4|11\rangle$
  $|\phi\rangle = \beta_1|00\rangle + \beta_2|01\rangle + \beta_3|10\rangle + \beta_4|11\rangle$
  where $\sum_{i=1}^{4} |\alpha_i|^2 = \sum_{i=1}^{4} |\beta_i|^2 = 1$
  - Vector notation: $\psi = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$, etc.
- **In general**, $|\psi\rangle$, $|\phi\rangle$ are **not** product states
- General inner product: $\langle\phi|\psi\rangle \equiv \sum_{i=1}^{4} \beta_i^* \alpha_i$
  - Vector notation: (ϕ^*, ψ)

The EPR state – an example of a 2-Qbit general state

- The EPR state is $|EPR\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$.
  - If the first Qbit is $|0\rangle$, so is the second Qbit, etc.
  - $P(|00\rangle : |EPR\rangle) = P(|11\rangle : |EPR\rangle) = \frac{1}{2}$
    $P(|01\rangle : |EPR\rangle) = P(|10\rangle : |EPR\rangle) = 0$
  - Therefore we can't factorize $|EPR\rangle$ as $|s_1 s_2\rangle$
    - Otherwise $0 = P(|01\rangle : |EPR\rangle) = P(|0\rangle : s_1)P(|1\rangle : s_1)$
    - Then either $P(|0\rangle : s_1) = 0$ or $P(|1\rangle : s_1) = 0$
    - Implying either $P(|00\rangle : |EPR\rangle) = 0$ or $P(|11\rangle : |EPR\rangle) = 0$
  - We say that the EPR state is correlated.
- Vector representation of the EPR state is $|0\rangle|0\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Physical implementation of a 2-Qbit EPR state

- Consider the decay of a π^0-meson into two photons $\pi^0 \to \gamma\gamma$.
- If the pion has a total angular momentum $\mathbf{J} = 0$

$$|\Psi_{\gamma\gamma}\rangle = \frac{1}{\sqrt{2}} \left(|H\rangle_1 |H\rangle_2 + |V\rangle_1 |V\rangle_2 \right),$$

  where $|H\rangle$, $|V\rangle$ are horizontally and vertically polarized photons.
- If photon-1 is horizontally polarized, so is photon-2, etc.

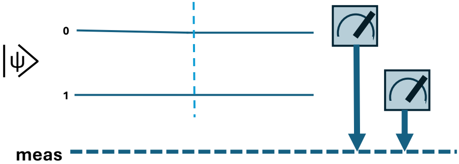
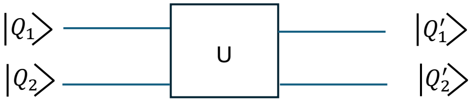


Figure: If the input state is $|\Psi_{\gamma\gamma}\rangle$ then the theoretical measurement is probabilities: $\{HH$: 50%, VV: 50%$\}$

2-Qbit **product** gates – pictorial representation

- Think of this as 2 **uncorrelated** stationary electrons with spin[1]
- Each electron is in its own state, $|Q_i\rangle = \alpha_i|0\rangle + \beta_i|1\rangle$.
- Analogy with a Boolean gate - $|Q_i\rangle$ and $|Q_i'\rangle$ are either $|0\rangle$ or $|1\rangle$



[1] By forced separation, we avoid Fermi statistics.

2-Qbit matrix representation of "product" gates

- $|Q_1 Q_2\rangle \to |v\,,w\rangle = \mathbf{v} \otimes \mathbf{w}$ and $U \to \mathbf{A} \otimes \mathbf{B}$. $U|Q_1 Q_2\rangle \to |\mathbf{A}v\,,\mathbf{B}w\rangle$
- Then

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} \otimes \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \mathbf{A} \otimes \mathbf{B} \left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right)$$

- With the 4 basis states $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \to \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ v_2 w_1 \\ v_2 w_2 \end{pmatrix}$$

- and $\mathbf{A} \otimes \mathbf{B}$ becomes

$$\begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & 0 & A_{11} & A_{12} \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & 0 & 0 \\ 0 & 0 & B_{11} & B_{12} \\ B_{21} & B_{22} & 0 & 0 \\ 0 & 0 & B_{21} & B_{22} \end{pmatrix}$$

Probability-Factorization and Product Gates

- Recall:
  Prob. of measuring $|s_3 s_4\rangle$ if system is in state $|s_1 s_2\rangle$, is $|\langle s_3 s_4|s_1 s_2\rangle|^2$
- Interpret by saying the two Qbits are uncorrelated:
  - $P(|s_3 s_4\rangle : |s_2 s_1\rangle) = P(|s_3\rangle : |s_1\rangle)P(|s_4\rangle : |s_2\rangle)$
- What if a product gate $\mathbf{M} = \mathbf{A} \otimes \mathbf{B}$ is applied to $|s_1 s_2\rangle$?
- $P_{\mathbf{M}}(|s_3 s_4\rangle : |s_2 s_1\rangle)$ =prob. of $|s_3 s_4\rangle$ after applying $\mathbf{M}$ to $|s_1 s_2\rangle$.

$$\begin{aligned} P_{\mathbf{M}}(|s_3 s_4\rangle : |s_2 s_1\rangle) &= \left(|\langle s_3|\mathbf{A}|s_1\rangle|^2\right)\left(|\langle s_4|\mathbf{B}|s_2\rangle|^2\right) \\ &= P_{\mathbf{A}}(|s_3\rangle : |s_1\rangle)P_{\mathbf{B}}(|s_4\rangle : |s_2\rangle) \end{aligned}$$

- This is interpreted as following:
  - Assume two *uncorrelated* input Qbits.
  - Then a product gate produces uncorrelated output Qbits.
  - Think of the product gate acting independently on each Qbit.

2-Qbit general states with 2-Qbit product gates

- Pictorial representations sometimes "hard to picture".
- Example: 2-photon EPR state

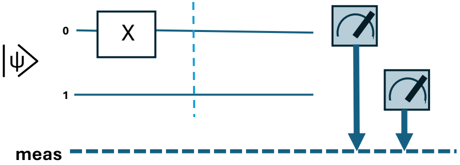


Figure: If the input state is $|\Psi_{\gamma\gamma}\rangle$ then the theoretical measurement is probabilities: $\{HV\colon 50\%,\ VH\colon 50\%\}$

- Matrix representations: $|H\rangle \to |0\rangle$, $|V\rangle \to |1\rangle$; $|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$;
  - $\mathbf{U}_{\mathbf{X} \otimes \mathbf{I}_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

2-Qbit **general** gates – Pictorial representation

- Most general input and output (no Boolean counterpart)

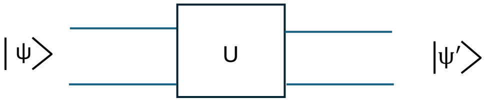


Figure: $|\psi\rangle = \alpha_1|00\rangle + \alpha_2|01\rangle + \alpha_3|10\rangle + \alpha_4|11\rangle$
$|\psi^{'}\rangle = \beta_1|00\rangle + \beta_2|01\rangle + \beta_3|10\rangle + \beta_4|11\rangle$

- Alternative picture for product states

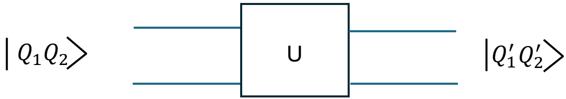


Figure: $|Q_1Q_2\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle)$
$|Q_1'Q_2'\rangle = (\alpha_1'|0\rangle + \beta_1'|1\rangle) \otimes (\alpha_2'|0\rangle + \beta_2'|1\rangle)$

2-Qbit matrix representation of "general" gates

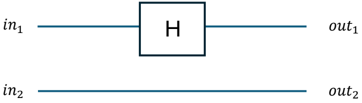
- **Less ambiguous than the pictorial representation**
- Write $|\psi^{'}\rangle = \mathbf{U}|\psi\rangle$.
- In component notation

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ U_{21} & U_{22} & U_{23} & U_{24} \\ U_{31} & U_{32} & U_{33} & U_{34} \\ U_{41} & U_{42} & U_{43} & U_{44} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$$

- $\mathbf{U}$ is unitary (i.e. $\mathbf{U}^{-1} = \mathbf{U}^{\dagger}$)
- In general, $\mathbf{U}$ does not have the form of a product gate.

Another product gate: The Hadamard $\otimes$ Identity gate

$$\mathbf{U}_{\mathbf{Had}\otimes\mathbf{I}_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$



$$\mathbf{v}_{out} = \left(\mathbf{U}_{\mathbf{Had}\otimes\mathbf{I}_2}\right)\mathbf{v}_{in}$$

EXAMPLE

$$\mathbf{v}_{in} = |0\rangle \otimes |0\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then

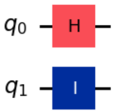
$$\mathbf{v}_{out} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |0\rangle$$

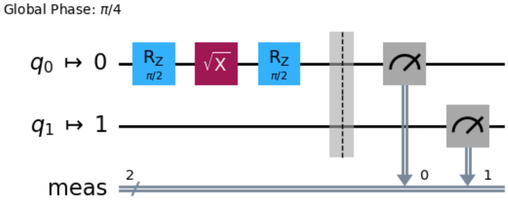
Run the Hadamard $\otimes$ Identity circuit at IBM

- Create the circuit with input Qbits are $|0\rangle$ and $|0\rangle$
- Execute 1000 times
- Print results

```
... PREAMBLE AS BEFORE ...
  ##  First create a circuit locally
  #   1. Create a 2-qubit circuit
  qc = QuantumCircuit(2)
  #   2a. Apply Hadamard gate to qbit 0 [WE CALLED THIS QBIT 1]
  qc.h(0)
  #  2b. Apply identity gate to qbit 1 [WE CALLED THIS QBIT 2]
  qc.id(1)
  #  2c. Display this meta-circuit
  display(qc.draw("mpl"))
  #   3. Measure
  qc.measure_all()

...SET UP IBM HARDWARE AS BEFORE AND DRAW THE TRANSPILED CIRCUIT
  display(isa_circuit.draw("mpl", idle_wires=False))
  ##  Running and analyzing the circuit
  #   8. Create the program to be run on the IBM cloud, which analyzes outputs of quantum circuit
  sampler = Sampler(mode=backend)
  #   9. Run the circuit with 1000 shots
  job = sampler.run([isa_circuit], shots=1000)
  print("Job ID:", job.job_id())
  #   10. Get results (waits until job finishes)
  result = job.result()
  #   11. Extract counts
  counts = result[0].data.meas.get_counts()
  #  12.  Change to probabilities and print
  total = sum(counts.values())
  probs = {k: v / total for k, v in counts.items()}
  print("Probabilities:", probs)
```

Results from the Hadamard $\otimes$ Identity run



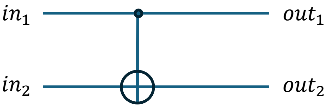



```
Job ID: d7b9sbh5a5qc73dn7b20
Probabilities: {'01': 0.489, '00': 0.508, '10': 0.003}
```

- '01' : 0.489 *means 0.489 is the fraction of outcomes with* **Output Qbit #1 has** ***z*****-value 0 and Output Qbit #2 has** ***z*****-value 1**
- We see that about half the results are $q0 = |0\rangle$, half are $q0 = |1\rangle$.
- In each of those results, $q1 = |0\rangle$
- A **small** fraction (0.003) differ.

A general gate: The CNOT gate

$$\mathbf{U_{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



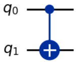
$$\text{"Truth table" is} \begin{pmatrix} \mathbf{in}_1 & \mathbf{in}_2 & \mathbf{out}_1 & \mathbf{out}_2 \\ |0\rangle & |0\rangle & |0\rangle & |0\rangle \\ |0\rangle & |1\rangle & |0\rangle & |1\rangle \\ |1\rangle & |0\rangle & |1\rangle & |1\rangle \\ |1\rangle & |1\rangle & |1\rangle & |0\rangle \end{pmatrix}$$

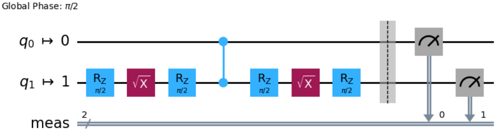
Run the CNOT circuit at IBM

- Create the circuit with input Qbits are $|0\rangle$ and $|0\rangle$
- Execute 10000 times
- Print results

```
......

#  Create a 2-qubit circuit
qc = QuantumCircuit(2)

#  Perform a CNOT (controlled-X) gate on qubit 1, controlled by qubit 0
qc.cx(0, 1)

......

#  Run the circuit with the number of shots
job = sampler.run([isa_circuit], shots=10000)

......
```

Results from the CNOT run






```
Job ID: d7co1qjklj2c73f0q9p0
Probabilities: {'00': 0.9454, '10': 0.052, '01': 0.0015, '11': 0.0011}
```

- The transpiled circuit, in matrix notation is

$$\mathbf{U}_{\mathbf{CNOT}} = \mathbf{U}_{\mathbf{I}_1 \otimes \mathbf{Had}} \mathbf{U}_{\mathbf{CZ}} \mathbf{U}_{\mathbf{I}_1 \otimes \mathbf{Had}} = \mathbf{U}_{\mathbf{I}_1 \otimes \mathbf{Had}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \mathbf{U}_{\mathbf{I}_1 \otimes \mathbf{Had}}$$

- '00' : 0.9454 *means 0.9454 is the fraction of outcomes with*

  Output Qbit #1 has z-value 0 and Output Qbit #2 has z-value 0
- We see that about 5% of the results come from other Qbit combos.
- About 5% of results are *wrong*.

Another general gate: 2-Qbit EPR gate (*Bell* gate) – matrix representation

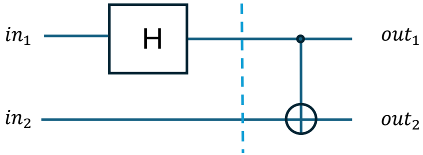
- Input the product state (2 independent Qbits) $\mathbf{v}_{in} = |0\rangle|0\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.
- Construct gate with matrix $\mathbf{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$.
- Apply the gate to the input-state: $\mathbf{v}_{out} = \mathbf{U}\mathbf{v}_{in}$.
- $\mathbf{v}_{out} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} (|0\rangle|0\rangle + |1\rangle|1\rangle)$.
- This is the EPR state $|EPR\rangle$ (Sometimes called the (*Bell* state))

The EPR gate – pictorial representation

- There are **not** single physical components corresponding to each possible gate (**U**).
- Instead, we build gates from standard components.
  - One standard gate followed by another, by another, looks like (in matrix representation) $\mathbf{U} = \mathbf{U}_1\mathbf{U}_2..\mathbf{U}_N$

$$\mathbf{U}_{\mathbf{EPR}} = \mathbf{U}_{\mathbf{CNOT}}\mathbf{U}_{\mathbf{Had}\otimes\mathbf{I}_2}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$



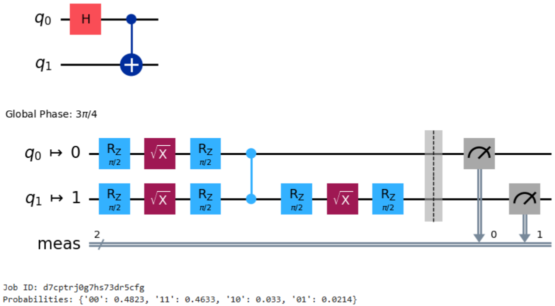
We previously saw that $U_{\mathbf{EPR}} : |0\rangle|0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$.

Run the EPR circuit at IBM

- Create the circuit with input Qbits are $|0\rangle$ and $|0\rangle$
- Execute 10000 times
- Print results

```
......

##  First create a circuit locally
# Create a new circuit with two qubits
qc = QuantumCircuit(2)

# Add a Hadamard gate to qubit 0
qc.h(0)

# Perform a controlled-X gate on qubit 1, controlled by qubit 0
qc.cx(0, 1)

......

# Run the circuit with the number of shots
job = sampler.run([isa_circuit], shots=10000)

......
```

Results from the EPR run



- '00' : .4823, '11': .4633 means *0.4823 is fraction of outcomes with*
  Output Qbit #1 has z-value 0 and Output Qbit #2 has z-value 0
  and *0.4633 is the fraction of outcomes with*
  Output Qbit #1 has z-value 1 and Output Qbit #2 has z-value 1
- The states $|0\rangle|0\rangle$ and $|1\rangle|1\rangle$ are about equally likely
- We see that about 5% of the results come from other Qbit combos.
- About 5% of results are *wrong*.

EPR with rotation

- Recall

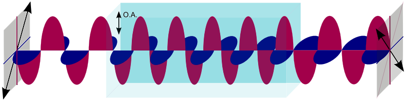


Figure: Half-wave plate. Different refractive indices for vertical and horizontal polarizations.

In code, this is `qc.z(q)` , operating on qbit q.

The matrix representation is $\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- Now rotate by π/x: $U = \mathbf{R}(\pi/x)\mathbf{T}\mathbf{R}(-\pi/x)$

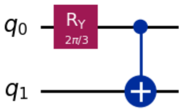
  `qc.ry(2*np.`pi/x`,0)` $\rightarrow \left(\cos\left(\frac{\pi}{x}\right)|0\rangle - \sin\left(\frac{\pi}{x}\right)|1\rangle\right)$.

EPR with rotation – results

- Hadamard gate → rotation gate; don't print transpiled circuit

```
# Add a rotation gate to qubit 0
# "np.pi" is the Python library value of pi
qc.ry(2*np.pi/3,0)
```

- New circuit should produce $\left(\cos\left(\frac{\pi}{3}\right)|0\rangle|0\rangle - \sin\left(\frac{\pi}{3}\right)|1\rangle|1\rangle\right)$




```
Job ID: d7e5sc95a5qc73dqig8g
Probabilities: {'11': 0.7205, '00': 0.257, '01': 0.0175, '10': 0.005}
```

- '00' : .257, '11' : .7205 means *0.257 is fraction of outcomes with* **Output Qbit #1 has** z**-value 0 and Output Qbit #2 has** z**-value 0** and *0.7205 is the fraction of outcomes with* **Output Qbit #1 has** z**-value 1 and Output Qbit #2 has** z**-value 1**
- Prob. of $|0\rangle|0\rangle$ is 0.25; Prob. of $|1\rangle|1\rangle$ is 0.75
- We see that about 2% of the results come from other Qbit combos.