Quantum Computing – The quantum theory of Bell and CHSH

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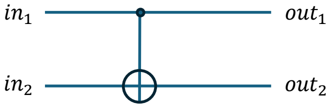
Overview

- We discussed Bell's theorem without doing QM
- We identified entanglement as the key new ingredient of QM
- Some examples of the practical exploitation of entanglement
  - Quantum computing
  - Quantum teleportation and networks
  - The no-cloning theorem and cryptography
- This presentation:
  - Re-cast Bell's theorem in the language of quantum circuits
  - Then discuss a related experiment: The CHSH game
- More details can be found in Wright kecture 3 `https://www.cs.cmu.edu/~odonnell/quantum15/QuantumComputationScribeNotesByRyanODonnellAndJohnWright.pdf`

A review of the Hadamard and CNOT gates

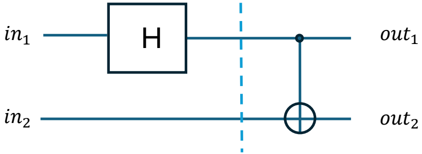
- Hadamard gate

- CNOT gate



$$\text{“Truth table” is } \begin{pmatrix} \textbf{in}_1 & \textbf{in}_2 & \textbf{out}_1 & \textbf{out}_2 \\ |0\rangle & |0\rangle & |0\rangle & |0\rangle \\ |0\rangle & |1\rangle & |0\rangle & |1\rangle \\ |1\rangle & |0\rangle & |1\rangle & |1\rangle \\ |1\rangle & |1\rangle & |1\rangle & |0\rangle \end{pmatrix}$$

A review of the EPR circuit

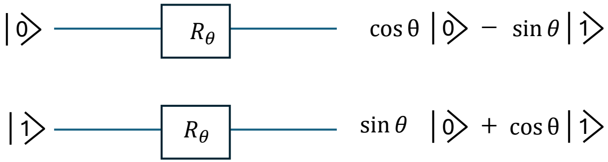


$$\mathbf{U_{EPR}} = \mathbf{U_{CNOT}}\mathbf{U_{Had \otimes I_2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

- So $U_{\mathbf{EPR}} : |0\rangle|0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$.
- Interpretation: This circuit produces an entangled state.
  - If observer 1 measures $|0\rangle$, so does observer 2.
  - If observer 1 measures $|1\rangle$, so does observer 2.

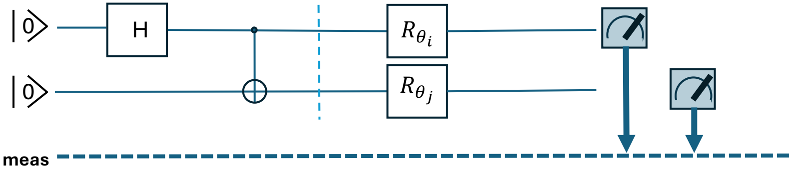
A review of rotation gates

The rotation gate R_θ in the z-basis represents a rotation around the y-axis of -2θ.



The Bell experiment using quantum circuits

- A "factory" produces an EPR pair (before dotted line)
- The first and second qbits of the EPR pair are separated
- Each qbit passed into a detector
- The detector is a R_{θ_i}-gate, followed by a z-measuremnt.
- There are three settings; R_{θ_1}, R_{θ_2}, R_{θ_3}



Predictions of the Bell circuits

- Each rotation acts on its qbit.

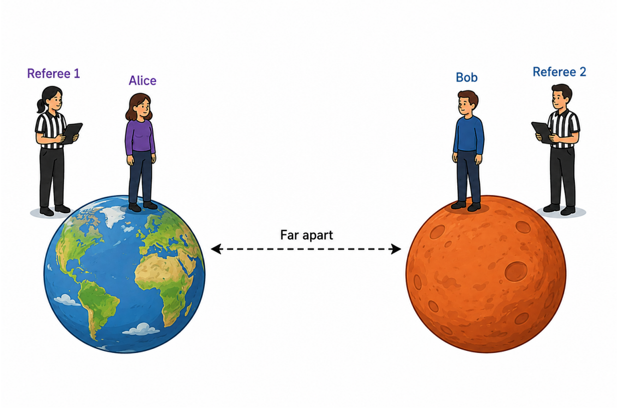
$$\left(R_{\theta_i} \otimes R_{\theta_j}\right)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) = \frac{1}{\sqrt{2}}\left(\cos\theta_i|0\rangle - \sin\theta_i|1\rangle\right) \otimes \left(\cos\theta_j|0\rangle - \sin\theta_j|1\rangle\right) + \frac{1}{\sqrt{2}}\left(\sin\theta_i|0\rangle + \cos\theta_i|1\rangle\right) \otimes \left(\sin\theta_j|0\rangle + \cos\theta_j|1\rangle\right)$$
$$= \frac{1}{\sqrt{2}}\left(\cos\Delta_{ij}|00\rangle + \sin\Delta_{ij}|01\rangle - \sin\Delta_{ij}|10\rangle + \cos\Delta_{ij}|11\rangle\right)$$

where $\Delta_{ij} \equiv \theta_i - \theta_j$.

- The probability that both qbits are the same is $P_{ij} = \cos^2\Delta_{ij}$.
- Take the average of the probabilities for all 9 measurements.
  - $P_{\text{av}}^{\text{QM}} = \frac{1}{9}\left(P_{11} + P_{12} + P_{13} + P_{21} + P_{22} + P_{23} + P_{31} + P_{32} + P_{33}\right)$
- Take $\theta_1 = 0, \theta_2 = \frac{\pi}{3}, \theta_3 = \frac{2\pi}{3}$.
  - Then $P_{\text{av}}^{\text{QM}} = \frac{1}{2} < \frac{5}{9}$
  - REMEMBER: The Bell inequality is $P \geq \frac{5}{9}$.
  - So $P_{\text{av}}^{\text{QM}}$ violates the Bell inequality.

The CHSH game: Setup

- Published by Clauser, Horne, Shimony, Holt in 1969.
- A generalization of Bell's theorem.



- Referee 1 randomly picks a bit "x" (0 or 1) and tells Alice its value
- Referee 2 randomly picks a bit "y" (0 or 1) and tells Bob its value

"Randomly" means 50/50

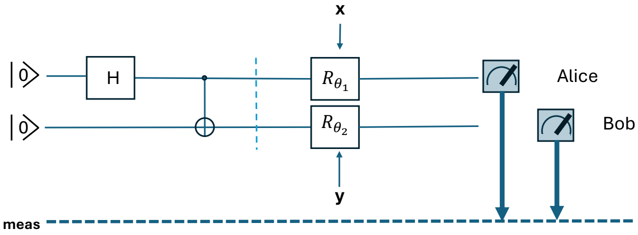
The CHSH game: The game

- Alice and Bob are a team
- Alice and Bob agree on a strategy for picking bits "a" and "b"
  - Alice picks her bit "a" (0 or 1) based on what referee 1 tells her
  - Bob picks his bit "b" (0 or 1) based on what referee 2 tells him
- Bob and Alice can't communicate before they pick their bits
- Later, they communicate:
  - If $(x, y) = (1, 1)$ then the team scores if and only if $a \neq b$.
  - If $(x, y) \neq (1, 1)$ then the team scores if and only if $a = b$.
- The team with the winning strategy has the highest average score.

The CSHS game: Classical limits

- Strategy 1: Bob and Alice always set $(a, b) = (0, 0)$
  - The case $(x, y) = (1, 1)$ occurs only 25% of the time.
  - When that happens, $a = 0 = b$ and the team does **not** score.
  - The rest of the time, the team scores because $a = 0 = b$.
  - The team's average score is 0.75.
- Any deterministic strategy:
  - Alice must set $a(x)$ to one of:
    $a(x) = 0$ <u>or</u> $a(x) = 1$ <u>or</u> $a(x) = x$ <u>or</u> $a(x) = (1 - x)$.
  - Bob must set $b(y)$ to one of:
    $b(y) = 0$ <u>or</u> $b(y) = 1$ <u>or</u> $b(y) = y$ <u>or</u> $b(y) = (1 - y)$.
  - Compute the average score for each of the 16 combinations.
  - **NO DETERMINISTIC STRATEGY BEATS 0.75**
- A randomized strategy:
  - Probability distributions of the deterministic strategies.
  - **So still bounded by 0.75**

The CSHS game: Quantum advantage



- If $x = 0$ then $\theta_1 = 0$; if $x = 1$ then $\theta_1 = \frac{\pi}{4}$
- If $y = 0$ then $\theta_2 = \frac{\pi}{8}$; if $y = 1$ then $\theta_2 = -\frac{\pi}{8}$

x	y	θ_1	θ_2	$\Delta_{12} = \theta_1 - \theta_2$	Pr[win]
0	0	0	$\pi/8$	$-\pi/8$	$\cos^2(-\pi/8) = \cos^2(\pi/8)$
0	1	0	$-\pi/8$	$\pi/8$	$\cos^2(\pi/8)$
1	0	$\pi/4$	$\pi/8$	$\pi/8$	$\cos^2(\pi/8)$
1	1	$\pi/4$	$-\pi/8$	$3\pi/8$	$\sin^2(3\pi/8) = \cos^2(\pi/8)$

- The average win probability is $\cos^2 \frac{\pi}{8} \approx 0.85 > \frac{3}{4}$.
  - THIS IS LARGER THAN THE CLASSICAL BOUND.